The T-equivariant integral cohomology ring of E_6/T

Takashi SATO

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Overview

- Introduction of the GKM theory
- Two theorems on GKM fiber bundles
- Apply these theorems to $\mathrm{Spin}(10) \cdot T^1/T \to E_6/T \to EIII$.

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Preceding studies

- Fukukawa, Ishida, and Masuda determined $H_T^*(G/T)$ for any Lie group G of classical type (except of type C) in 2011.
- Fukukawa determined $H_T^*(G_2/T)$ in 2012.
- I determined $H_T^*(F_4/T)$ in 2013.

A manifold with a torus action

M: a closed, connected manifold
T: a torus
Assume that T acts on M smoothly and M^T is a finite set.
$H_T^*(M) = H^*(ET \times_T M)$: the equivariant integral cohomology ring of M.
We suppose that the restriction map

$$i^*: H_T^*(M) \to H_T^*(M^T) \qquad \text{is injective.}$$

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We suppose that the restriction map

$$i^* : H_T^*(M) \to H_T^*(M^T) \cong \prod_{p \in M^T} H^*(BT) \text{ is injective.}$$

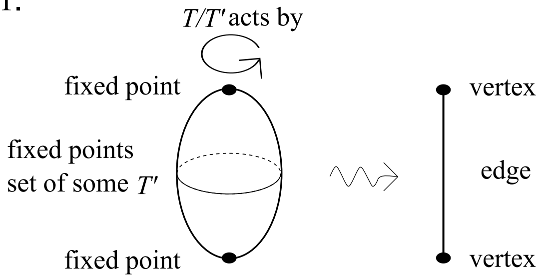
We want to determine the image of i^* in $\prod_{p \in M^T} H^*(BT)$.

A graph M_1/T

M_1: the union of the fixed point set of some subtori of T of codim. 1
Assume that $\dim M_1 = 2$. $\rightsquigarrow$ Regard M_1 as a graph.

- The vertex set is M^T
- The edge set is the set of invariant 2-spheres

An invariant 2-sphere is a part of fixed point set of a subtorus T' of codimension 1.



This graph has much geometrical information on M.

GKM graphs and GKM functions

G: a compact, connected Lie group, T: a maximal torus of G.
T acts on G/T by the left multiplication.

Definition 1 (The GKM graph of G/T)

The vertex set: $(G/T)^T \cong W(G)$
$w, w' \in W(G)$ are adjacent $\Leftrightarrow$ $\exists\alpha$: a root s.t. $w' = \sigma_\alpha w$
The edge ww' is labeled by $(\pm)\alpha$.

Definition 2 (GKM functions)

A set map $f \in \mathrm{Map}(W(G), H^*(BT)) = \prod_{W(G)} H^*(BT)$ is called a GKM function if it satisfies the following condition.

For any edge ww' whose label is α, $f(w) - f(w') \in (\alpha) \subset H^*(BT)$

Example: the GKM graph of $SU(3)/T$

T: the standard maximal torus of $SU(3)$, $T \subset T^3 \subset U(3)$
t_1, t_2, t_3: the standard basis of $H^2(BT^3)$ ($t_1 + t_2 + t_3 = 0$ on $H^2(BT)$)

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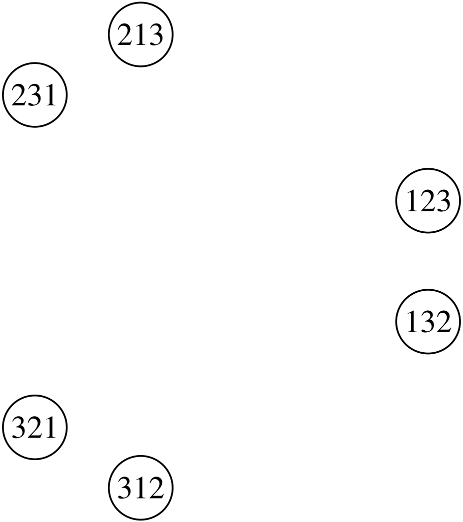

