Decomposing real moment-angle complexes

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Our object

- Let K be a simplicial complex on the vertex set $[m] = \{1, \ldots, m\}$.
- Let $(\underline{X}, \underline{A}) = \{(X_i, A_i)\}_{i \in [m]}$ be a sequence of pairs of spaces.

Definition

The polyhedral product $Z_K(\underline{X}, \underline{A})$ is defined as

$$Z_K(\underline{X}, \underline{A}) = \bigcup_{\sigma \in K} D(\sigma) \quad (\subset X_1 \times \cdots \times X_m)$$

where $D(\sigma) = Y_1 \times \cdots \times Y_m$ for $Y_i = \begin{cases} X_i & i \in \sigma \\ A_i & i \notin \sigma. \end{cases}$

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Our object is the real moment-angle complex

$$\mathbb{Z}_K = Z_K(D^1, S^0)$$

which is fundamental in studying the real version of quasitoric manifolds (called small covers), right-angled Coxeter groups and etc.

Motivation

Generalizing the decomposition

$$\Sigma(X \times Y) \simeq \Sigma X \vee \Sigma Y \vee \Sigma(X \wedge Y),$$

Bahri, Bendersky, Cohen & Gitler decomposed $\Sigma Z_K(\underline{X}, \underline{A})$. As a special case, we have:

Theorem (Bahri, Bendersky, Cohen & Gitler '10)

There is a homotopy equivalence

$$\Sigma Z_K(C\underline{X}, \underline{X}) \simeq \Sigma \bigvee_{\emptyset \neq I \subset [m]} |\Sigma K_I| \wedge \widehat{X}^I$$

where K_I is the maximum subcomplex of K on the vertex set I and $\widehat{X}^I = \bigwedge_{i \in I} X_i$.

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Question

When does this decomposition desuspend?

If the decomposition desuspends, $Z_K(C\underline{X}, \underline{X})$ must be a suspension.
So at least we need the following property for K.

Definition

K is called Golod if all products in $\widetilde{H}^*(Z_K(D^2, S^1))$ are trivial.

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There is a homological characterization of CM complexes.

Theorem (Reisner '76)

K is CM $\iff \widetilde{H}_*(\mathrm{lk}_K(\sigma)) = 0$ *for* $* < \dim \mathrm{lk}_K(\sigma)$ *and* $\forall \sigma \in K$.

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- The Alexander dual of K is defined as

$$K^\vee = \{\sigma \subset [m] \,|\, [m] - \sigma \notin K\}.$$

Theorem (Herzog, Reiner & Welker '99)
If $K^\vee$ is CM, K is Golod.

CM complexes are pure, and sequentially Cohen-Macaulay (SCM) complexes are a nonpure generalization of CM complexes.

- Let $K^{[d]}$ be the subcomplex of K generated by d-dim faces.

Theorem-Definition (Duval '96)

K is SCM $\iff$ $K^{[d]}$ is CM for $\forall d \geq 0$.

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Corollary

K is CM $\iff$ K is SCM and pure.

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Theorem (Herzog, Reiner & Welker '99)

If $K^{\vee}$ is SCM, K is Golod.

Question

Does the decomposition of $\Sigma Z_K(C\underline{X}, \underline{X})$ desuspend if $K^{\vee}$ is SCM?

There are implications of simplicial complexes:

$$\text{shifted} \Rightarrow \text{vertex-decomposable} \Rightarrow \text{shellable} \Rightarrow \text{SCM}$$

The desuspension of the decomposition of $\Sigma Z_K(C\underline{X}, \underline{X})$ was proved when $K^\vee$ is

- shifted by Grbić & Theriault '12, Iriye & K '12, and
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Finally we have:

Theorem (Iriye & K '13)

If $K^\vee$ is SCM and each X_i is a connected finite CW-complex, then

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...but the theorem does not apply to real moment-angle complexes, for which I will explain a new method.

Result

Theorem
If $K^\vee$ is SCM, then

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In showing the previous result, it was proved:

Lemma (Iriye & K '13)

If $K^\vee$ is SCM, then for $\emptyset \neq \forall I \subset [m]$, $|\Sigma K_I|$ has the homotopy type of a wedge of spheres.

Hence we get:

Corollary

If $K^\vee$ is SCM, Z_K has the homotopy type of a wedge of spheres.

Stratification

To prove the main theorem, we

- describe the stratification of Z_K in terms of the cubical subdivision of a simplicial complex in the book of Buchstaber and Panov, and
- show the triviality of strata when $K^\vee$ is SCM.

Stratification

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- describe the stratification of Z_K in terms of the cubical subdivision of a simplicial complex in the book of Buchstaber and Panov, and
- show the triviality of strata when $K^\vee$ is SCM.

