Smooth structures on moment-angle complexes for simplicial posets
(joint work with Mikiya Masuda)

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K : abstract simplicial complex on $[m] := \{1, \dots, m\}$
i.e. $K \subseteq 2^{[m]}$ such that $\tau \subset \sigma \in K \Longrightarrow \tau \in K$.

$$|K| = \bigcup \Delta^\sigma,$$

where

$$\Delta^\sigma := \left\{ \sum_{i \in \sigma} a_i e_i \mid a_i \geq 0, \sum_{i \in \sigma} a_i = 1 \right\} \subset \mathbb{R}^m.$$

We set

$$(\mathbb{D}, S^1)^\sigma := \prod_{i=1}^{m} Y_i,$$

where

$$Y_i = \begin{cases} \mathbb{D} & \text{if } i \in \sigma, \\ S^1 & \text{if } i \notin \sigma. \end{cases}$$

$$\mathcal{Z}_K := \bigcup_{\sigma \in K} (\mathbb{D}, S^1)^\sigma \subseteq \mathbb{D}^m$$ moment-angle complex

Any abstract simplicial complex K is a poset, and hence it can be regarded as a category.

$$\mathrm{Ob}(K) := \{\sigma \in K\},$$

For $\sigma, \tau \in K$,

$$\mathrm{Hom}_K(\sigma, \tau) := \begin{cases} \{(\sigma, \tau)\} & \text{if } \sigma \subseteq \tau, \\ \emptyset & \text{otherwise.} \end{cases}$$

We have 2 functors from K to Top.

1. $\Delta^K : \sigma \mapsto \Delta^\sigma$, $(\sigma, \tau) \mapsto (\Delta^\sigma \hookrightarrow \Delta^\tau)$,
2. $(\mathbb{D}, S^1)^K : \sigma \mapsto (\mathbb{D}, S^1)^\sigma$, $(\sigma, \tau) \mapsto ((\mathbb{D}, S^1)^\sigma \hookrightarrow (\mathbb{D}, S^1)^\tau)$.

$|K|$ and $\mathcal{Z}_K$ are nothing but the colimits of Δ^K and $(\mathbb{D}, S^1)^K$.

Definition

A simplicial poset $\mathcal{P}$ is a finite partially ordered set $(\mathcal{P}, \subseteq)$ such that

1. $\mathcal{P}$ has the unique minimal element $\emptyset$,
2. every interval
$$[\emptyset, \sigma] := \{\tau \in \mathcal{P} \mid \emptyset \subseteq \tau \subseteq \sigma\}$$
is a simplex (as a poset).

We give a numbering

$$V^0 : \mathcal{P}^0 \to [m] \quad \text{injective}$$

for vertices in $\mathcal{P}$ and extend it to the whole of $\mathcal{P}$:

$$V(\sigma) := \{V^0(v) \mid v \in \mathcal{P}^0, v \subseteq \sigma\} \subseteq \{1, \dots, m\}.$$

$V : (\mathcal{P}, \subseteq) \to (2^{[m]}, \subseteq)$ is a functor.

We have 2 functors $(2^{[m]}, \subseteq) \to \mathrm{Top}$:

1. $\Delta^{2^{[m]}} : I \mapsto \Delta^I \subset \mathbb{R}^m$, $(I, J) \mapsto (\Delta^I \hookrightarrow \Delta^J)$,
2. $(\mathbb{D}, S^1)^{2^{[m]}} : I \mapsto (\mathbb{D}, S^1)^I$, $(I, J) \mapsto ((\mathbb{D}, S^1)^I \hookrightarrow (\mathbb{D}, S^1)^J)$.

By composing with $V : (\mathcal{P}, \subseteq) \to (2^{[m]}, \subseteq)$, we have 2 functors $(\mathcal{P}, \subseteq) \to \mathrm{Top}$.

1. $|\mathcal{P}| := \operatorname{colim} \Delta^{2^{[m]}} \circ V$ geometric realization.
2. $\mathcal{Z}_\mathcal{P} := \mathcal{Z}_{\mathcal{P},V} := \operatorname{colim} (\mathbb{D}, S^1)^{2^{[m]}} \circ V$ moment-angle complex.

$$(S^1)^m \curvearrowright \mathcal{Z}_\mathcal{P}.$$

Theorem (Buchstaber-Panov)

K : abstract simplicial complex on $[m]$, $|K| \approx S^{n-1}$.
Then, $\mathcal{Z}_K$ is a topological manifold of dimension $m+n$.

