Examples of quasitoric manifolds as special unitary manifolds

—Joint work with Wei Wang

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Toric Topology 2014 in Osaka
January 25, 2014

Outline

- Notions and Background
- Motivation and Problem
- Main results
- Proof

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Notions–unitary manifold

Unitary manifold

A **unitary manifold** is an oriented closed smooth manifold whose tangent bundle admits a stably complex structure. Namely, there exists a bundle map

$$J : TM \oplus \mathbb{R}^k \longrightarrow TM \oplus \mathbb{R}^k$$

such that $J^2 = -1$.
A unitary manifold is said to be **special** if the first Chern class vanishes.

Milnor showed that

Theorem (Milnor)

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Notions–quasitoric manifold

Definition

A **quasitoric manifold** M^{2n} is a closed smooth manifold with an effective action of T^n such that

1) M^{2n} is locally iso. to the *standard* T^n*-repre. on* $\mathbb{C}^n$;

2) its orbit space M^{2n}/T^n is a *simple convex polytope*.

RK. A quasitoric manifold is the topological version of a nonsingular compact toric variety, introduced by Davis–Januszkiewicz [Duke Math. J., 1991]

Standard T^n-representation

Standard T^n-representation on $\mathbb{C}^n$ defined by

$$(z_1, ..., z_n) \longmapsto (g_1 z_1, ..., g_n z_n)$$

whose orbit space is the positive cone in $\mathbb{R}^n$.

Simple convex polytopes

A convex n-polytope is the convex hull of some finite points in $\mathbb{R}^n$.

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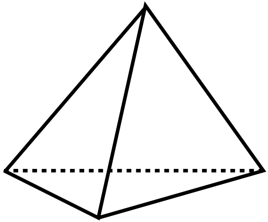

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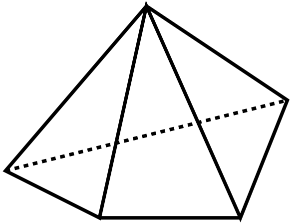

Not simple

Example

- $S^1 \curvearrowright \mathbb{C}P^1$ defined by $[z_0 : z_1] \longmapsto (z_0 : gz_1]$, gives the 1-simplex as its orbit space.

- More generally, $T^n \curvearrowright \mathbb{C}P^n$ defined by

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Two key points for theory of quasitoric manifolds

–Geometric topology

- **Characteristic function:** Each quasitoric manifold $\pi : M^{2n} \longrightarrow P^n$ determines

$$\lambda : \mathcal{F}(P^n) \longrightarrow \mathbb{Z}^n$$

mapping n facets meeting at each vertex to a basis of $\mathbb{Z}^n$, where $\mathcal{F}(P^n)$:=all facets of P^n.

- **Reconstruction:** Up to equivariant diffeomorphism, M^{2n} can be recovered by the pair (P^n, λ), denoted by $M(P^n, \lambda)$.

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Two key points for theory of quasitoric manifolds

$\pi : M^{2n} \longrightarrow P^n$: a quasitoric manifold over P^n.

–Algebraic topology

- **Equivariant cohomology:** $H^*_{T^n}(M^n) \cong R(P^n)$ where $R(P^n)$ is the Reisner-Stanley face ring of P^n:

$$R(P^n) = \mathbb{Z}[F_1, ..., F_l]/I$$

$I = (F_{i_1} \cdots F_{i_r} | F_{i_1} \cap \cdots \cap F_{i_r} = \emptyset)$ is an ideal, and each F_i is a facet (ie., codim-one face) of P^n.

- **Betti numbers:** $(b_0, b_2, ..., b_{2n}) = (h_0, h_1, ..., h_n)$ where $(h_0, h_1, ..., h_n)$ is the h-vector of P^n
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Quasitoric manifolds as examples of unitary toric mflds–Buchstaber–Ray's work

In their 1998 paper [Russ. Math. Surv. 53 (1998), 371–373], Buchstaber and Ray studied the cobordism of quasitoric manifolds. They first showed

Buchstaber–Ray

Each omnioriented quasitoric manifold is a unitary manifold.

Remark: An **omniorientation** of a quasitoric manifold $\pi : M^{2n} \longrightarrow P^n$ is a collection of all orientations of $M^{2n}, \pi^{-1}(F) = M_F, F \in \mathcal{F}(P^n)$.

