Quasitoric manifolds and toric origami manifolds

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Table of contents

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- the action of T^n on M is locally standard;
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2. $\mathbb{C}P^2 \# \mathbb{C}P^2$

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For $n > 1$, T^n-action on $S^{2n} \subseteq \mathbb{C}^n \oplus \mathbb{R}$ given by

$$(t_1, \ldots, t_n) \cdot (z_1, \ldots, z_n, r) = (t_1 z_1, \ldots, t_n z_n, r)$$

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Properties of quasitoric manifolds

1. There is a one-to-one correspondence

$$\{\text{quasitoric manifolds } M\} \stackrel{1:1}{\longleftrightarrow} \{\text{characteristic pairs } (P, \boldsymbol{\lambda})\}.$$

2. Every omniorientation of a quasitoric manifold M determines a stably almost complex structure.

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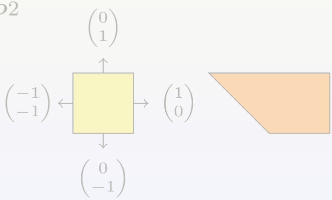

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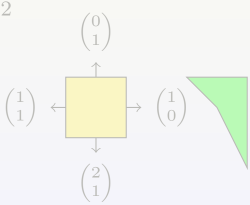

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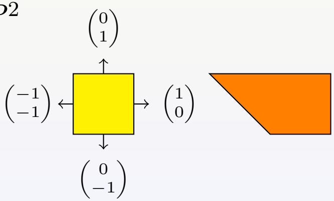

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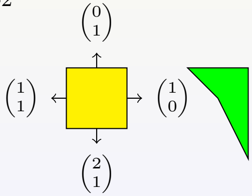

Symplectic manifolds

A *symplectic manifold* M is a manifold equipped with a *symplectic form* $\omega \in \Omega^2(M)$ that is closed ($d\omega = 0$) and non-degenerate.

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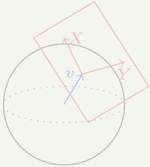

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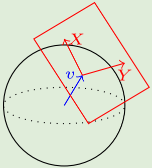

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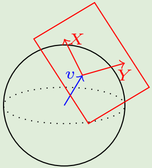

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Folded symplectic manifolds

A *folded symplectic form* on a $2n$-dim'l manifold M is a closed 2-form ω

- ω^n vanishes transversally on a submanifold Z,
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The submanifold Z is called the *folding hypersurface* or *fold*.

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Theorem [Cannas da Silva, 2010]

Let M be a $2n$-dim'l manifold with a stably almost complex structure J. Then M admits a folded symplectic form consistent with J in any degree 2 cohomology class.

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Origami manifolds

Let (M, ω) be a folded symplectic manifold with a fold Z.
Let $i\colon Z \hookrightarrow M$ be the inclusion.
Then $i^*\omega$ has a 1-dimensional kernel on Z.

Definition [Cannas da Silva-Guillemin-Pires, IMRN 2011]

ω is called an *origami form* if this line field is the vertical bundle of an oriented S^1-fiber bundle $Z \xrightarrow{\pi} B$ over a compact base B.

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Toric origami manifolds

The action of a Lie group G on an origami manifold (M, ω) is *Hamiltonian* if it admits a *moment map* $\mu\colon M \to \mathfrak{g}^*$ satisfying the conditions:

- μ collects Hamiltonian functions, i.e., $d\langle \mu, X\rangle = \iota_{X^\#}\omega$, $\forall X \in \mathfrak{g} := \mathrm{Lie}(G)$, where $X^\#$ is the vector field generated by X;
- μ is equivariant with respect to the given action of G on M and the coadjoint action of G on the dual vector space $\mathfrak{g}^*$.

A *toric origami manifold* is a compact connected origami manifold (M, ω) equipped with an effective Hamiltonian action of a torus T with $\dim T = \frac{1}{2}\dim M$.

NOTE: If $Z = \emptyset$, a toric origami manifold is a toric symplectic manifold.

