Buchstaber invariant, 2-surfaces and matroids

Nikolay Erokhovets

Lomonosov Moscow State University

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Toric topology

Canonical correspondence

simplicial complex K $\dim K = n-1$ number of vertices $= m$	$\longrightarrow$	moment-angle complex $\mathcal{Z}_K$ $\dim \mathcal{Z}_K = m+n$ canonical T^m-action
Combinatorics of K	$\longleftrightarrow$	Topology of $\mathcal{Z}_K$

K-power

$A \subset X$ – a pair of topological spaces.

$$(X,A)^K = \bigcup_{\sigma \in K} X^\sigma \times A^{[m]\setminus\sigma} \subset X^m,$$

where $X^\sigma \times A^{[m]\setminus\sigma} = X_1 \times \cdots \times X_m$, $X_i = \begin{cases} X, & i \in \sigma \\ A, & i \in [m] \setminus \sigma \end{cases}$

Special cases

$D^2 = \{\boldsymbol{z} \in \mathbb{C} \colon |\boldsymbol{z}| \leqslant 1\}$, $S^1 = \{\boldsymbol{z} \in \mathbb{C} \colon |\boldsymbol{z}| = 1\}$,

$(D^2, S^1)^K$ – a moment-angle complex $\mathcal{Z}_K$.

$D^1 = \{x \in \mathbb{R} \colon |x| \leqslant 1\}$, $S^0 = \{\pm 1\}$

$(D^1, S^1)^K$ – a real moment-angle complex $\mathbb{R}\mathcal{Z}_K \subset \mathcal{Z}_K$.

There are canonical coordinate actions:
$T^m = (S^1)^m$ on $\mathcal{Z}_K$, and $(S^0)^m \simeq \mathbb{Z}_2^m$ on $\mathbb{R}\mathcal{Z}_K$.

Definition

A Buchstaber invariant $s(K)$ is the maximal dimension r of toric subgroups $H \subset T^m$, $H \simeq T^r$, that act freely on $\mathcal{Z}_K$.
A real Buchstaber invariant $s_{\mathbb{R}}(K)$ is the maximal dimension r of subgroups $H_2 \subset \mathbb{Z}_2^m$ that act freely on $\mathbb{R}\mathcal{Z}_K$.

$$s(\Delta^{n-1}) = s_{\mathbb{R}}(\Delta^{n-1}) = 0,$$

$$1 \leqslant s(K) \leqslant s_{\mathbb{R}}(K) \leqslant m-n, \qquad K \neq \Delta^{n-1}$$

Let $K = \partial(P^n)^*$

$s(P) = m-n \Leftrightarrow \exists$ quasitoric manifold $M^{2n} = \mathcal{Z}_P/T^{m-n},\ M/T^n = P.$

$s_{\mathbb{R}}(P) = m-n \Leftrightarrow \exists$ small cover $N^n = \mathbb{R}\mathcal{Z}_P/\mathbb{Z}_2^{m-n},\ N/\mathbb{Z}_2^n = P.$

Buchstaber problem

Problem (V. M. Buchstaber, 02)

To find an EFFECTIVE combinatorial description of $s(K)$.

Modifications (V. M. Buchstaber, 12)

Problem'

For any n to calculate $s(K)$ for all simplicial complexes with $\dim K = n - 1$.

Problem"

For any r to characterize combinatorially simplicial complexes K with $s(K) = r$.

Two descriptions of a toric subgroup $\Rightarrow$

(S) $s(K)$ is the maximal r that admits a matrix $S \in \mathbb{Z}^{m\times r}$ such that for any $\sigma \in K$, the rows $\{S^i : i \in [m] \setminus \sigma\}$ span $\mathbb{Z}^r$;

(Λ) $s(K)$ is the maximal r that admits a characteristic mapping $\Lambda\colon [m] \to \mathbb{Z}^{m-r}$ such that for any simplex $\sigma \in K$ the vectors $\Lambda(\sigma)$ form part of a basis in $\mathbb{Z}^{m-r}$.

Two descriptions of a linear subspace in $\mathbb{Z}_2^m$ $\Rightarrow$

(S_2) $s_{\mathbb{R}}(K)$ is the maximal r that admits a matrix $S \in \mathbb{Z}_2^{m\times r}$ such that for any $\sigma \in K$, the rows $\{S^i : i \in [m] \setminus \sigma\}$ span $\mathbb{Z}_2^r$;

(Λ_2) $s_{\mathbb{R}}(K)$ is the maximal r that admits a characteristic mapping $\Lambda\colon [m] \to \mathbb{Z}_2^{m-r}$ such that for any simplex $\sigma \in K$ the vectors $\Lambda(\sigma)$ are linearly independent.