There are 2^{m+1} omniorientations where m is the number of all facets of P^n.

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Theorem (Buchstaber–Ray, 1998)

The unitary cobordism class of each unitary manifold contains an omnioriented quasitoric manifold as its representative.

In other words, each class of Ω_{2n}^U is represented by an omnioriented quasitoric manifold, where Ω_{2n}^U is the abelian group formed by the unitary cobordism classes of all $2n$-dimensional unitary manifolds.

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Buchstaber-Panov-Ray's work

Furthermore, in their paper [Toric Genera, Internat. Math. Res. Notices 2010, 3207–3262], Buchstaber-Panov-Ray investigated when an omnioriented quasitoric manifold is a special unitary manifold.

Proposition (Buchstaber–Panov–Ray, 2010)

Let $M(P^n, \lambda)$ be a quasitoric manifold. Then $M(P^n, \lambda)$ with an omniorientation is a special unitary manifold if and only if for each facet $F \in \mathcal{F}(P^n)$, the sum of all entries of $\lambda(F)$ is exactly 1.

Then they showed

Proposition (Buchstaber–Panov–Ray, 2010)

Suppose that $M(P^n, \lambda)$ with an omniorientation is a special unitary manifold. When $n < 5$, $M(P^n, \lambda)$ represents the zero element in Ω^U_{2n}.

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Buchstaber-Panov-Ray's conjecture

Buchstaber–Panov–Ray Conjecture (2010)

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Motivation

Motivation of this talk

To consider the Buchstaber–Panov–Ray conjecture.

Main result

We shall construct some examples of specially omnioriented quasitoric manifolds that are not cobordant to zero in Ω_*^U, which give the negative answer to the above conjecture in almost all possible dimensional cases.

Our main result is stated as follows.

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For each $n \geq 5$ with only $n \neq 6$, there exists a $2n$-dimensional specially omnioriented quasitoric manifold M^{2n} which represents a nonzero element in Ω_*^U.

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Proof

Our strategy is as follows: Related to the unoriented cobordism theory

- Milnor's work tells us that there is an epimorphism

$$F_* : \Omega_*^U \longrightarrow \mathfrak{N}_*^2$$

where $\mathfrak{N}_*$ denotes the ring produced by the unoriented cobordism classes of all smooth closed manifolds, and $\mathfrak{N}_*^2 = \{\alpha^2 | \alpha \in \mathfrak{N}_*\}$.

- This actually implies that there is a covering homomorphism

$$H_n : \Omega_{2n}^U \longrightarrow \mathfrak{N}_n$$

which is induced by $\theta_n \circ F_n$, where $\theta_n : \mathfrak{N}_n^2 \longrightarrow \mathfrak{N}_n$ is defined by mapping $\alpha^2 \longmapsto \alpha$.

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An explanation for $H_n : \Omega_{2n}^U \longrightarrow \mathfrak{N}_n$

Theorem (Buchstaber–Ray)

Each class of $\mathfrak{N}_n$ contains an n-dimensional small cover as its representative, where a small cover is also introduced by Davis and Januszkiewicz, and it is the real analogue of a quasitoric manifold.

Davis and Januszkiewicz tell us that each quasitoric manifold M^{2n} over a simple convex polytope P^n always admits a natural conjugation involution τ whose fixed point set M^τ is just a small cover over P^n. Thus, τ induces a homomorphism $\phi_n^\tau : \Omega_{2n}^U \longrightarrow \mathfrak{N}_n$, which exactly agrees with the above homomorphism $H_n : \Omega_{2n}^U \longrightarrow \mathfrak{N}_n$.

An approach: to construct the examples of specially omnioriented quasitoric manifolds whose images under ϕ^τ are nonzero in $\mathfrak{N}_*$.

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An approach: to construct the examples of specially omnioriented quasitoric manifolds whose images under ϕ^τ are nonzero in $\mathfrak{N}_*$.

Examples of specially omnioriented quasitoric manifolds

Throughout the following

for a k-dimensional simplex Δ^k, $\Delta_i^{(k)}, i = 1, ..., k+1$ mean the $k+1$ facets of Δ^k,

for a product $P = \Delta^{k_1} \times \cdots \times \Delta^{k_r}$ of simplices, $F_{k_i,j}$ means that the facet

$$\Delta^{k_1} \times \cdots \times \Delta^{k_{i-1}} \times \Delta_j^{(k_i)} \times \Delta^{k_{i+1}} \times \cdots \times \Delta^{k_r}$$

of P.