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Examples

1. $T = (S^1)^2$ acts on $(\mathbb{C}P^2, \omega)$ by
$$(t_1, t_2) \cdot [z_1, z_2, z_3] = [t_1 z_1, t_2 z_2, z_3]$$
with moment map
$$\mu(z_1, z_2, z_3) = \left(\frac{|z_1|^2}{|z_1|^2 + |z_2|^2 + |z_3|^2}, \frac{|z_2|^2}{|z_1|^2 + |z_2|^2 + |z_3|^2} \right).$$
2. $T = (S^1)^2$ acts on $(S^4 \subset \mathbb{C}^2 \oplus \mathbb{R}, \omega_{\mathbb{C}^n} \oplus 0)$ by
$$(t_1, t_2) \cdot (z_1, z_2, r) = (t_1 z_1, t_2 z_2, r)$$
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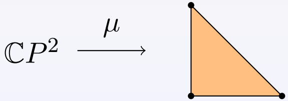
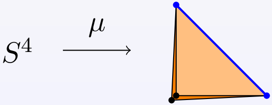
$$\mu(z_1, z_2, r) = (|z_1|^2, |z_2|^2).$$





Origami templates

An *origami template* consists of a pair $(\mathcal{P}, \mathcal{F})$, where $\mathcal{P}$ is a collection of Delzant polytopes in $\mathbb{R}^n$ and $\mathcal{F}$ is a collection of facets and pairs of facets of polytopes in $\mathcal{P}$ s.t.

(O1) $\forall \{F, F'\} \in \mathcal{F}$, if $F \subset P$ and $F' \subset P'$, then $\exists\, \mathcal{U} \underset{\text{open}}{\subset} \mathbb{R}^n$ s.t. $\mathcal{U} \cap P = \mathcal{U} \cap P'$;

(O2) if $F \in \mathcal{F}$ or $\{F, F'\} \in \mathcal{F}$, then neither F nor any of its neighboring facets occur elsewhere in $\mathcal{F}$;

(O3) the topological space, denoted $|(\mathcal{P}, \mathcal{F})|$, constructed from the disjoint union $\sqcup P_j$, $P_j \in \mathcal{P}$, by identifying facet pairs in $\mathcal{F}$ is connected.

Theorem (Cannas da Silva-Guillemin-Pires, 2011)

$$\{\textit{toric origami manifolds}\} \overset{1:1}{\longleftrightarrow} \{\textit{origami templates}\}$$

NOTE: $M/T \underset{\text{homeo}}{\cong} |(\mathcal{P}, \mathcal{F})|$ as a manifold with corners.

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Remarks

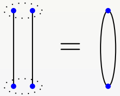
Non-orientable toric origami manifolds[1]

$\mathbb{R}P^2$
with $\mathcal{F} = \{$a red dot$\}$

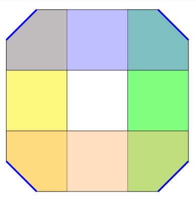
Klein bottle
with $\mathcal{F} = \{$two red dots$\}$

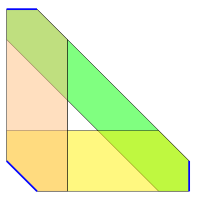
The black dot is corresponding to a fixed point, and each red dot is corresponding to the point whose isotropy subgroup is $\mathbb{Z}_2$.

Non-simply connected origami templates



torus
$\mathcal{P} = \{$two intervals$\}$
$\mathcal{F} = \{$two pairs of blue dots$\}$





[1] An origami template is *oriented* if the polytopes in $\mathcal{P}$ come with an orientation and $\mathcal{F}$ consists solely of pairs of facets which belong to polytopes with opposite orientations.

Motivation

For an orientable toric origami manifold M,

- if $M^T \neq \emptyset$, M is a torus manifold[2].
- if $|(\mathcal{P}, \mathcal{F})|$ is (combinatorially) a simple polytope, M is a quasitoric manifold.

Question

What about the converse?

[2] A *torus manifold* is a $2n$-dim'l oriented compact connected smooth manifold M with an effective action of $T = T^n$ with $M^T \neq \emptyset$.