Theorem (Panov-Lü)

$\mathcal{P}$: simplicial poset, $|\mathcal{P}| \approx S^{n-1}$.
$V : \mathcal{P} \to 2^{[m]}$ numbering.
Then, $\mathcal{Z}_{\mathcal{P},V}$ is a topological manifold of dimension $m+n$.

Theorem (Panov-Ustinovsky)

If K is star-shaped sphere, then

1. $\mathcal{Z}_K$ admits a smooth structure invariant under the action of $(S^1)^m$,
2. $\mathcal{Z}_K$ admits a complex structure invariant under the action of $(S^1)^m$ if $m+n$ is even.

Theorem (I.)

$\mathcal{Z}_\mathcal{P}$ admits a complex structure invariant under the action of $(S^1)^m$ if and only if $\mathcal{P}$ is a star-shaped sphere (in particular, simplicial complex) and $m+n$ is even.

Problem

Find a necessary and sufficient condition for K (or $\mathcal{P}$) so that the corresponding moment-angle complex admits a smooth structure invariant under the natural torus action.

1. A necessary condition for $\mathcal{P}$ so that $\mathcal{Z}_{\mathcal{P}}$ admits a structure of a smooth $(S^1)^m$-manifold.
   - An example of $\mathcal{P}$ such that $\mathcal{Z}_{\mathcal{P}}$ admits a structure of a topological $(S^1)^m$-manifold but not smooth $(S^1)^m$-manifold.
2. A sufficient condition for $\mathcal{P}$ so that $\mathcal{Z}_{\mathcal{P}}$ admits a structure of a smooth $(S^1)^m$-manifold.
3. Uniqueness of such a structure on $\mathcal{Z}_{\mathcal{P}}$ up to equivariant diffeomorphisms.

Purpose

Want : If $\mathcal{Z}_{\mathcal{P}}$ is a smooth $(S^1)^m$-manifold, then $\mathcal{Z}_{\mathcal{P}}/(S^1)^m$ is a manifold with boundary, i.e. $\mathcal{Z}_{\mathcal{P}}/(S^1)^m$ is locally homeomorphic to an open subset of the upper half space of $\mathbb{R}^n$.

Suppose $\mathcal{Z}_{\mathcal{P}}$ admits a structure of a smooth $(S^1)^m$-manifold (fix).

$$\mathcal{Z}_{\mathcal{P}} = \bigsqcup_{\sigma \in \mathcal{P}} ((\mathbb{D}, S^1)^{V(\sigma)} \times \{\sigma\}) / \sim .$$

For $\sigma \in \mathcal{P}$,

$$p_\sigma := [(z_1, \ldots, z_m), \sigma] \in \mathcal{Z}_{\mathcal{P}},$$

where

$$z_i = \begin{cases} 0 & \text{if } i \in V(\sigma) \\ 1 & \text{otherwise.} \end{cases}$$

The isotropy subgroup at p_σ is $(S^1)^\sigma := (S^1, \{1\})^{V(\sigma)} \subseteq (S^1)^m$.

$\mathcal{Y}_\sigma$: the connected component of $\mathcal{Z}_{\mathcal{P}}^{(S^1)^\sigma}$ which contains p_σ (submanifold).

Lemma

$\mathcal{Y}_\sigma \subseteq \mathcal{Y}_\tau$ if and only if $\sigma \supseteq \tau$.

We set

$$\mathcal{Y}_\sigma^0 := \mathcal{Y}_\sigma \setminus \bigcup_{\sigma \subsetneq \tau} \mathcal{Y}_\tau.$$

Lemma

1. $\{\mathcal{Y}_\sigma^0\}_{\sigma \in \mathcal{P}}$ is a partition of $\mathcal{Z}_\mathcal{P}$ i.e.

$$\mathcal{Z}_\mathcal{P} = \bigsqcup_{\sigma \in \mathcal{P}} \mathcal{Y}_\sigma^0.$$

2. $p \in \mathcal{Y}_\sigma^0 \implies$ the isotropy subgroup at p is $(S^1)^\sigma$.
3. codim $\mathcal{Y}_\sigma = 2\dim(S^1)^\sigma$.