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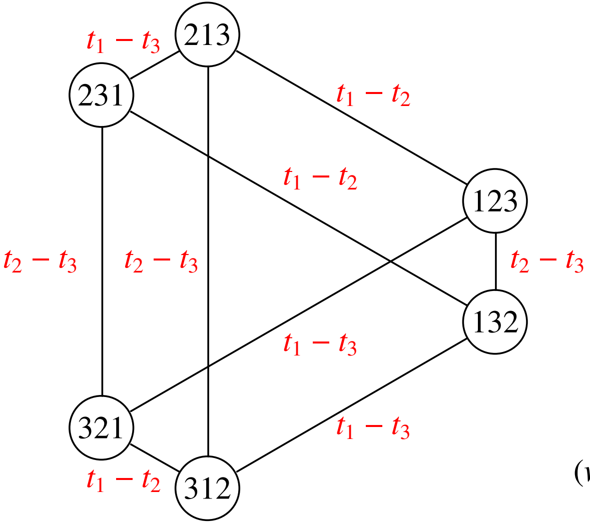

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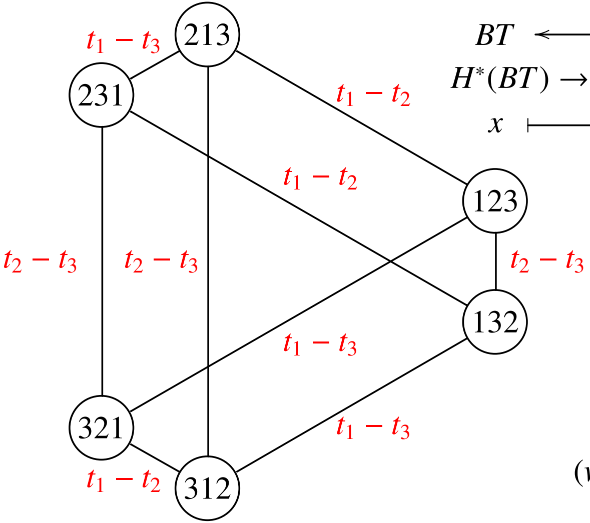

$$BT \longleftarrow BT \times M^T$$

$$H^*(BT) \to H_T^*(M) \to H_T^*(M^T)$$

$$x \longmapsto \text{const. func. } x$$

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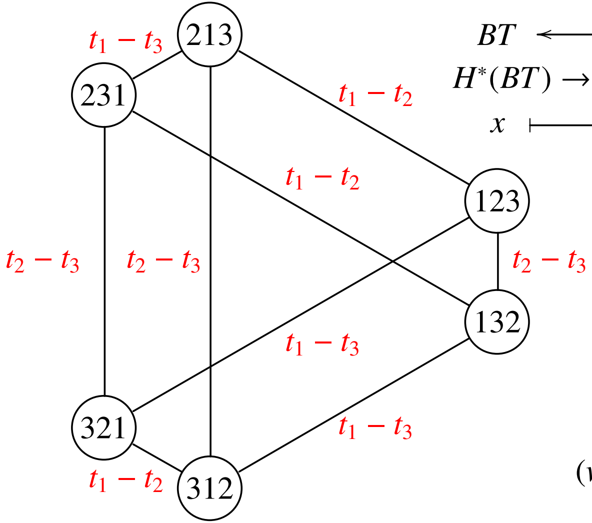

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$$\tau_i(w) = w(t_i)$$

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$(w' = \sigma_\alpha w)$

The GKM theory

Let $\mathcal{G}$ be the GKM graph of G/T.

Definition 3 (the "cohomology" ring of $\mathcal{G}$)

$H^*(\mathcal{G})$ is the subring of $\mathrm{Map}(W(G), H^*(BT))$ which consists of all GKM functions.