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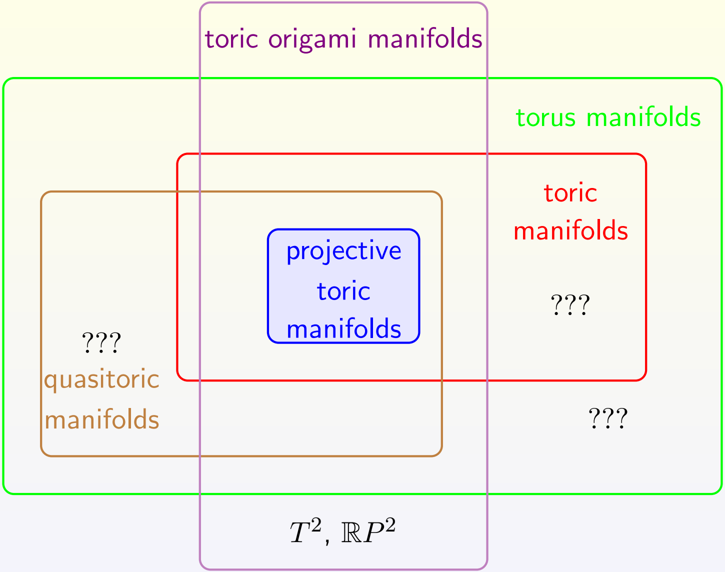
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T^2, $\mathbb{R}P^2$

Not all torus manifolds admit toric origami forms.

From the following facts

- $S^4 \times S^4$ is simply connected,
- $(S^4 \times S^4)/T^4 = |(\mathcal{P}, \mathcal{F})|$ has four facets,
- $(S^4 \times S^4)/T^4 = |(\mathcal{P}, \mathcal{F})|$ has four vertices,

we can show that $S^4 \times S^4$ cannot be a toric origami manifold.

Theorem (Masuda-P)

Let S^{2n_i} $(n_i \geq 1)$ be the standard $2n_i$-sphere with the standard T^{n_i}-sphere with the standard T^{n_i}-action for $i = 1, \ldots, k$. Then $\prod_{i=1}^k S^{2n_i}$ with the product action of $\prod_{i=1}^k T^{n_i}$ admits a toric origami form if and only if $n_i = 1$ except for one i.

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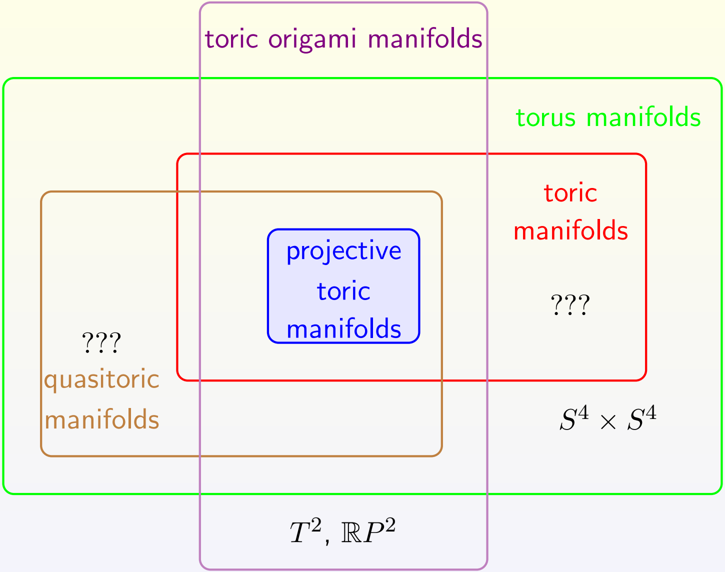
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All 4-dimensional quasitoric manifolds are equivariantly diffeomorphic to some toric origami manifolds.

- There is a one-to-one correspondence

$$\{\text{simply conn. cpt sm. 4-mfd } M \text{ with effective } T^2-\text{action}\}$$
$$\overset{1:1}{\longleftrightarrow} \{\text{unimodular sequences } v_1, \dots, v_d\},$$

where v_i and v_{i+1} form a basis of $\mathbb{Z}^2$ for $i = 1, \dots, d$ and $v_{d+1} = v_1$.

- For $d \geq 4$, there is v_j $(1 \leq j \leq d)$ which satisfies

$\epsilon_{j-1}v_{j-1} + \epsilon_j v_{j+1} + a_j v_j = 0$ (with $a_j = \pm 1$ or $0, \epsilon_j = \det(v_j, v_{j+1})$).

- Use induction on d.