Suppose $p \in \mathcal{Y}_\sigma^0$.
As $(S^1)^\sigma$-representations,

$$T_p\mathcal{Z}_\mathcal{P} \cong \underbrace{T_p\mathcal{Y}_\sigma^0}_{\text{trivial}} \oplus \underbrace{(T_p\mathcal{Z}_\mathcal{P}/T_p\mathcal{Y}_\sigma^0)}_{\text{faithful, } \dim = 2\dim(S^1)^\sigma}$$

$$\cong T_p((S^1)^m \cdot p) \oplus \underbrace{(T_p\mathcal{Y}_\sigma^0/T_p((S^1)^m \cdot p))}_{\text{trivial}} \oplus (T_p\mathcal{Z}_\mathcal{P}/T_p\mathcal{Y}_\sigma^0).$$

Therefore

$$(T_p\mathcal{Z}_\mathcal{P}/T_p((S^1)^m \cdot p))/(S^1)^\sigma \approx \mathbb{R}^{n-\dim(S^1)^\sigma} \times (\mathbb{R}_{\geq 0})^{\dim(S^1)^\sigma}$$

$$\approx \begin{cases} \mathbb{R}^n & \text{if } \sigma = \emptyset, \\ \{x \in \mathbb{R}^n \mid x_n \geq 0\} & \text{otherwise.} \end{cases}$$

Lemma

If $\mathcal{Z}_\mathcal{P}$ has a structure of a smooth $(S^1)^m$-manifold, then $\mathcal{Z}_\mathcal{P}/(S^1)^m$ is a manifold with boundary.

On the other hand,

$$\underbrace{\mathcal{Z}_{\mathcal{P}}/(S^1)^m}_{\text{manifold with boundary}} \approx C(|\mathcal{P}|) \quad \text{cone of } |\mathcal{P}|.$$

Lemma

The boundary $\partial(\mathcal{Z}_{\mathcal{P}}/(S^1)^m)$ corresponds to $|\mathcal{P}|$.

Therefore, $|\mathcal{P}|$ should be a manifold.

Lemma

$|\mathcal{P}|$ is a homology sphere.

Proof.

This follows from the l.e.s. for $(C(|P|), |P|)$ and the Lefschetz duality. □

1. $C(|\mathcal{P}|)$ is a manifold with boundary $|\mathcal{P}|$.
2. $|\mathcal{P}|$ is a homology sphere.

Therefore

1. the suspension $\Sigma|\mathcal{P}|$ is a closed manifold,
2. the double suspension $\Sigma^2|\mathcal{P}|$ is a sphere.

Lemma

X : (topological) closed manifold
Suppose ΣX is a sphere. Then, X is a (homotopy) sphere.

Applying the Lemma twice, we have that $|\mathcal{P}|$ should be a sphere of dimension $n-1$.

The same argument works for $\mathcal{Y}_\sigma$.

Lemma

$\partial(\mathcal{Y}_\sigma/(S^1)^m) \approx |\mathrm{link}\,(\sigma;\mathcal{P})| \approx S^{n-2-\dim\sigma}$.

Definition

$(\mathcal{P},\subseteq)$: simplicial poset

For $\sigma\in\mathcal{P}$,

$$\mathrm{link}\,(\sigma;\mathcal{P}) := \{\tau\in\mathcal{P} \mid \sigma\subseteq\tau\}$$

is a simplicial poset with the minimal element σ.

Remark

$\mathrm{link}\,(\emptyset;\mathcal{P}) = \mathcal{P}$.

Theorem (Necessary condition)

$\mathcal{P}$: simplicial poset.
Suppose $\mathcal{Z}_{\mathcal{P}}$ admits a structure of a smooth $(S^1)^m$-manifold. Then, $\mathcal{P}$ satisfies the following ; for each $\sigma \in \mathcal{P}$, $|\operatorname{link}(\sigma, \mathcal{P})|$ is a sphere.