Definition

For $i = 0, \ldots, m$, we define

$$Z_K^i = \bigcup_{I \subset [m],\, |I|=i} Z_{K_I}$$

where Z_{K_I} lies in $\{(x_1, \ldots, x_m) \in (D^1)^m \,|\, x_j = -1 \text{ for } j \notin I\}$.

Then we get a stratification

$$* = Z_K^0 \subset Z_K^1 \subset \cdots \subset Z_K^{m-1} \subset Z_K^m = Z_K.$$

Let us recall the cubical subdivision of K.

- For $\sigma \subset \tau \subset [m]$, put

$$C_{\sigma\subset\tau} = \{(x_1, \ldots, x_m) \in (D^1)^m \mid x_i = -1\ (i \in \sigma), +1\ (i \notin \tau)\}$$

  which is a $(|\tau| - |\sigma|)$-dimensional face of $(D^1)^m$.

All faces of $(D^1)^m$ not including $(+1, \ldots, +1)$ are given by $C_{\sigma\subset\tau}$.

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- A piecewise linear map

$$i_c : |\mathrm{Sd}\Delta^{m-1}| \to (D^1)^m, \quad \sigma \mapsto C_{\sigma \subset \sigma}$$

is an embedding, where $\emptyset \neq \sigma \subset [m]$ are vertices of $\mathrm{Sd}\Delta^{m-1}$.

So $i_c(|\mathrm{Sd}\Delta^{m-1}|)$ is the union of all faces of $(D^1)^m$ not including $(+1, \ldots, +1)$, which is regarded as the cubical subdivision of Δ^{m-1}.

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- Define the embedding $\mathrm{Cone}(i_c) : |\mathrm{Cone}(\mathrm{Sd}\Delta^{m-1})| \to (D^1)^m$ as the extension of i_c which sends the cone point to $(+1, \dots, +1)$.

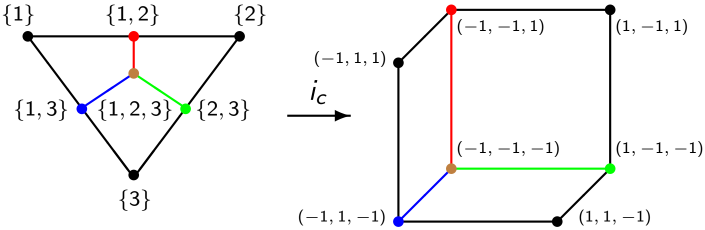

Figure : The embedding $i_c : |\mathrm{Sd}\Delta^2| \to (D^1)^3$

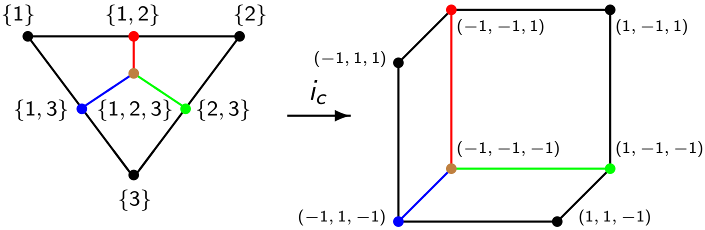

Figure : The embedding $i_c : |\mathrm{Sd}\Delta^2| \to (D^1)^3$

- Define the embeddings

$$i_c : |\mathrm{Sd}K| \to (D^1)^m, \quad \mathrm{Cone}(i_c) : |\mathrm{Cone}(\mathrm{Sd}K)| \to (D^1)^m$$

  as the restriction of the above embeddings.

These are regarded as the cubical subdivisions of K and its cone.

By definition, we have

$$Z_K^m = \bigcup_{\substack{\sigma \subset \tau \subset [m] \\ \tau - \sigma \in K}} C_{\sigma \subset \tau}, \quad Z_K^{m-1} = \bigcup_{\substack{\emptyset \neq \sigma \subset \tau \subset [m] \\ \tau - \sigma \in K}} C_{\sigma \subset \tau}$$

and

$$\mathrm{Cone}(i_c)(|\mathrm{Cone}(\mathrm{Sd} K)|) = \bigcup_{\sigma \subset \tau \in K} C_{\sigma \subset \tau}, \quad i_c(|\mathrm{Sd} K|) = \bigcup_{\emptyset \neq \sigma \subset \tau \in K} C_{\sigma \subset \tau}.$$

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Then the map $\mathrm{Cone}(i_c) : |\mathrm{Cone}(\mathrm{Sd}K)| \to (D^1)^m$ descends to

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On the other hand, since

$$\begin{aligned} Z_K^m - Z_K^{m-1} &= \bigcup_{\sigma\subset\tau\in K} C_{\sigma\subset\tau} - \bigcup_{\emptyset\neq\sigma\subset\tau\in K} C_{\sigma\subset\tau} \\ &= \mathrm{Cone}(i_c)(|\mathrm{Cone}(\mathrm{Sd}K)|) - i_c(|\mathrm{Sd}K|), \end{aligned}$$

the above map is a relative homeomorphism.