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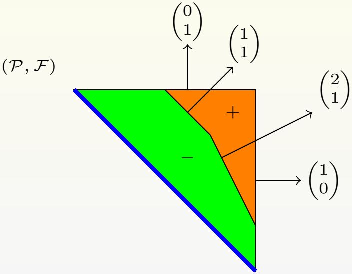
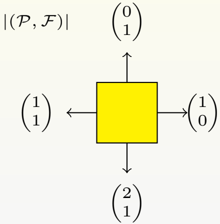
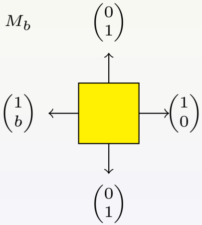




Remarks

Not all omniorientations of a quasitoric manifold determine toric origami forms.

Let M_b be an omnioriented quasitoric manifold which is corresponding to a unimodular sequence $(1,0)$, $(0,1)$, $(1,b)$, $(0,1)$. Then, the omniorientation of M_b admits a toric origami form if and only if $b = \pm 1$ or $b = \pm 2$.



Observation

Let $F_1, \ldots, F_n, F_{n+1}$ be the facets of Δ^n.

1. By cutting the face $F_1 \cap \cdots \cap F_k$, we get a new polytope $P \approx \Delta^{n-k+1} \times \Delta^{k-1}$ whose facets $G, F_1', \ldots, F_k', F_{k+1}', \ldots, F_{n+1}'$, where $G \approx \Delta^{n-k} \times \Delta^{k-1}$, $F_1', \ldots, F_k' \approx \Delta^{n-k+1} \times \Delta^{k-2}$, and $F_{k+1}', \ldots, F_{n+1}' \approx \Delta^{n-k} \times \Delta^{k-1}$.
2. Let P_1 be the polytope obtained from Δ^n by cutting a vertex $F_1 \cap \cdots \cap F_n$ of Δ^n. Then facets of P_1 are of the form

$$G_1, F_{n+1}' \approx \Delta^{n-1} \,\&\, F_1', \ldots, F_n' \approx \mathrm{vc}(\Delta^{n-1}).$$

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3. By cutting a face $G_1 \cap F'_1 \cap \cdots \cap F'_k$ from P_1, we get a new polytope $P_2 \approx \mathrm{vc}(\Delta^{n-k} \times \Delta^k)$ whose facets are

$$F''_{n+1} \approx \Delta^{n-1}, G_2 \approx \Delta^{n-k-1} \times \Delta^k, G'_1 \approx \Delta^{n-k} \times \Delta^{k-1},$$
$$F''_1, \ldots, F''_k \approx \mathrm{vc}(\Delta^{n-k} \times \Delta^{k-1}), F''_{k+1}, \ldots, F''_n \approx \mathrm{vc}(\Delta^{n-k-1} \times \Delta^k).$$

Observation

Let $F_1, \ldots, F_n, F_{n+1}$ be the facets of Δ^n.

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$$F_{n+1}'' \approx \Delta^{n-1}, G_2 \approx \Delta^{n-k-1} \times \Delta^k, G_1' \approx \Delta^{n-k} \times \Delta^{k-1},$$
$$F_1'', \ldots, F_k'' \approx \mathrm{vc}(\Delta^{n-k} \times \Delta^{k-1}), F_{k+1}'', \ldots, F_n'' \approx \mathrm{vc}(\Delta^{n-k-1} \times \Delta^k).$$

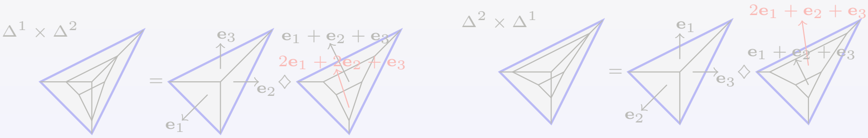
Now, let $M_{r,s}$ be a quasitoric manifold over $\Delta^n \times \Delta^m$ whose characteristic matrix is of the form $\Lambda_{r,s}$.

$$\Lambda_{r,s} = \left(\begin{array}{c|c|cc}
 & & 1 & 2 \\
 & & \vdots & \vdots \\
 & & 1 & 2 \;\}s \\
E_n & & 1 & 0 \\
 & & \vdots & \vdots \\
 & & 1 & 0 \\
\hline
 & & 1 & 1 \\
 & & \vdots & \vdots \\
 & & 1 \;\}r & 1 \\
 & E_m & 0 & 1 \\
 & & \vdots & \vdots \\
 & & 0 & 1
\end{array}\right)$$

Lemma

A quasitoric manifold $M_{m,n}$ over $\Delta^n \times \Delta^m$ is equivariantly homeomorphic to a toric origami manifold.