Example

$\mathcal{P}$: simplicial poset, $|\mathcal{P}|$: homology sphere but not sphere.
$\mathcal{Z}_{\Sigma^2 \mathcal{P}}$ is a topological manifold, but it does NOT carry the smooth structure invariant under the natural torus action.

Proposition

$\mathcal{P}$: simplicial poset.
$B\mathcal{P}$: barycentric subdivision of $\mathcal{P}$.
Then, $B\mathcal{P}$ is a simplicial complex.

Definition

$\mathcal{P}$ is called a PL sphere if $B\mathcal{P}$ is a PL sphere.

$$\{\mathcal{P} \mid \mathcal{P} \text{ is a PL sphere}\} \subseteq \{\mathcal{P} \mid \mathcal{P} \text{ satisfies the necessary condition}\}.$$

"=" holds if and only if the PL Poincaré Conjecture is true.

Theorem

$\mathcal{P}$: simplicial poset, PL sphere.
Then, $\mathcal{Z}_{\mathcal{P}}$ admits a structure of a smooth $(S^1)^m$-manifold and it is unique up to equivariant diffeomorphisms.

Theorem (Lickorish)

K : simplicial complex, PL sphere.
Then, K can be obtained from $\partial\Delta^n$ by stellar subdivisions and their inverses.

Definition

$\mathcal{P}$: simplicial poset, PL sphere.

$$\ell(\mathcal{P}) := \text{the minimal length of chains of certain stellar subdivisions and their inverses between } \mathcal{P} \text{ and } \partial\Delta^n.$$

"certain" means
$(\mathcal{P} = \mathcal{P}_0, \mathcal{P}_1, \ldots, \mathcal{P}_\ell = \partial\Delta^n)$: chain,
$\mathcal{P}_{i+1}$ is obtained from $\mathcal{P}_i$ by a (inverse) stellar subdivision along $\sigma \in \mathcal{P}_i$
$\Longrightarrow V_{\text{link}\,(\sigma;\mathcal{P})}$ is a numbering.

$\mathcal{P}$: simplicial poset. $V : \mathcal{P} \to 2^{[m]}$: numbering.

For $\sigma \in \mathcal{P}$ and $\tau \in \mathrm{link}\,(\sigma;\mathcal{P})$, put

$$V_{\mathrm{link}\,(\sigma;\mathcal{P})}(\tau) := V(\tau) \setminus V(\sigma) \in 2^{[m]\setminus V(\sigma)}.$$

The functor $V_{\mathrm{link}\,(\sigma;\mathcal{P})} : \mathrm{link}\,(\sigma;\mathcal{P}) \to 2^{[m]\setminus V(\sigma)}$ may not be a numbering, but a coloring.

Proposition

$\mathcal{Y}_\sigma \approx \mathrm{colim}\ \Delta^{[m]\setminus V(\sigma)} \circ V_{\mathrm{link}\,(\sigma;\mathcal{P})}$ a partial quotient over $\mathrm{link}\,(\sigma;\mathcal{P})$.

The following is the case when $\ell(\mathcal{P}) = 0$.

Theorem (Bosio-Meersseman)

X : smooth $(S^1)^m$-manifold.
Suppose

$$X \approx S^{2n-1} \times (S^1)^{m-n} \quad \text{equivariant homeomorphic.}$$

Then,

$$X \cong S^{2n-1} \times (S^1)^{m-n} \quad \text{equivariant diffeomorphic.}$$

Definition

$\mathcal{P}_i$: simplicial poset, $i = 1, 2$.
$V_i : \mathcal{P}_i \to 2^{[m]}$: numbering.
$f : \mathcal{Z}_{\mathcal{P}_1} \to \mathcal{Z}_{\mathcal{P}_2}$: equivariant homeomorphism.
Then, f induces an isomorphism

$$f_* : \mathcal{P}_1 \to \mathcal{P}_2$$

Theorem (strong version of the previous Theorem)