Theorem 1 (Harada-Henriques-Holm, 2004)

If G is not of type C, then the equivariant cohomology ring and the cohomology ring of the GKM graph $\mathcal{G}$ are isomorphic.

$$H_T^*(G/T) \cong H^*(\mathcal{G})$$

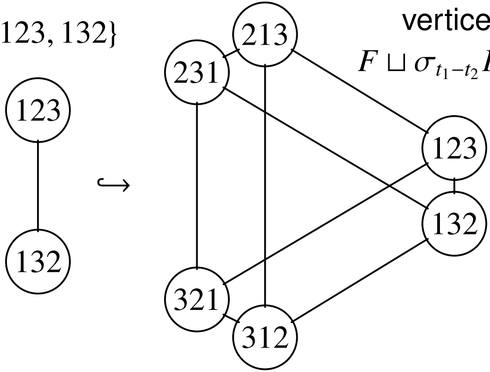
Goresky, Kottwitz, and MacPherson showed Theorem 1 tensoring with $\mathbb{C}$ in 1998.

Analysis of fiber bundles by the GKM theory

$F = \{123, 132\}$

vertices

$F \sqcup \sigma_{t_1 - t_2} F \sqcup \sigma_{t_1 - t_3} F$



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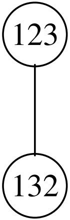

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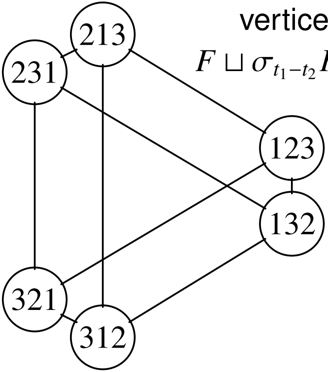

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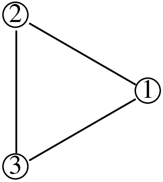

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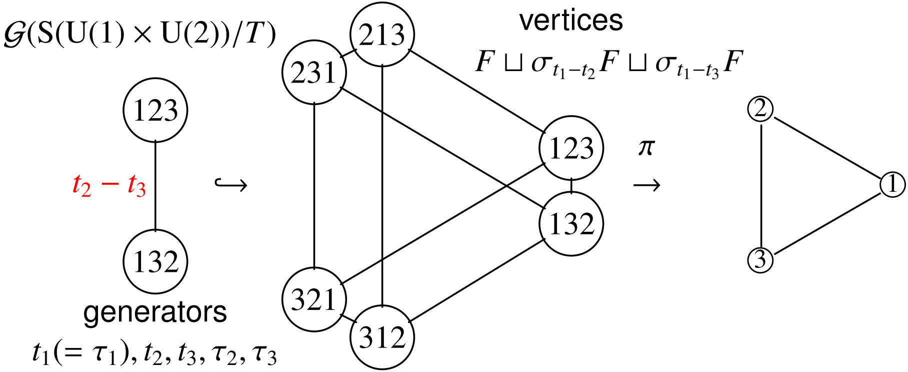

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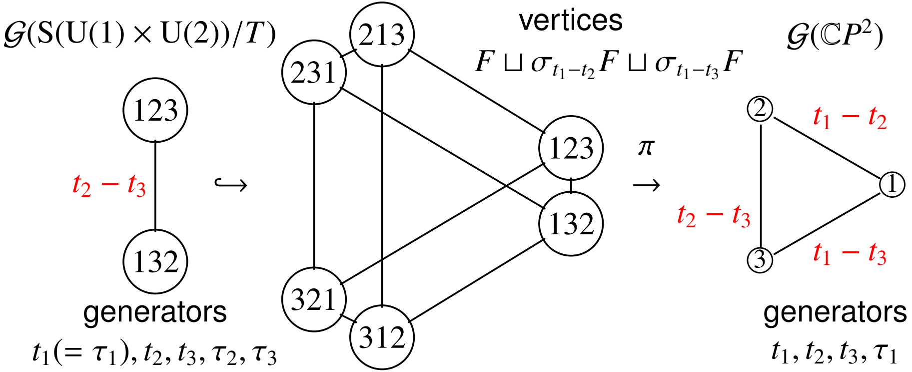