1. Let $F_1, \dots, F_{n+m+1}$ be the facets of Δ^{n+m} and let the corresponding outward normal vectors are $\mathbf{e}_1, \dots, \mathbf{e}_{n+m}, -\mathbf{e}_1 - \cdots - \mathbf{e}_{n+m}$.
2. Let P be the polytope obtained by cutting $F_1 \cap \cdots \cap F_{n+m}$ and then by cutting $G \cap F_1' \cap \cdots \cap F_n'$, where $G, F_1', \dots, F_{n+m}'$ are new facets obtained from the first cutting.
3. The origami template which we want is the pair of $\mathcal{P} = \{\Delta^{n+m}, P\}$ and $\mathcal{F} = \{(F_{n+m+1}, F_{n+m+1}'')\}$.



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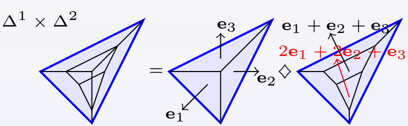
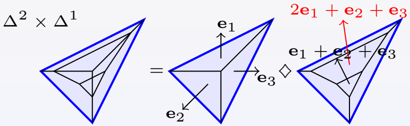
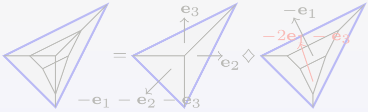




Lemma

A quasitoric manifold $M_{m,1}$ over $\Delta^n \times \Delta^m$ is equivariantly homeomorphic to a toric origami manifold.

1. Let $F_1, \dots, F_{n+m+1}$ be the facets of Δ^{n+m} and let the corresponding outward normal vectors are $\mathbf{e}_1, \dots, \mathbf{e}_{n+m}, -\mathbf{e}_1 - \cdots - \mathbf{e}_{n+m}$.
2. Let P be the polytope obtained from Δ^{n+m} by cutting $F_2 \cap \cdots \cap F_{n+m+1}$ and then by cutting $G \cap F_2' \cap \cdots \cap F_n' \cap F_{n+m+1}'$, where G and $F_1', \dots, F_n'$ are new facets obtained by the first cutting.
3. $\mathcal{P} = \{\Delta^{n+m}, P\}$ and $\mathcal{F} = (F_1, F_1'')$.



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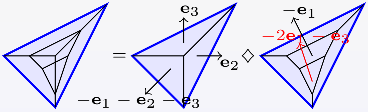

Lemma

Two quasitoric manifolds $M_{r,s}$ and $M_{m-r+1,n-s+1}$ over $\Delta^n \times \Delta^m$ are weakly equivariantly diffeomorphic.

Define an isomorphism $\Theta\colon \mathbb{Z}^{n+m} \to \mathbb{Z}^{n+m}$ by

$$\begin{array}{ccc}
\mathbf{e}_1 & \mapsto & -\sum_{i=1}^{n+1} \mathbf{e}_i - \sum_{j=r+1}^{m} \mathbf{e}_{n+j} \\
\mathbf{e}_{n+1} & \mapsto & 2\mathbf{e}_1 + 2\sum_{i=s+1}^{n} \mathbf{e}_i + \sum_{j=1}^{m} \mathbf{e}_{n+j} \\
\mathbf{e}_2, \ldots, \mathbf{e}_s, \mathbf{e}_{n+r+1}, \ldots, \mathbf{e}_{n+m} & \mapsto & \mathbf{e}_2, \ldots, \mathbf{e}_s, \mathbf{e}_{n+r+1}, \ldots, \mathbf{e}_{n+m} \\
\mathbf{e}_{s+1}, \ldots, \mathbf{e}_n, \mathbf{e}_{n+2}, \ldots, \mathbf{e}_{n+r} & \mapsto & -\mathbf{e}_{s+1}, \ldots, -\mathbf{e}_n, -\mathbf{e}_{n+2}, \ldots, -\mathbf{e}_{n+r}
\end{array}$$

Proposition

Any quasitoric manifold over $\Delta^n \times \Delta^m$ with $n, m \leq 2$ is weakly equivariantly homeomorphic to a toric origami manifold.