Lemma

1. $s(K) \geqslant r \iff s_{\mathbb{R}}(K) \geqslant r$ *for* $r = 1, 2, 3$.
2. $s(K) = s_{\mathbb{R}}(K)$ *for* $\dim K = 0, 1, 2$.

The proof is based on the following fact

Lemma

For a matrix $A \in \{0,1\}^{r \times r}$, $r = 1, 2, 3$, *the equality* $\det A = 1 \mod 2$ *implies* $\det A = \pm 1$. *For* $r \geqslant 4$ *this is not true.*

Generalized chromatic number

A chromatic number $\gamma(K)$ = minimal r such that $\exists$ a coloring of $\mathrm{vert}(K)$ in r colors with i, j – different color if $\{i, j\} \in K$.
(=non-degenerate simplicial mapping $K \to \Delta^{r-1}$)

$s_{\mathbb{R}}(K)$ [resp. $s(K)$] = minimal r such that $\exists$ a coloring of $\mathrm{vert}(K)$ in vectors in $\mathbb{Z}_2^{m-r}$ [resp. $\mathbb{Z}^{m-r}$] such that vectors corresponding to vertices of any simplex are linearly independent [resp. form part of a basis].

Problem': Polytopes

- $s(P^0) = 0$;
- $s(P^1) = 1$;
- $s(P^2) = m - 2$;
- $s(P^3) = m - 3$ due to the 4-colors theorem;
- $s(P^4)$ can be less than $m - 4$ (polytopes dual to cyclic).

Problem': Simplicial complexes

- If $\dim K = 0$ then $s(K) = m - 1$.

Theorem (Ayzenberg, 09)

If $\dim K = 1$, *then* $s(K) = m - \lceil \log_2(\gamma(K) + 1) \rceil$.
In general case $s(K) \leqslant m - \lceil \log_2(\gamma(K) + 1) \rceil$.

Theorem (E, 13)

If $\dim K = 2$, *then*

$$m - 1 - \lceil \log_2(\gamma(K)) \rceil \leqslant s(K) \leqslant m - \lceil \log_2(\gamma(K) + 1) \rceil.$$

In particular, if $\gamma(K) = 2^k$, *then* $s(K) = m - k - 1$.

Moreover

$$s(K) \leqslant m - 1 - \left\lceil \log_2 \max \left\{ \frac{\gamma(K)+1}{2}, \gamma(\mathrm{lk}_K v) + 1 \right\}_{v \in [m]} \right\rceil$$

Corollary (A. Ayzenberg, M. Masuda & Y. Fukukawa,09)

For the 2-skeleton of an $(m-1)$-simplex we have

$$s(\Delta^{m-1}_{(2)}) = m - 1 - \lceil \log_2 m \rceil .$$

2-surfaces

Denote $\mathcal{H}(M) = \frac{7+\sqrt{49-24\chi(M)}}{2}$.
Classics: for any triangulation K of compact closed 2-surface M with m vertices

$$\gamma(K) \leqslant [\mathcal{H}(M)]; \quad m \geqslant \lceil \mathcal{H}(M) \rceil$$

Corollary

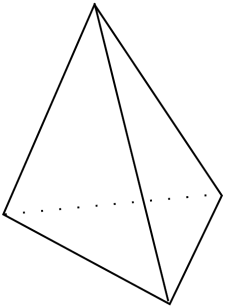
For any triangulation of compact closes 2-surface K

$$s(K) \geqslant m - 1 - \lceil \log_2 [\mathcal{H}(K)] \rceil \geqslant \lceil \mathcal{H}(K) \rceil - 1 - \lceil \log_2 [\mathcal{H}(K)] \rceil.$$

Sphere

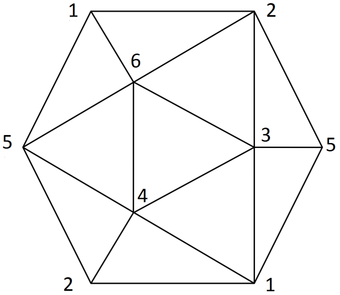
$\mathcal{H}(S^2) = 4$ and $\gamma(K) \leqslant 4 \Rightarrow s(K) = m - 3 \geqslant 1$ for any triangulation K.

$s(K) \geqslant 1$ and $s(K) = 1 \Leftrightarrow K = K_0 = \partial\Delta^3$.