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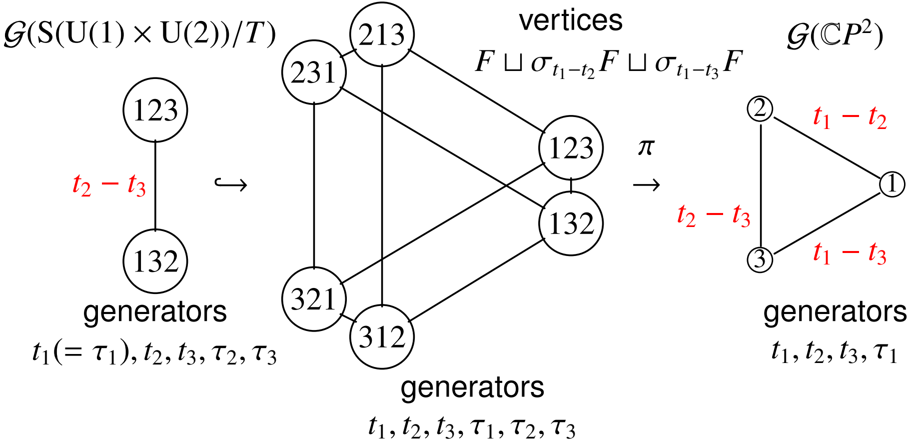
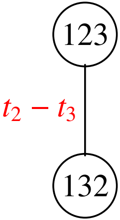


Analysis of fiber bundles by the GKM theory

$\mathcal{G}(\mathrm{S}(\mathrm{U}(1)\times\mathrm{U}(2))/T)$



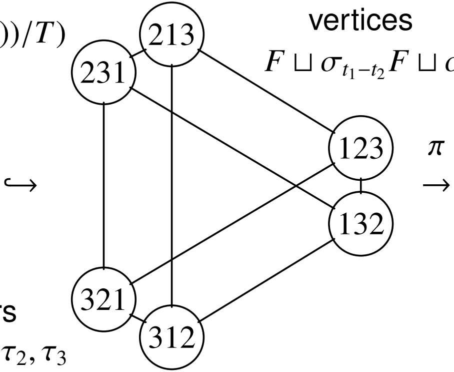
generators

$t_1(=\tau_1), t_2, t_3, \tau_2, \tau_3$

relations

$t_2 + t_3 = \tau_2 + \tau_3,$

$t_2 t_3 = \tau_2\tau_3,$



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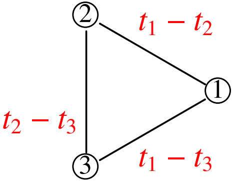

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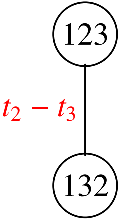

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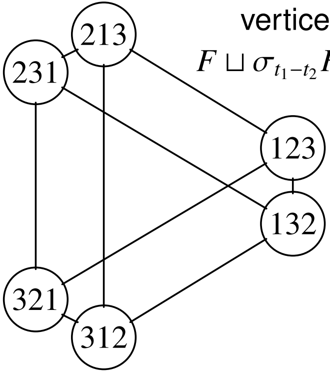

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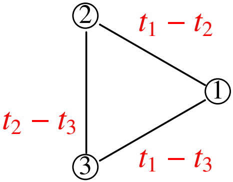

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$e_1(t) = e_1(\tau),$

$e_2(t) = e_2(\tau),$

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Two theorems on GKM fiber bundles

For a subset of $\Phi(G)$, their reflections generate a subgroup of $W(G)$. Then we obtain maps between GKM graphs $\mathcal{F} \to \mathcal{G} \to \mathcal{B}$.

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Theorem 2

Assume that the generators $\{g_\lambda\}$ of $H^*(\mathcal{F})$ extend to $\{\tilde{g}_\lambda\}$ of $H^*(\mathcal{G})$ and that there is a seq. of vertices $v_1, v_2, \ldots, v_N$ of $\mathcal{B}$ $(v_i \neq v_j)$ and there is a seq. $\varphi_1 = 1, \varphi_2, \ldots, \varphi_N \in H^*(\mathcal{B})$, which satisfy the following conditions;

$$\varphi_i(v_j) = 0 \ (j < i) \quad \text{and} \quad \varphi_i(v_i) = \prod \alpha,$$

where the product is taken over all roots α appearing as the labels of edges between v_i and v_j $(j < i)$. Then the cohomology $H^*(\mathcal{G})$ is generated by $\{\tilde{g}_\lambda\}$ as an $H^*(\mathcal{B})$-algebra.