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\mathbf{e}_2,\ldots,\mathbf{e}_s,\mathbf{e}_{n+r+1},\ldots,\mathbf{e}_{n+m} & \mapsto & \mathbf{e}_2,\ldots,\mathbf{e}_s,\mathbf{e}_{n+r+1},\ldots,\mathbf{e}_{n+m} \\
\mathbf{e}_{s+1},\ldots,\mathbf{e}_n,\mathbf{e}_{n+2},\ldots,\mathbf{e}_{n+r} & \mapsto & -\mathbf{e}_{s+1},\ldots,-\mathbf{e}_n,-\mathbf{e}_{n+2},\ldots,-\mathbf{e}_{n+r}
\end{array}$$

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Any quasitoric manifold over $\Delta^n \times \Delta^m$ with $n, m \leq 2$ is weakly equivariantly homeomorphic to a toric origami manifold.

Theorem [Choi & P & Suh]

Any quasitoric manifold over $\Delta^n \times \Delta^1$ is homeomorphic to

- two-stage generalized Bott manifolds,
- $M_{1,1}$ over $\Delta^n \times \Delta^1$, or
- $M_{1,1}$ over $\Delta^1 \times \Delta^n$.

$M_{1,1} \cong_{eq} M_{1,n}$ over $\Delta^n \times \Delta^1$ & $M_{1,1} \cong_{eq} M_{n,1}$ over $\Delta^1 \times \Delta^n$.

Proposition

Any quasitoric manifold over $\Delta^n \times \Delta^1$ is homeomorphic to a toric origami manifold.

Denote

$$\mathbf{1}_n = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}_{n\times 1}, \mathbf{2}_m = \begin{pmatrix} 2 \\ \vdots \\ 2 \end{pmatrix}_{m\times 1}.$$

Consider a quasitoric manifold M over $\prod_{i=1}^{h} \Delta^{n_i}$ whose reduced characteristic matrix is

$$\begin{pmatrix} \mathbf{1}_{n_1} & \mathbf{2}_{n_1} & \cdots & \mathbf{2}_{n_1} \\ \mathbf{1}_{n_2} & \mathbf{1}_{n_2} & \cdots & \mathbf{2}_{n_2} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{1}_{n_h} & \mathbf{1}_{n_h} & \cdots & \mathbf{1}_{n_h} \end{pmatrix}.$$

Proposition

M admits a toric origami form.

Let $F_1, \ldots, F_n, F_{n+1}$ be the facets of Δ^n whose outward normal vectors are $\mathbf{e}_1, \ldots, \mathbf{e}_n, -\mathbf{e}_1 - \cdots - \mathbf{e}_n$, where $n = \sum_{i=1}^h n_i$.

1. Let P_1 be the polytope obtained from Δ^n by cutting $F_1 \cap \cdots \cap F_n$. Then, $P_1 \approx \mathrm{vc}(\Delta^n)$ and we get new facets $G_1, F_1^{(1)}, \ldots, F_{n+1}^{(1)}$,
2. Let P_2 be the polytope obtained from P_1 by cutting $G_1 \cap F_1^{(1)} \cap \cdots F_{n_1}^{(1)}$. Then, $P_2 \approx \mathrm{vc}(\Delta^{n-n_1} \times \Delta^{n_1})$ and we get new facets $G_2, G_1^{(1)}, F_1^{(2)}, \ldots, F_{n+1}^{(2)}$,
3. Let P_3 be the polytope obtaind from P_2 by cutting $G_2 \cap F_{n_1+1}^{(2)} \cap \cdots F_{n_1+n_2}^{(2)}$. Then $P_3 \approx \mathrm{vc}(\Delta^{n-n_1-n_2} \times \Delta^{n_2} \times \Delta^{n_1})$ and we get new facets $G_3, G_1^{(2)}, G_2^{(1)}, F_1^{(3)}, \ldots, F_{n+1}^{(3)}$,

   $\vdots$

4. Let P_h be the polytope obtained from P_{h-1} by cutting $G_{h-1} \cap F_{n-n_h+1}^{(h-1)} \cap \cdots \cap F_n^{(h-1)}$. Then $P_h \approx \mathrm{vc}(\Delta^{n_h} \times \cdots \times \Delta^{n_1})$ and we get new facets $G_h, G_{h-1}^{(1)}, \ldots, G_1^{(h-1)}, F_1^{(h)}, \ldots, F_{n+1}^{(h)}$.
5. $\mathcal{P} = \{\Delta^n, P_h\}$, $\mathcal{F} = (F_{n+1}, F_{n+1}^{(h)})$.

