Quasitoric manifolds and toric origami manifolds

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Quasitoric manifolds

A *quasitoric manifold* is a closed smooth manifold $M$ of dim. $2n$ with a smooth $T^n$ action such that

- the action of $T^n$ on $M$ is locally standard;
- the orbit space $M/T^n$ can be identified with a simple polytope.

**Example**

1. projective non-singular toric varieties
2. $CP^2 \# CP^2$
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**Non-example**

For $n > 1$, $T^n$-action on $S^{2n} \subseteq \mathbb{C}^n \oplus \mathbb{R}$ given by

$$(t_1, \ldots, t_n) \cdot (z_1, \ldots, z_n, r) = (t_1 z_1, \ldots, t_n z_n, r)$$

is locally standard, but it cannot give rise to a quasitoric manifold.
Quasitoric manifolds

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is locally standard, but it cannot give rise to a quasitoric manifold.
Properties of quasitoric manifolds

1. There is a one-to-one correspondence

\[
\{\text{quasitoric manifolds } M \} \xleftrightarrow{1:1} \{\text{characteristic pairs } (P, \lambda)\}.
\]

2. Every omniorientation of a quasitoric manifold \( M \) determines a stably almost complex structure.
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A **symplectic manifold** $\mathcal{M}$ is a manifold equipped with a **symplectic form** $\omega \in \Omega^2(\mathcal{M})$ that is closed ($d\omega = 0$) and non-degenerate.

**Example**

The unit sphere $S^2$ in $\mathbb{R}^3$ is a symplectic manifold.

$$\omega_v(X, Y) = v \cdot (X \times Y) = \det(v, X, Y)$$
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Folded symplectic manifolds

A *folded symplectic form* on a $2n$-dim’l manifold $M$ is a closed 2-form $\omega$

- $\omega^n$ vanishes transversally on a submanifold $Z$,
- $\omega^n|_Z$ has maximal rank.

The submanifold $Z$ is called the *folding hypersurface* or *fold.*
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**Example**

1. $(\mathbb{R}^{2d}, x_1 dx_1 \wedge dy_1 + \sum_{i=2}^{d} dx_i \wedge dy_i), Z = \{x_1 = 0\}$.
2. $(S^{2n} \subset \mathbb{C}^n \oplus \mathbb{R}, \omega_{\mathbb{C}^n} \oplus 0), Z = S^{2n-1} \subset \mathbb{C}^n \oplus \{0\}$
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\textbf{Theorem [Cannas da Silva, 2010]}

Let $M$ be a $2n$-dim’l manifold with a stably almost complex structure $J$. Then $M$ admits a folded symplectic form consistent with $J$ in any degree 2 cohomology class.

\textbf{NOTE:} Omnioriented quasitoric manifolds admit folded symplectic forms.
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\textbf{NOTE:} Omnioriented quasitoric manifolds admit folded symplectic forms.
Origami manifolds

Let \((M, \omega)\) be a folded symplectic manifold with a fold \(Z\).

Let \(i: Z \hookrightarrow M\) be the inclusion.

Then \(i^* \omega\) has a 1-dimensional kernel on \(Z\).

**Definition** [Cannas da Silva-Guillemin-Pires, IMRN 2011]

\(\omega\) is called an *origami form* if this line field is the vertical bundle of an oriented \(S^1\)-fiber bundle \(Z \xrightarrow{\pi} B\) over a compact base \(B\).
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**Example (continued)**

1. \(\omega = x_1 dx_1 \wedge dy_1 + \sum_{i=2}^d dx_i \wedge dy_i\) on \(\mathbb{R}^{2d}\) is not origami.
2. \(\omega_{\mathbb{C}^n} \oplus 0\) on \(S^{2n} \subset \mathbb{C}^n \oplus \mathbb{R}\) is origami because the line field is the vertical bundle of the Hopf bundle \(\pi: S^{2n-1} \rightarrow \mathbb{C}P^{n-1}\).
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Toric origami manifolds

The action of a Lie group $G$ on an origami manifold $(M, \omega)$ is *Hamiltonian* if it admits a *moment map* $\mu: M \to \mathfrak{g}^*$ satisfying the conditions:

- $\mu$ collects Hamiltonian functions, i.e., $d\langle \mu, X \rangle = \iota_X \# \omega$, $\forall X \in \mathfrak{g} := \text{Lie}(G)$, where $X\#$ is the vector field generated by $X$;
- $\mu$ is equivariant with respect to the given action of $G$ on $M$ and the coadjoint action of $G$ on the dual vector space $\mathfrak{g}^*$.