More generally, we have:

Proposition

The map

$$\mathrm{Cone}(i_c) : \coprod_{I\subset[m],\ |I|=i} (|\mathrm{Cone}(\mathrm{Sd}K_I)|, |\mathrm{Sd}K_I|) \to (Z_K^i, Z_K^{i-1})$$

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Corollary

Z_K^i is obtained from Z_K^{i-1} by attaching cones to $i_c(|\mathrm{Sd} K_I|) \subset Z_K^{i-1}$ for $\forall I \subset [m]$ with $|I| = i$.

Corollary

If $i_c : |\mathrm{Sd} K_I| \to Z_K^{|I|-1}$ is null homotopic for all $\emptyset \neq I \subset [m]$, then

$$Z_K \simeq \bigvee_{\emptyset \neq I \subset [m]} |\Sigma K_I|.$$

SCM case

Lemma

If $K^\vee$ is SCM, so is $(K_I)^\vee$ for any $\emptyset \neq I \subset [m]$.

Then it is sufficient to show that $i_c : |\mathrm{Sd}K| \to Z_K^{m-1}$ is null homotopic.

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- Let $\widehat{K}$ be the simplicial complex obtained from K by adding all missing faces.

Lemma

The map $i_c : |\mathrm{Sd}K| \to Z_K^{m-1}$ factors as

$$|\mathrm{Sd}K| \xrightarrow{\text{incl}} |\mathrm{Sd}\widehat{K}| \to Z_K^{m-1}.$$

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Proposition

If $K^\vee$ is SCM, then for each prime p, there is a simplicial complex Δ such that

$$K \subset \Delta \subset \widehat{K} \quad \text{and} \quad |\Delta|_{(p)} \simeq *.$$

In particular, the map $|\mathrm{Sd} K| \xrightarrow{\text{incl}} |\mathrm{Sd} \widehat{K}|$ is null homotopic.

Generalization

Define $Z_K^i(C\underline{X}, \underline{X}) \subset Z_K(C\underline{X}, \underline{X})$ similarly to $Z_K^i \subset Z_K$. Then there is a stratification

$$* = Z_K^0(C\underline{X}, \underline{X}) \subset Z_K^1(C\underline{X}, \underline{X}) \subset \cdots \subset Z_K^m(C\underline{X}, \underline{X}) = Z_K(C\underline{X}, \underline{X}).$$

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The composite

$$\begin{aligned} |\mathrm{Cone}(\mathrm{Sd}K)| \times X_1 \times \cdots \times X_m &\xrightarrow{i_c \times 1} (D^1)^m \times X_1 \times \cdots \times X_m \\ &\xrightarrow{\mathrm{perm}} (D^1 \times X_1) \times \cdots \times (D^1 \times X_m) \\ &\xrightarrow{\mathrm{proj}} CX_1 \times \cdots \times CX_m \end{aligned}$$

descends to a relative homeomorphism

$$(|\mathrm{Cone}(\mathrm{Sd}K)|, |\mathrm{Sd}K|) \times (X, F) \to (Z_K^m(C\underline{X},\underline{X}), Z_K^{m-1}(C\underline{X},\underline{X}))$$

where $X = X_1 \times \cdots \times X_m$ and F is the fat wedge of $X_1, \ldots, X_m$.

We can get an analogous relative homemorphism for the pair

$$(Z_K^i(C\underline{X}, \underline{X}), Z_K^{i-1}(C\underline{X}, \underline{X}))$$

$(i = 1, \ldots, m)$. Then we obtain that $Z_K^i(C\underline{X}, \underline{X})$ is constructed from $Z_K^{i-1}(C\underline{X}, \underline{X})$ by attaching certain spaces, where the attaching maps are explicitly described as above.

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Using the previous result of Iriye & K, we can prove that the attaching maps are null homotopic when $K^\vee$ is SCM. Therefore we obtain:

Theorem

If $K^\vee$ is SCM and each X_i is a CW-complex whose components are finite complexes, then

$$Z_K(C\underline{X}, \underline{X}) \simeq \bigvee_{\emptyset \neq I \subset [m]} |\Sigma K_I| \wedge \widehat{X}^I.$$