Denote

$$\mathbf{1}_n = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}_{n\times 1}, \mathbf{0}_m = \begin{pmatrix} 2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{m\times 1}, (2) = \begin{pmatrix} 2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{n_1\times 1}.$$

Consider a quasitoric manifold M over $\prod_{i=1}^{h} \Delta^{n_i}$ whose reduced characteristic matrix is

$$\begin{pmatrix} \mathbf{1}_{n_1} & (2) & \cdots & (2) \\ \mathbf{1}_{n_2} & \mathbf{1}_{n_2} & \cdots & \mathbf{0}_{n_2} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{1}_{n_h} & \mathbf{1}_{n_h} & \cdots & \mathbf{1}_{n_h} \end{pmatrix}.$$

Proposition

M admits a toric origami form.

Bundle structures on polytopes

Let $\tilde{\Delta} = \cap_{j=1}^{\tilde{N}} \{x \in \tilde{\mathfrak{t}}^* \mid \langle \tilde{\eta}_j, x \rangle \leq \tilde{\kappa}_j\}$ and $\hat{\Delta} = \cap_{i=1}^{\hat{N}} \{x \in \hat{\mathfrak{t}}^* \mid \langle \hat{\eta}_i, x \rangle \leq \hat{\kappa}_i\}$ be simple polytopes. We say that a simple polytope $\Delta \subset \mathfrak{t}^*$ is a *bundle with fiber* $\tilde{\Delta}$ *over the base* $\hat{\Delta}$ if there exists a short exact sequence

$$0 \to \tilde{\mathfrak{t}} \xrightarrow{\iota} \mathfrak{t} \xrightarrow{\pi} \hat{\mathfrak{t}} \to 0$$

so that

- Δ is combinatorially equivalent to the product $\tilde{\Delta} \times \hat{\Delta}$.
- If $\tilde{\eta}_j{}'$ denotes the outward conormal to the facet $\tilde{F}_j'$ of Δ which corresponds to $\tilde{F}_j \times \hat{\Delta} \subset \tilde{\Delta} \times \hat{\Delta}$, then $\tilde{\eta}_j' = \iota(\tilde{\eta}_j)$ for all $1 \leq j \leq \tilde{N}$.
- If $\hat{\eta}_i{}'$ denotes the outward conormal to the facet $\hat{F}_i'$ of Δ which corresponds to $\tilde{\Delta} \times \hat{F}_i \subset \tilde{\Delta} \times \hat{\Delta}$, then $\pi(\hat{\eta}_i{}') = \hat{\eta}_i$ for all $1 \leq i \leq \hat{N}$.

The facets $\tilde{F}_1', \ldots, \tilde{F}_{\tilde{N}}'$ are called *fiber facets* and the facets $\hat{F}_1', \ldots, \hat{F}_{\hat{N}}'$ are called *base facets*.

Bundle structures on origami templates

Let $(\mathcal{P}, \mathcal{F})$, $(\tilde{\mathcal{P}}, \tilde{\mathcal{F}})$, and $(\hat{\mathcal{P}}, \hat{\mathcal{F}})$ be origami templates defined in $\mathfrak{t}^*$, $\tilde{\mathfrak{t}}^*$, and $\hat{\mathfrak{t}}^*$, respectively.

Definition

An origami template $(\mathcal{P}, \mathcal{F})$ is a bundle with fiber $(\tilde{\mathcal{P}}, \tilde{\mathcal{F}})$ over the base $\hat{\Delta}$ (respectively, a bundle with fiber $\tilde{\Delta}$ over the base $(\hat{\mathcal{P}}, \hat{\mathcal{F}})$) if there exists a short exact sequence

$$0 \to \tilde{\mathfrak{t}} \xrightarrow{\iota} \mathfrak{t} \xrightarrow{\pi} \hat{\mathfrak{t}} \to 0$$

so that

- each $P \in \mathcal{P}$ is a bundle with fiber $\tilde{P} \in \tilde{\mathcal{P}}$ over the base $\hat{\Delta}$ (respectively, a bundle with fiber $\tilde{\Delta}$ over a base $\hat{P} \in \hat{\mathcal{P}}$.
- if F occurs in $\mathcal{F}$, then there $\tilde{F}$ (respectively, $\hat{F}$) must occur in $\tilde{\mathcal{F}}$ (respectively, $\hat{\mathcal{F}}$) such that F is a bundle with fiber $\tilde{F}$ over the base $\hat{\Delta}$ (respectively, a bundle with fiber $\tilde{\Delta}$ over a base $\hat{F}$).