A *toric origami manifold* is a compact connected origami manifold $(M, \omega)$ equipped with an effective Hamiltonian action of a torus $T$ with $\dim T = \frac{1}{2} \dim M$.

**NOTE:** If $Z = \emptyset$, a toric origami manifold is a toric symplectic manifold.
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Examples

1. $T = (S^1)^2$ acts on $(\mathbb{C}P^2, \omega)$ by

$$(t_1, t_2) \cdot [z_1, z_2, z_3] = [t_1 z_1, t_2 z_2, z_3]$$

with moment map

$$\mu(z_1, z_2, z_3) = \left( \frac{|z_1|^2}{|z_1|^2 + |z_2|^2 + |z_3|^2}, \frac{|z_2|^2}{|z_1|^2 + |z_2|^2 + |z_3|^2} \right).$$

2. $T = (S^1)^2$ acts on $(S^4 \subset \mathbb{C}^2 \oplus \mathbb{R}, \omega_{\mathbb{C}^n} \oplus 0)$ by

$$(t_1, t_2) \cdot (z_1, z_2, r) = (t_1 z_1, t_2 z_2, r)$$

with moment map

$$\mu(z_1, z_2, r) = (|z_1|^2, |z_2|^2).$$
Origami templates

An origami template consists of a pair $(\mathcal{P}, \mathcal{F})$, where $\mathcal{P}$ is a collection of Delzant polytopes in $\mathbb{R}^n$ and $\mathcal{F}$ is a collection of facets and pairs of facets of polytopes in $\mathcal{P}$ s.t.

1. \( \forall \{F, F'\} \in \mathcal{F}, \text{ if } F \subset P \text{ and } F' \subset P', \text{ then } \exists U \subset \text{open } \mathbb{R}^n \text{ s.t. } U \cap P = U \cap P'; \)
2. if \( F \in \mathcal{F} \) or \( \{F, F'\} \in \mathcal{F} \), then neither \( F \) nor any of its neighboring facets occur elsewhere in \( \mathcal{F} \);
3. the topological space, denoted \( |(\mathcal{P}, \mathcal{F})| \), constructed from the disjoint union \( \bigsqcup P_j, P_j \in \mathcal{P} \), by identifying facet pairs in \( \mathcal{F} \) is connected.

Theorem (Cannas da Silva-Guillemin-Pires, 2011)

\{ toric origami manifolds \} \overset{1:1}{\longleftrightarrow} \{ origami templates \}

NOTE: \( M/T \cong |(\mathcal{P}, \mathcal{F})| \) as a manifold with corners.
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Remarks

Non-orientable toric origami manifolds

\[ \mathbb{R}P^2 \]
with \( \mathcal{F} = \{ \text{a red dot} \} \)

Klein bottle
with \( \mathcal{F} = \{ \text{two red dots} \} \)

The black dot is corresponding to a fixed point, and each red dot is corresponding to the point whose isotropy subgroup is \( \mathbb{Z}_2 \).

Non-simply connected origami templates

\[ \mathcal{P} = \{ \text{two intervals} \} \]
\[ \mathcal{F} = \{ \text{two pairs of blue dots} \} \]

An origami template is **oriented** if the polytopes in \( \mathcal{P} \) come with an orientation and \( \mathcal{F} \) consists solely of pairs of facets which belong to polytopes with opposite orientations.
Motivation

For an orientable toric origami manifold $M$,
- if $M^T \neq \emptyset$, $M$ is a torus manifold$^2$.
- if $|(\mathcal{P},\mathcal{F})|$ is (combinatorially) a simple polytope, $M$ is a quasitoric manifold.

Question

What about the converse?

---

$^2$A torus manifold is a $2n$-dim'l oriented compact connected smooth manifold $M$ with an effective action of $T = T^n$ with $M^T \neq \emptyset$. 
For an orientable toric origami manifold $M$,
- if $M^T \neq \emptyset$, $M$ is a torus manifold\(^2\).
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What about the converse?

\(^2\)A torus manifold is a $2n$-dim'l oriented compact connected smooth manifold $M$ with an effective action of $T = T^{n}$ with $M^T \neq \emptyset$. 
projective toric manifolds

quasitoric manifolds

$T^2, \mathbb{R}P^2$

toric origami manifolds

torus manifolds

???

???

Toric origami manifolds
Not all torus manifolds admit toric origami forms.