Examples of specially omnioriented quasitoric manifolds

Example I

Let $P^{4l+5} = \Delta^2 \times \Delta^{4l+3}$ with $l \geq 0$. Define a characteristic function $\lambda^{(2,0,\ldots,0)}$ on P^{4l+5} in the following way. First let us fix an ordering of all facets of P^{4l+5} as follows

$$F_{2,1}, F_{2,2}, F_{2,3}, F_{4l+3,1}, \ldots, F_{4l+3,4l+3}, F_{4l+3,4l+4}.$$

Then we construct the characteristic matrix $\Lambda^{(2,0,\ldots,0)}$ of the required characteristic function $\lambda^{(2,0,\ldots,0)}$ on the above ordered facets as follows:

$$\Lambda^{(2,0,\ldots,0)} = \begin{pmatrix} I_2 & \widetilde{\mathbf{1}}_2 & & \\ & J_1 & I_{4l+3} & \widetilde{\mathbf{1}}_{4l+3} \end{pmatrix}$$

We obtain the **special unitary mfd** $M(P^{4l+5}, \lambda^{(2,0,\ldots,0)})$.

Examples of specially omnioriented quasitoric manifolds

Example II

Let $P^{8l+11} = \Delta^4 \times \Delta^2 \times \Delta^{8l+5}$ with $l \geq 0$. In a similar way as above, fix an ordering of all facets of P^{8l+11} as follows:

$$F_{4,1}, F_{4,2}, F_{4,3}, F_{4,4}, F_{4,5}, F_{2,1}, F_{2,2}, F_{2,3}, F_{8l+5,1}, \ldots, F_{8l+5,8l+5}, F_{8l+5,8l+6}.$$

Then we define a characteristic function $\lambda^{(4,2,0,\ldots,0)}$ on the above ordered facets of P^{8l+11} by the following characteristic matrix

$$\Lambda^{(4,2,0,\ldots,0)} = \begin{pmatrix} I_4 & \widetilde{\mathbf{1}}_4 & & & & \\ & & I_2 & \widetilde{\mathbf{1}}_2 & & \\ & J_1 & & J_2 & I_{8l+5} & \widetilde{\mathbf{1}}_{8l+5} \end{pmatrix}$$

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Stong manifolds

Lemma

The images of $M(P^{4l+5}, \lambda_2^{(2,0,\ldots,0)})$ and $M(P^{8l+11}, \lambda_2^{(4,2,0,\ldots,0)})$ under the mape

$$\Omega^U_{2n} \longrightarrow \mathfrak{N}_n$$

are exactly Stong manifolds $\mathbb{R}P(2, \underbrace{0, \ldots, 0}_{4l+3})$ and $\mathbb{R}P(4, 2, \underbrace{0, \ldots, 0}_{8l+4})$.

Definition

A **Stong manifold** is defined as the real projective space bundle denoted by $\mathbb{R}P(n_1, \ldots, n_k)$ of the bundle $\gamma_1 \oplus \cdots \oplus \gamma_k$ over $\mathbb{R}P^{n_1} \times \cdots \times \mathbb{R}P^{n_k}$, where γ_i is the pullback of the canonical bundle over the i-th factor $\mathbb{R}P^{n_i}$.

The Stong manifold $\mathbb{R}P(n_1, \ldots, n_k)$ has dimension $n_1 + \cdots + n_k + k - 1$.

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Indecomposable Stong manifolds

Theorem (Stong)

A Stong manifold $\mathbb{R}P(n_1, ..., n_k)$ is indecomposable if and only if

$$\binom{n-1}{n_1} + \cdots + \binom{n-1}{n_k} \equiv 1 \mod 2$$

where $n = n_1 + \cdots + n_k + k - 1$.

$$\Downarrow$$

$\mathbb{R}P(2, \underbrace{0, ..., 0}_{4l+3})$ and $\mathbb{R}P(4, 2, \underbrace{0, ..., 0}_{8l+4})$ are indecomposable

so they represent nonzero elements in $\mathfrak{N}_*$.