Lemma

A quasitoric manifold over a cube $(\Delta^1)^3$ is homeomorphic to a toric origami manifold.

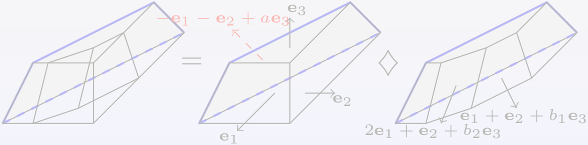
Let M be a quasitoric manifold over $(\Delta^1)^3$. Up to homeomorphism, by S. Hasui, the reduced characteristic matrix of M is one of the following

$$\begin{pmatrix} 1 & 0 & 0 \\ a_1 & 1 & 0 \\ a_2 & a_3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \\ b_1 & b_2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & c_1 \\ 1 & 1 & c_2 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

.

(i) The first case gives rise to a Bott manifold which admits a symplectic toric structure.

(ii)



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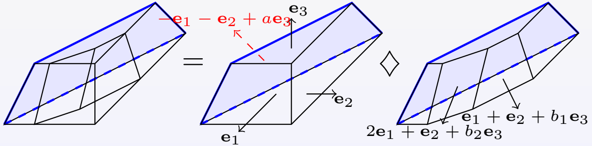
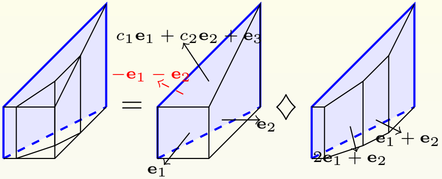
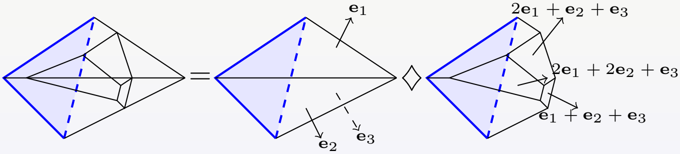


(iii)



$=$
$c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + \mathbf{e}_3$
$-\mathbf{e}_1 - \mathbf{e}_2$
$\mathbf{e}_2$
$\mathbf{e}_1$
$\diamondsuit$
$\mathbf{e}_1 + \mathbf{e}_2$
$2\mathbf{e}_1 + \mathbf{e}_2$

(iv)



$=$
$\mathbf{e}_1$
$\mathbf{e}_2$
$\mathbf{e}_3$
$\diamondsuit$
$2\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$
$2\mathbf{e}_1 + 2\mathbf{e}_2 + \mathbf{e}_3$
$\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$

Remarks

Up to weakly equivariant homeomorphism, for a quasitoric manifold M over $(\Delta^1)^3$, if M has no bundle structure, then the reduced characteristic matrix of M is one of the following

$$\begin{pmatrix}1 & 2 & 2\\ 1 & 1 & 2\\ 1 & 1 & 1\end{pmatrix}, \begin{pmatrix}1 & 2 & 1\\ 0 & 1 & 1\\ 2 & 2 & 1\end{pmatrix}, \begin{pmatrix}1 & 1 & 1\\ 0 & 1 & 2\\ 2 & 1 & 1\end{pmatrix}, \begin{pmatrix}1 & 2 & 2\\ 0 & 1 & 2\\ 1 & 1 & 1\end{pmatrix}.$$

Future

Prove or disprove the following

1. any quasitoric manifold with $\beta_2 = 2$ is equivariantly homeomorphic to a toric origami manifold.
2. any quasitoric manifold over $\prod_{i=1}^{h} \Delta^{n_i}$ is equivariantly homeomorphic to a toric origami manifold.
3. any toric manifold is equivariantly homeomorphic to a toric origami manifold.

Thank you for your attention!