From the following facts

- $S^4 \times S^4$ is simply connected,
- $(S^4 \times S^4)/T^4 = |(P, F)|$ has four facets,
- $(S^4 \times S^4)/T^4 = |(P, F)|$ has four vertices,

we can show that $S^4 \times S^4$ cannot be a toric origami manifold.

**Theorem (Masuda-P)**

Let $S^{2n_i}$ ($n_i \geq 1$) be the standard $2n_i$-sphere with the standard $T^{n_i}$-sphere with the standard $T^{n_i}$-action for $i = 1, \ldots, k$. Then $\prod_{i=1}^{k} S^{2n_i}$ with the product action of $\prod_{i=1}^{k} T^{n_i}$ admits a toric origami form if and only if $n_i = 1$ except for one $i$. 
Not all torus manifolds admit toric origami forms.

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we can show that \( S^4 \times S^4 \) cannot be a toric origami manifold.

**Theorem (Masuda-P)**

Let \( S^{2n_i} \) \( (n_i \geq 1) \) be the standard 2\( n_i \)-sphere with the standard \( T^{n_i} \)-sphere with the standard \( T^{n_i} \)-action for \( i = 1, \ldots, k \). Then \( \prod_{i=1}^{k} S^{2n_i} \) with the product action of \( \prod_{i=1}^{k} T^{n_i} \) admits a toric origami form if and only if \( n_i = 1 \) except for one \( i \).
Quasitoric and toric origami

- Toric origami manifolds
- Quasitoric manifolds
- Torus manifolds
- Projective toric manifolds

- $T^2, \mathbb{R}P^2$
- $S^4 \times S^4$
All 4-dimensional quasitoric manifolds are equivariantly diffeomorphic to some toric origami manifolds.

- There is a one-to-one correspondence

\[
\left\{ \text{simply conn. cpt sm. 4-mfd } M \text{ with effective } T^2 \text{ action} \right\} \\
\overset{1:1}{\leftrightarrow} \left\{ \text{unimodular sequences } v_1, \ldots, v_d \right\},
\]

where \( v_i \) and \( v_{i+1} \) form a basis of \( \mathbb{Z}^2 \) for \( i = 1, \ldots, d \) and \( v_{d+1} = v_1 \).

- For \( d \geq 4 \), there is \( v_j \) (\( 1 \leq j \leq d \)) which satisfies

\[
\epsilon_{j-1}v_{j-1} + \epsilon_j v_{j+1} + a_j v_j = 0 \quad (\text{with } a_j = \pm 1 \text{ or } 0, \ \epsilon_j = \det(v_j, v_{j+1})).
\]

- Use induction on \( d \).
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- Use induction on \(d\).

**Theorem (Masuda-P)**

Any simply connected compact smooth 4-manifold \(M\) with an effective smooth action of \(T^2\) is equivariantly diffeomorphic to a toric origami manifold.
All 4-dimensional quasitoric manifolds are equivariantly diffeomorphic to some toric origami manifolds.

- There is a one-to-one correspondence

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Any simply connected compact smooth 4-manifold \(M\) with an effective smooth action of \(T^2\) is equivariantly diffeomorphic to a toric origami manifold.
Example $\mathbb{CP}^2 \# \mathbb{CP}^2$
Remarks

Not all omniorientations of a quasitoric manifold determine toric origami forms.

Let $M_b$ be an omnioriented quasitoric manifold which is corresponding to a unimodular sequence $(1, 0), (0, 1), (1, b), (0, 1)$. Then, the omniorientation of $M_b$ admits a toric origami form if and only if $b = \pm 1$ or $b = \pm 2$. 

\[
\begin{array}{c}
M_b \\
(1) \\
(b) \\
(0)
\end{array}
\quad
\begin{array}{c}
(0) \\
1 \\
(1)
\end{array}
\]
Observation

Let $F_1, \ldots, F_n, F_{n+1}$ be the facets of $\Delta^n$.

1. By cutting the face $F_1 \cap \cdots \cap F_k$, we get a new polytope $P \approx \Delta^{n-k+1} \times \Delta^{k-1}$ whose facets $G, F'_1, \ldots, F'_k, F'_{k+1}, \ldots, F'_{n+1}$, where $G \approx \Delta^{n-k} \times \Delta^{k-1}$, $F'_1, \ldots, F'_k \approx \Delta^{n-k+1} \times \Delta^{k-2}$, and $F'_{k+1}, \ldots, F'_{n+1} \approx \Delta^{n-k} \times \Delta^{k-1}$.