Projective plane

$\mathcal{H}(\mathbb{R}P^2) = 6$ and $\gamma(K) \leqslant 6 \Rightarrow m - 4 \leqslant s(K) \leqslant m - 3$.
There is a unique triangulation K_0 with $m = 6$. $\gamma(K_0) = 6$.

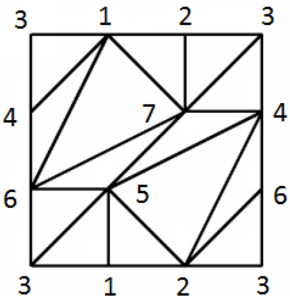


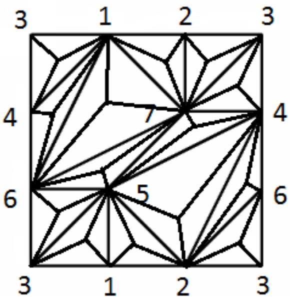
1	**(1,0,0)**
2	**(0,1,0)**
3	**(1,1,0)**
4	**(0,0,1)**
5	**(1,0,1)**
6	**(0,1,1)**

Characteristic map $\Rightarrow s(K_0) = m - 3 = 3$. $K \neq K_0 \Rightarrow m \geqslant 7$ and $s(K) \geqslant m - 4 \geqslant 3$. So $s(K) \geqslant 3$ and $s(K_0) = 3$.

Torus

$\mathcal{H}(T^2) = 7$ and $\gamma(K) \leqslant 7 \Rightarrow m - 4 \leqslant s(K) \leqslant m - 3$.
There is a unique triangulation K_0 with $m = 7$. $\gamma(K_0) = 7$





No characteristic map to $\mathbb{Z}_2^3 \Rightarrow s(K_0) = m - 4 = 3$.
$K_0 \longrightarrow K_1$ stellar subdivision of all triangles. Still $\gamma(K_1) = 7$, but $s(K_1) = m - 3 = 18$. So $s(K) \geqslant 3$ and $s(K) = 3 \Leftrightarrow K = K_0$.

Matroids

Combinatorial analogs of vector configurations.

Definition

Abstract simplicial complex M is called a matroid if for any $\sigma_1, \sigma_2 \in M$ with $|\sigma_1| < |\sigma_2|$ there is $i \in \sigma_2 \setminus \sigma_1$ such that $\sigma_1 \cup \{i\} \in M$.

Maximal simplices $=$ bases, minimal non-simplices $=$cycles.
A binary matroid is a collection of all the linearly independent subsets in the configuration of m vectors in $\mathbb{Z}_2^k$.

Alexander dual

Definition

Let $m \geqslant n+2$. An Alexander dual simplicial complex $\widehat{K}$ is

$$\widehat{K} = \{\sigma \subset [m] \colon [m] \setminus \sigma \notin K\}.$$

Maximal simplices of $\widehat{K}$ = complements to minimal non-simplices of K.

$$K_1 \subset K_2 \Longleftrightarrow \widehat{K}_2 \subset \widehat{K}_1.$$

K is **k-neighborly** if $\sigma \in K$ for all $\sigma \subset [m]$ with $|\sigma| \leqslant k$.

Theorem

Let $m \geqslant n+2$. Then $s_{\mathbb{R}}(K) \geqslant r$ if and only if there is an $(r-1)$-neighborly subcomplex $\widehat{M} \subset \widehat{K}$ such that $(\sigma_1 \cap \sigma_2) \cup \{i,j\} \in \widehat{M}$ for any two distinct maximal simplices $\sigma_1, \sigma_2 \in \widehat{M}$ and possibly coinciding vertices $i,j \notin \sigma_1 \cup \sigma_2$.

Idea

Theorem is based on the following fact from the matroid theory: *$\mathcal{C} = \{C_k \subset [m]\}$ is a collection of minimal non-simplices of a binary matroid if and only if*

- $\varnothing \notin \mathcal{C}$;
- $C_i \not\subset C_j$ for $i \neq j$;
- *for any distinct $C_1, C_2 \in \mathcal{C}$ and any $\{i, j\} \subset C_1 \cap C_2$, possibly $i = j$, there is $C_3 \in \mathcal{C}$ such that $C_3 \subset C_1 \cup C_2 \setminus \{i, j\}$.*

Questions

1. To proof that $s(P^3) = m - 3$ without 4-colors theorem.
2. To find an exact formula for $s(K)$ for $\dim K = 2$.
3. To simplify the criterion for $s_{\mathbb{R}}(K) \geqslant r$ in terms of combinatorics of K.

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Thank You for Your Attention!