For a PL sphere $\mathcal{P}$ and a numbering V,

1. $^\exists\mathcal{T}$: invariant smooth structure on $\mathcal{Z}_\mathcal{P}$.
2. $f : \mathcal{Z}_{\mathcal{P}_1} \to \mathcal{Z}_{\mathcal{P}_2}$: equivariant homeomorphism.
   Then, $^\exists f' : (\mathcal{Z}_{\mathcal{P}_1}, \mathcal{T}_1) \to (\mathcal{Z}_{\mathcal{P}_2}, \mathcal{T}_2)$: equivariant diffeomorphism such that
   $$f'_* = f_*.$$

Assume that the Theorem holds for $\mathcal{P}$ such that

$$\begin{cases} n < n_0, \\ n = n_0 \quad \text{and} \quad \ell(\mathcal{P}) < \ell_0. \end{cases}$$

$\mathcal{P}$: simplicial poset, PL sphere of dimension $n_0 - 1$, $\ell(\mathcal{P}) = \ell_0$.
$V : \mathcal{P} \to 2^{[m]}$: numbering.
$\mathcal{P}'$: obtained from $\mathcal{P}$ by stellar subdivision along σ.
$V' : \mathcal{P}' \to 2^{[m+1]}$: numbering obtained from V. Suppose $\ell(\mathcal{P}') = \ell_0 - 1$.

${}^{\exists}\mathcal{T}'$: invariant smooth structure on $\mathcal{Z}_{\mathcal{P}'}$ (by induction hypothesis)
For $v_\sigma \in \mathcal{P}'^0$: extra vertex, $\mathcal{Y}_{v_\sigma}$ is a submanifold.

$$\mathcal{Y}_{v_\sigma} \approx \mathcal{Z}_{\mathrm{link}\,(v_\sigma;\mathcal{P}')} \approx \mathcal{Z}_{\partial\sigma} \times \mathcal{Z}_{\mathrm{link}\,(\sigma;\mathcal{P})}$$
$$\mathcal{Y}_{v_\sigma} \cong S^{2\dim\sigma - 1} \times (\mathcal{Z}_{\mathrm{link}\,(\sigma;\mathcal{P})}, {}^{\exists}\mathcal{T}'') \quad \text{by induction hypothesis.}$$

U : invariant tubular nbd. of $\mathcal{Y}_{v_\sigma}$. $U \cong \mathcal{Y}_{v_\sigma} \times \mathbb{D}$.

Lemma

$\mathcal{Z}_{\mathcal{P}} \approx ((\mathcal{Z}_{\mathcal{P}'} \setminus U) \cup ((S^1)^{v_\sigma} \times B^\sigma \times \mathcal{Z}_{\mathrm{link}\,(\sigma;\mathcal{P})}))/(S^1)^{v_\sigma}$.

An almost same argument works for the case of inverse stellar subdivision (omit).

$\mathcal{P}_i$: simplicial poset, PL sphere, $\ell(\mathcal{P}_i) = \ell_0$ for $i = 1, 2$.
$\mathcal{T}_i$: invariant smooth structure on $\mathcal{Z}_{\mathcal{P}_i}$.
$f : \mathcal{Z}_{\mathcal{P}_1} \to \mathcal{Z}_{\mathcal{P}_2}$: equivariant homeomorphism.
$\mathcal{P}'_i$: obtained from $\mathcal{P}_i$ by stellar (or inverse stellar) subdivision along the "same" simplex (via f_*), $\ell(\mathcal{P}'_i) = \ell_0 - 1$.

$\mathcal{T}'_i$: invariant smooth structure on $\mathcal{Z}_{\mathcal{P}'_i}$ obtained from $\mathcal{T}_i$ by the equivariant surgery.

1. f induces an equivariant homeomorphism $g : \mathcal{Z}_{\mathcal{P}'_1} \to \mathcal{Z}_{\mathcal{P}'_2}$.
2. g can be replaced by an equivariant diffeomorphism $g' : (\mathcal{Z}_{\mathcal{P}'_1}, \mathcal{T}'_1) \to (\mathcal{Z}_{\mathcal{P}'_2}, \mathcal{T}'_2)$ such that $g_* = g'_*$.
3. g' induces an equivariant diffeomorphism $f' : (\mathcal{Z}_{\mathcal{P}_1}, \mathcal{T}_1) \to (\mathcal{Z}_{\mathcal{P}_2}, \mathcal{T}_2)$ such that $f_* = f'_*$.