2. Let $P_1$ be the polytope obtained from $\Delta^n$ by cutting a vertex $F_1 \cap \cdots \cap F_n$ of $\Delta^n$. Then facets of $P_1$ are of the form $G_1, F'_{n+1} \approx \Delta^{n-1}$ & $F'_1, \ldots, F'_n \approx \text{vc}(\Delta^{n-1})$. 
Observation

Let $F_1, \ldots, F_n, F_{n+1}$ be the facets of $\triangle^n$.

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$$G_1, F'_{n+1} \approx \Delta^{n-1} \& F'_1, \ldots, F'_n \approx \text{vc}(\Delta^{n-1}).$$

3. By cutting a face $G_1 \cap F'_1 \cap \cdots \cap F'_k$ from $P_1$, we get a new polytope $P_2 \approx \text{vc}(\Delta^{n-k} \times \Delta^k)$ whose facets are

$$F''_{n+1} \approx \Delta^{n-1}, G_2 \approx \Delta^{n-k-1} \times \Delta^k, G'_1 \approx \Delta^{n-k} \times \Delta^{k-1},$$

$$F''_1, \ldots, F''_k \approx \text{vc}(\Delta^{n-k} \times \Delta^{k-1}), F''_{k+1}, \ldots, F''_n \approx \text{vc}(\Delta^{n-k-1} \times \Delta^k).$$
Observation

Let $F_1, \ldots, F_n, F_{n+1}$ be the facets of $\Delta^n$.

1. By cutting the face $F_1 \cap \cdots \cap F_k$, we get a new polytope $P \approx \Delta^{n-k+1} \times \Delta^{k-1}$ whose facets $G, F_1', \ldots, F_k', F_{k+1}', \ldots, F_{n+1}'$, where $G \approx \Delta^{n-k} \times \Delta^{k-1}$, $F_1', \ldots, F_k' \approx \Delta^{n-k+1} \times \Delta^{k-2}$, and $F_{k+1}', \ldots, F_{n+1}' \approx \Delta^{n-k} \times \Delta^{k-1}$.

2. Let $P_1$ be the polytope obtained from $\Delta^n$ by cutting a vertex $F_1 \cap \cdots \cap F_n$ of $\Delta^n$. Then facets of $P_1$ are of the form

   \[ G_1, F_{n+1}' \approx \Delta^{n-1} \& F_1', \ldots, F_n' \approx \text{vc}(\Delta^{n-1}). \]

3. By cutting a face $G_1 \cap F_1' \cap \cdots \cap F_k'$ from $P_1$, we get a new polytope $P_2 \approx \text{vc}(\Delta^{n-k} \times \Delta^k)$ whose facets are

   \[ F_{n+1}'' \approx \Delta^{n-1}, G_2 \approx \Delta^{n-k-1} \times \Delta^k, G_1' \approx \Delta^{n-k} \times \Delta^{k-1}, \]

   \[ F_1'', \ldots, F_k'' \approx \text{vc}(\Delta^{n-k} \times \Delta^{k-1}), F_{k+1}'', \ldots, F_n'' \approx \text{vc}(\Delta^{n-k-1} \times \Delta^k). \]
Now, let $M_{r,s}$ be a quasitoric manifold over $\Delta^n \times \Delta^m$ whose characteristic matrix is of the form $\Lambda_{r,s}$.

$$\Lambda_{r,s} = \begin{pmatrix}
E_n & 1 & 2 \\
& \vdots & \vdots \\
& \vdots & \vdots \\
1 & 2 & \\
1 & 0 & s \\
1 & 0 \\
\end{pmatrix}
\begin{pmatrix}
E_m & 1 & 1 \\
& \vdots & r \\
& \vdots & \vdots \\
1 & 1 & \\
0 & 1 & \\
0 & 1 \\
\end{pmatrix}$$
Lemma

A quasitoric manifold $M_{m,n}$ over $\Delta^n \times \Delta^m$ is equivariantly homeomorphic to a toric origami manifold.

1. Let $F_1, \ldots, F_{n+m+1}$ be the facets of $\Delta^{n+m}$ and let the corresponding outward normal vectors are $e_1, \ldots, e_{n+m}, -e_1 - \cdots - e_{n+m}$.

2. Let $P$ be the polytope obtained by cutting $F_1 \cap \cdots \cap F_{n+m}$ and then by cutting $G \cap F_1' \cap \cdots \cap F_{n}'$, where $G, F_1', \ldots, F_{n+m}'$ are new facets obtained from the first cutting.