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Lemmas

Let $\alpha^{8l+10} = \{M(P^{4l+5}, \lambda^{(2,0,\ldots,0)})\}$ and $\beta^{16l+22} = \{M(P^{8l+11}, \lambda^{(4,2,0,\ldots,0)})\}$

Lemma 1

α^{8l+10} and β^{16l+22} form a subring of Ω_*^U

$$\mathbb{Z}[\alpha^{8l+10}, \beta^{16l+22} | l \geq 0]$$

which contains nonzero classes of dimension $\neq 2, 4, 6, 8, 12, 14, 16, 24$.

It remains to consider $2n = 14, 16, 24$ (i.e., $n = 7, 8, 12$).

The case $n = 7$

Consider the polytope $P^7 = \Delta^4 \times \Delta^3$. Then we may define a characteristic function $\lambda^{<7>}$ on the ordered facets of P^7 by the following characteristic matrix

$$\begin{pmatrix} 1 & & & & 1 & & & & \\ & 1 & & & -1 & & & & \\ & & 1 & & 1 & & & & \\ & & & 1 & -1 & & & & \\ & & & & 1 & 1 & & & 1 \\ & & & & & & 1 & & -1 \\ & & & & & & & 1 & 1 \end{pmatrix},$$

which gives a special unitary manifold $M(P^7, \lambda^{<7>})$. A direct calculation gives the Chern number $\langle c_3 c_4, [M(P^7, \lambda^{<7>})] \rangle = -2 \neq 0$, which implies that this specially omnioriented quasitoric manifold $M(P^7, \lambda^{<7>})$ is not cobordant to zero in Ω_*^U.

The case $n = 8$

Consider the polytope $P^8 = \Delta^3 \times \Delta^5$ with a characteristic function $\lambda^{<8>}$ on the ordered facets of P^8 by

$$\begin{pmatrix} 1 & & & 1 & & & & & & \\ & 1 & & -1 & & & & & & \\ & & 1 & 1 & & & & & & \\ & & & -1 & 1 & & & & & 1 \\ & & & 1 & & 1 & & & & -1 \\ & & & & & & 1 & & & 1 \\ & & & & & & & 1 & & -1 \\ & & & & & & & & 1 & 1 \end{pmatrix},$$

which also gives a special unitary manifold $M(P^8, \lambda^{<8>})$. One has the Chern number $\langle c_4^2, [M(P^8, \lambda^{<8>})] \rangle = 4 \neq 0$. So $M(P^8, \lambda^{<8>})$ is not cobordant to zero in Ω_*^U.

The case $n = 12$

Consider the polytope $P^{12} = \Delta^3 \times \Delta^9$ with a characteristic function $\lambda^{<12>}$ on the ordered facets of P^{12} by the matrix

$$\begin{pmatrix}
1 & & & 1 & & & & & & & & & & \\
 & 1 & & -1 & & & & & & & & & & \\
 & & 1 & 1 & & & & & & & & & & \\
 & & & -1 & 1 & & & & & & & & & 1 \\
 & & & 1 & & 1 & & & & & & & & -1 \\
 & & & & & & 1 & & & & & & & 1 \\
 & & & & & & & 1 & & & & & & -1 \\
 & & & & & & & & 1 & & & & & 1 \\
 & & & & & & & & & 1 & & & & -1 \\
 & & & & & & & & & & 1 & & & 1 \\
 & & & & & & & & & & & 1 & & -1 \\
 & & & & & & & & & & & & 1 & 1
\end{pmatrix},$$

from which one obtains a special unitary manifold $M(P^{12}, \lambda^{<12>})$. Then one has that the Chern number $\langle c_6^2, [M(P^{12}, \lambda^{<12>})]\rangle = 64 \neq 0$. Thus $M(P^{12}, \lambda^{<12>})$ is not cobordant to zero in Ω_*^U.

Remark

We have done many tries to find a counterexample in the case $n = 6$, but failed.

It seems to be reasonable to the assertion as in the Buchstaber–Panov–Ray conjecture that **each 12-dimensional specially omnioriented quasitoric manifold is cobordant to zero in Ω_*^U** since each 6-dimensional orientable smooth closed manifold is always cobordant to zero in $\mathfrak{N}_*$.

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Thank You!