3. The origami template which we want is the pair of $\mathcal{P} = \{\Delta^{n+m}, P\}$ and $\mathcal{F} = \{(F_{n+m+1}, F''_{n+m+1})\}$. 

\[
\Delta^1 \times \Delta^2 = \begin{align*}
\begin{array}{c}
\Delta^1 \\
\text{e}_1 \\
\text{e}_2 \\
\text{e}_3 \\
\end{array}
\end{align*}
\begin{array}{c}
\Delta^2 \\
\text{2e}_1 + \text{e}_2 + \text{e}_3 \\
\text{e}_1 + \text{e}_2 + \text{e}_3 \\
\end{array} = \begin{array}{c}
\Delta^2 \\
\text{e}_3 \\
\text{e}_2 \\
\text{e}_1 \\
\end{array}
\begin{array}{c}
\Delta^1 \\
\text{2e}_1 + \text{e}_2 + \text{e}_3 \\
\text{e}_1 + \text{e}_2 + \text{e}_3 \\
\end{array}
\begin{array}{c}
\Delta^1 \\
\text{e}_3 \\
\text{e}_2 \\
\text{e}_1 \\
\end{array}
\begin{array}{c}
\Delta^2 \\
\text{2e}_1 + \text{e}_2 + \text{e}_3 \\
\text{e}_1 + \text{e}_2 + \text{e}_3 \\
\end{array}
\]
Lemma

A quasitoric manifold $M_{m,n}$ over $\Delta^n \times \Delta^m$ is equivariantly homeomorphic to a toric origami manifold.

1. Let $F_1, \ldots, F_{n+m+1}$ be the facets of $\Delta^{n+m}$ and let the corresponding outward normal vectors are $e_1, \ldots, e_{n+m}, -e_1 - \cdots - e_{n+m}$.

2. Let $P$ be the polytope obtained by cutting $F_1 \cap \cdots \cap F_{n+m}$ and then by cutting $G \cap F'_1 \cap \cdots \cap F'_{n}$, where $G, F'_1, \ldots, F'_{n+m}$ are new facets obtained from the first cutting.

3. The origami template which we want is the pair of $\mathcal{P} = \{\Delta^{n+m}, P\}$ and $\mathcal{F} = \{(F_{n+m+1}, F''_{n+m+1})\}$.

\[ \Delta^1 \times \Delta^2 = \begin{array}{c} e_3 \\ e_1 \\ e_2 \end{array}, \quad \Delta^2 \times \Delta^1 = \begin{array}{c} 2e_1 + e_2 + e_3 \\ e_1 \\ e_2 \\ e_3 \end{array} \]
Lemma

A quasitoric manifold $M_{m,1}$ over $\Delta^n \times \Delta^m$ is equivariantly homeomorphic to a toric origami manifold.

1. Let $F_1, \ldots, F_{n+m+1}$ be the facets of $\Delta^{n+m}$ and let the corresponding outward normal vectors are $e_1, \ldots, e_{n+m}, -e_1 - \cdots - e_{n+m}$.

2. Let $P$ be the polytope obtained from $\Delta^{n+m}$ by cutting $F_2 \cap \cdots \cap F_{n+m+1}$ and then by cutting $G \cap F'_2 \cap \cdots \cap F'_n \cap F'_{n+m+1}$, where $G$ and $F'_1, \ldots, F'_n$ are new facets obtained by the first cutting.

3. $\mathcal{P} = \{\Delta^{n+m}, P\}$ and $\mathcal{F} = (F_1, F''_1)$. 

\[ \begin{array}{c}
- e_1 - e_2 - e_3 \\
\end{array} \]
Lemma

A quasitoric manifold $M_{m,1}$ over $\Delta^n \times \Delta^m$ is equivariantly homeomorphic to a toric origami manifold.

1. Let $F_1, \ldots, F_{n+m+1}$ be the facets of $\Delta^{n+m}$ and let the corresponding outward normal vectors are $e_1, \ldots, e_{n+m}, -e_1 - \cdots - e_{n+m}$.

2. Let $P$ be the polytope obtained from $\Delta^{n+m}$ by cutting $F_2 \cap \cdots \cap F_{n+m+1}$ and then by cutting $G \cap F'_2 \cap \cdots \cap F'_n \cap F'_{n+m+1}$, where $G$ and $F'_1, \ldots, F'_n$ are new facets obtained by the first cutting.

3. $\mathcal{P} = \{\Delta^{n+m}, P\}$ and $\mathcal{F} = (F_1, F'_1)$. 

![Diagram](image-url)
**Lemma**

Two quasitoric manifolds $M_{r,s}$ and $M_{m-r+1,n-s+1}$ over $\Delta^n \times \Delta^m$ are weakly equivariantly diffeomorphic.

Define an isomorphism $\Theta: \mathbb{Z}^{n+m} \to \mathbb{Z}^{n+m}$ by

\[
\begin{align*}
\mathbf{e}_1 & \mapsto -\sum_{i=1}^{n+1} \mathbf{e}_i - \sum_{j=r+1}^{m} \mathbf{e}_{n+j} \\
\mathbf{e}_{n+1} & \mapsto 2\mathbf{e}_1 + 2\sum_{i=s+1}^{n} \mathbf{e}_i + \sum_{j=1}^{m} \mathbf{e}_{n+j} \\
\mathbf{e}_2, \ldots, \mathbf{e}_s, \mathbf{e}_{n+r+1}, \ldots, \mathbf{e}_{n+m} & \mapsto \mathbf{e}_2, \ldots, \mathbf{e}_s, \mathbf{e}_{n+r+1}, \ldots, \mathbf{e}_{n+m} \\
\mathbf{e}_{s+1}, \ldots, \mathbf{e}_n, \mathbf{e}_{n+2}, \ldots, \mathbf{e}_{n+r} & \mapsto -\mathbf{e}_{s+1}, \ldots, -\mathbf{e}_n, -\mathbf{e}_{n+2}, \ldots, -\mathbf{e}_{n+r}
\end{align*}
\]

**Proposition**

Any quasitoric manifold over $\Delta^n \times \Delta^m$ with $n, m \leq 2$ is weakly equivariantly homeomorphic to a toric origami manifold.
Lemma

Two quasitoric manifolds $M_{r,s}$ and $M_{m-r+1,n-s+1}$ over $\Delta^n \times \Delta^m$ are weakly equivariantly diffeomorphic.

Define an isomorphism $\Theta: \mathbb{Z}^{n+m} \to \mathbb{Z}^{n+m}$ by

$e_1 \mapsto -\sum_{i=1}^{n+1} e_i - \sum_{j=r+1}^{m} e_{n+j}$
$e_{n+1} \mapsto 2e_1 + 2\sum_{i=s+1}^{n} e_i + \sum_{j=1}^{m} e_{n+j}$
$e_2, \ldots, e_s, e_{n+r+1}, \ldots, e_{n+m} \mapsto e_2, \ldots, e_s, e_{n+r+1}, \ldots, e_{n+m}$
$e_{s+1}, \ldots, e_n, e_{n+2}, \ldots, e_{n+r} \mapsto -e_{s+1}, \ldots, -e_n, -e_{n+2}, \ldots, -e_{n+r}$

Proposition

Any quasitoric manifold over $\Delta^n \times \Delta^m$ with $n, m \leq 2$ is weakly equivariantly homeomorphic to a toric origami manifold.
Theorem [Choi & P & Suh]

Any quasitoric manifold over $\Delta^n \times \Delta^1$ is homeomorphic to
- two-stage generalized Bott manifolds,
- $M_{1,1}$ over $\Delta^n \times \Delta^1$, or
- $M_{1,1}$ over $\Delta^1 \times \Delta^n$.

$$M_{1,1} \cong_{eq} M_{1,n} \text{ over } \Delta^n \times \Delta^1 \text{ & } M_{1,1} \cong_{eq} M_{n,1} \text{ over } \Delta^1 \times \Delta^n.$$ 

Proposition

Any quasitoric manifold over $\Delta^n \times \Delta^1$ is homeomorphic to a toric origami manifold.
Denote

\[ 1_n = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}_{n \times 1}, \quad 2_m = \begin{pmatrix} 2 \\ \vdots \\ 2 \end{pmatrix}_{m \times 1}. \]

Consider a quasitoric manifold \( M \) over \( \prod_{i=1}^{h} \Delta_{n_i} \) whose reduced characteristic matrix is

\[
\begin{pmatrix}
1_{n_1} & 2_{n_1} & \cdots & 2_{n_1} \\
1_{n_2} & 1_{n_2} & \cdots & 2_{n_2} \\
\vdots & \vdots & \ddots & \vdots \\
1_{n_h} & 1_{n_h} & \cdots & 1_{n_h}
\end{pmatrix}.
\]

**Proposition**

\( M \) admits a toric origami form.
Let $F_1, \ldots, F_n, F_{n+1}$ be the facets of $\Delta^n$ whose outward normal vectors are $e_1, \ldots, e_n, -e_1 - \cdots - e_n$, where $n = \sum_{i=1}^{h} n_i$.

1. Let $P_1$ be the polytope obtained from $\Delta^n$ by cutting $F_1 \cap \cdots \cap F_n$. Then, $P_1 \approx \text{vc}(\Delta^n)$ and we get new facets $G_1, F_1^{(1)}, \ldots, F_n^{(1)}$.

2. Let $P_2$ be the polytope obtained from $P_1$ by cutting $G_1 \cap F_1^{(1)} \cap \cdots F_n^{(1)}$. Then, $P_2 \approx \text{vc}(\Delta^{n-n_1} \times \Delta^{n_1})$ and we get new facets $G_2, G_1^{(1)}, F_1^{(2)}, \ldots, F_n^{(2)}$.

3. Let $P_3$ be the polytope obtained from $P_2$ by cutting $G_2 \cap F_{n_1+1}^{(2)} \cap \cdots F_{n_1+n_2}^{(2)}$. Then $P_3 \approx \text{vc}(\Delta^{n-n_1-n_2} \times \Delta^{n_2} \times \Delta^{n_1})$ and we get new facets $G_3, G_1^{(2)}, G_2^{(1)}, F_1^{(3)}, \ldots, F_n^{(3)}$.

\[\vdots\]

4. Let $P_h$ be the polytope obtained from $P_{h-1}$ by cutting $G_{h-1} \cap F_{n-n_h+1}^{(h-1)} \cap \cdots \cap F_n^{(h-1)}$. Then $P_h \approx \text{vc}(\Delta^{n_h} \times \cdots \times \Delta^{n_1})$ and we get new facets $G_h, G_{h-1}, \ldots, G_1^{(h-1)}, F_1^{(h)}, \ldots, F_n^{(h)}$.

5. $\mathcal{P} = \{\Delta^n, P_h\}$, $\mathcal{F} = (F_{n+1}, F_n^{(h)}).$
Denote

\[ 1_n = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}_{n \times 1}, \quad 0_m = \begin{pmatrix} 2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{m \times 1}, \quad (2) = \begin{pmatrix} 2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{n_1 \times 1}. \]

Consider a quasitoric manifold \( M \) over \( \prod_{i=1}^{h} \Delta^{n_i} \) whose reduced characteristic matrix is

\[
\begin{pmatrix}
1_{n_1} & (2) & \cdots & (2) \\
1_{n_2} & 1_{n_2} & \cdots & 0_{n_2} \\
\vdots & \vdots & \ddots & \vdots \\
1_{n_h} & 1_{n_h} & \cdots & 1_{n_h}
\end{pmatrix}.
\]

**Proposition**

\( M \) admits a toric origami form.
Bundle structures on polytopes

Let \( \tilde{\Delta} = \cap_{j=1}^{\tilde{N}} \{ x \in \tilde{t}^* \mid \langle \tilde{\eta}_j, x \rangle \leq \kappa_j \} \) and \( \hat{\Delta} = \cap_{i=1}^{\hat{N}} \{ x \in \hat{t}^* \mid \langle \hat{\eta}_i, x \rangle \leq \hat{\kappa}_i \} \) be simple polytopes. We say that a simple polytope \( \Delta \subset t^* \) is a bundle with fiber \( \tilde{\Delta} \) over the base \( \hat{\Delta} \) if there exists a short exact sequence

\[
0 \to \tilde{t} \xrightarrow{\iota} t \xrightarrow{\pi} \hat{t} \to 0
\]

so that

- \( \Delta \) is combinatorially equivalent to the product \( \tilde{\Delta} \times \hat{\Delta} \).
- If \( \tilde{\eta}_j' \) denotes the outward conormal to the facet \( \tilde{F}_j' \) of \( \Delta \) which corresponds to \( \tilde{F}_j \times \hat{\Delta} \subset \tilde{\Delta} \times \hat{\Delta} \), then \( \tilde{\eta}_j' = \iota(\tilde{\eta}_j) \) for all \( 1 \leq j \leq \tilde{N} \).
- If \( \hat{\eta}_i' \) denotes the outward conormal to the facet \( \hat{F}_i' \) of \( \Delta \) which corresponds to \( \tilde{\Delta} \times \hat{F}_i \subset \tilde{\Delta} \times \hat{\Delta} \), then \( \pi(\hat{\eta}_i') = \hat{\eta}_i \) for all \( 1 \leq i \leq \hat{N} \).

The facets \( \tilde{F}_1', \ldots, \tilde{F}_{\tilde{N}}' \) are called fiber facets and the facets \( \hat{F}_1', \ldots, \hat{F}_{\hat{N}}' \) are called base facets.
Bundle structures on origami templates

Let \((P, F)\), \((\tilde{P}, \tilde{F})\), and \((\hat{P}, \hat{F})\) be origami templates defined in \(t^*, \tilde{t}^*, \) and \(\hat{t}^*\), respectively.

**Definition**

An origami template \((P, F)\) is a bundle with fiber \((\tilde{P}, \tilde{F})\) over the base \(\tilde{\Delta}\) (respectively, a bundle with fiber \(\hat{\Delta}\) over the base \((\hat{P}, \hat{F})\)) if there exists a short exact sequence

\[
0 \rightarrow \tilde{t} \xrightarrow{l} t \xrightarrow{\pi} \hat{t} \rightarrow 0
\]

so that

- each \(P \in P\) is a bundle with fiber \(\tilde{P} \in \tilde{P}\) over the base \(\tilde{\Delta}\) (respectively, a bundle with fiber \(\hat{\Delta}\) over a base \(\hat{P} \in \hat{P}\)).
- if \(F\) occurs in \(F\), then there \(\tilde{F}\) (respectively, \(\hat{F}\)) must occur in \(\tilde{F}\) (respectively, \(\hat{F}\)) such that \(F\) is a bundle with fiber \(\tilde{F}\) over the base \(\tilde{\Delta}\) (respectively, a bundle with fiber \(\hat{\Delta}\) over a base \(\hat{F}\)).
Lemma

A quasitoric manifold over a cube \((\Delta^1)^3\) is homeomorphic to a toric origami manifold.

Let \(M\) be a quasitoric manifold over \((\Delta^1)^3\). Up to homeomorphism, by S. Hasui, the reduced characteristic matrix of \(M\) is one of the following

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}, \quad 
\begin{pmatrix}
1 & 2 & 0 \\
1 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}, \quad 
\begin{pmatrix}
1 & 2 & c_1 \\
1 & 1 & c_2 \\
0 & 0 & 1 \\
\end{pmatrix}, \quad 
\begin{pmatrix}
1 & 2 & 2 \\
1 & 1 & 2 \\
1 & 1 & 1 \\
\end{pmatrix}
\]

(i) The first case gives rise to a Bott manifold which admits a symplectic toric structure.

(ii)
Lemma

A quasitoric manifold over a cube \((\Delta^1)^3\) is homeomorphic to a toric origami manifold.

Let \(M\) be a quasitoric manifold over \((\Delta^1)^3\). Up to homeomorphism, by S. Hasui, the reduced characteristic matrix of \(M\) is one of the following

\[
\begin{pmatrix}
1 & 0 & 0 \\
a_1 & 1 & 0 \\
a_2 & a_3 & 1
\end{pmatrix},
\begin{pmatrix}
1 & 2 & 0 \\
1 & 1 & 0 \\
b_1 & b_2 & 1
\end{pmatrix},
\begin{pmatrix}
1 & 2 & c_1 \\
1 & 1 & c_2 \\
0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
1 & 2 & 2 \\
1 & 1 & 2 \\
1 & 1 & 1
\end{pmatrix}
\].

(i) The first case gives rise to a Bott manifold which admits a symplectic toric structure.

(ii)
Remarks

Up to weakly equivariant homeomorphism, for a quasitoric manifold $M$ over $(\Delta^1)^3$, if $M$ has no bundle structure, then the reduced characteristic matrix of $M$ is one of the following

$$
\begin{pmatrix}
1 & 2 & 2 \\
1 & 1 & 2 \\
1 & 1 & 1
\end{pmatrix},
\begin{pmatrix}
1 & 2 & 1 \\
0 & 1 & 1 \\
2 & 2 & 1
\end{pmatrix},
\begin{pmatrix}
1 & 1 & 1 \\
0 & 1 & 2 \\
2 & 1 & 1
\end{pmatrix},
\begin{pmatrix}
1 & 2 & 2 \\
0 & 1 & 2 \\
1 & 1 & 1
\end{pmatrix}.
$$
Future

Prove or disprove the following

1. any quasitoric manifold with $\beta_2 = 2$ is equivariantly homeomorphic to a toric origami manifold.
2. any quasitoric manifold over $\prod_{i=1}^{h} \Delta^{n_i}$ is equivariantly homeomorphic to a toric origami manifold.
3. any toric manifold is equivariantly homeomorphic to a toric origami manifold.
Thank you for your attention!