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Especially $H^*(\mathcal{B})$ is generated by $\{\varphi_i\}$, and $H^*(\mathcal{G})$ is generated by $\{\varphi_i\}$ and $\{\tilde{g}_\lambda\}$ as an $H^*(BT)$-algebra.

Two theorems on GKM fiber bundles

Let $\{g_\lambda\}$, $\{\tilde{g}_\lambda\}$, $\{v_i\}$, and $\{\varphi_i\}$ be as above, and

- $I_{\mathcal{F}} := \ker(H^*(BT)[g_\lambda] \to H^*(\mathcal{F}))$,
- $I_{\mathcal{G}} := \ker(H^*(BT)[\varphi_i, \tilde{g}_\lambda] \to H^*(\mathcal{G}))$,
- $I_{\mathcal{B}} := \ker(H^*(BT)[\varphi_i] \to H^*(\mathcal{B}))$,

$$H^*(BT)[\varphi_i, \tilde{g}_\lambda] \to H^*(BT)[g_\lambda] \quad \tilde{g}_\lambda \mapsto g_\lambda,\ \varphi_i \mapsto \varphi_i(\mathcal{F}).$$

Theorem 3

Assume that the generators $\{r_{\lambda'}\}$ of the ideal $I_{\mathcal{F}}$ extend to $\{\tilde{r}_{\lambda'}\}$ of $I_{\mathcal{G}}$. Then the ideal $I_{\mathcal{G}}$ is generated by $\{\tilde{r}_{\lambda'}\}$ and $I_{\mathcal{B}}$ $(\subset H^*(BT)[\varphi_i, \tilde{g}_\lambda])$.

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The key idea: for $w \in W(G)$, a GKM (partial) function f, and a vertex v, we can define the following GKM (partial) function $w \cdot f$

$$(w \cdot f)(v) = w(f(w^{-1}v))$$

$\rightsquigarrow$ $\{w \cdot g_\lambda\}$ generates $H^*(w\mathcal{F})$ as an $H^*(BT)$-algebra.

The root system of E_6

Apply these theorems to
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The root systems of $\mathrm{Spin}(10)$ and E_6 are given as:

$$\Phi(\mathrm{Spin}(10)) = \{\pm(t_i + t_j), \pm(t_i - t_j) \mid 1 \leq i < j \leq 5\},$$

$$\Phi(E_6) = \left\{ \begin{array}{l} \pm (t_i + t_j),\ \pm(t_i - t_j), \\ \dfrac{1}{2}\left(t_8 - t_7 - t_6 + \displaystyle\sum_{i=1}^{5} \varepsilon_i t_i\right) \end{array} \middle| \ 1 \leq i < j \leq 5,\ \varepsilon_i = \pm 1,\ \prod_{i=1}^{5} \varepsilon_i = 1 \right\}$$

$W(E_6)$ decomposes into 27 cosets by $W(\mathrm{Spin}(10))$.
The base graph is 16-regular and called the Schläfli graph.

The cohomology of the fiber and base graph

e_i : the i-th elementary symmetric polynomial

According to Fukukawa, Ishida, and Masuda, $H^*_{T^5}(\mathrm{Spin}(10)/T^5)$ is as follows. $(H^*(BT^5) \cong \mathbb{Z}[t_i, \gamma \mid 1 \leq i \leq 5]/(2\gamma - e_1(t)))$

generators (as an $H^*(BT^5)$-algebra): τ_i, γ_j $(1 \leq i \leq 5, 1 \leq j \leq 4)$

relations: $2\gamma_j = e_j(\tau) - e_j(t), \; e_i(\tau^2) - e_i(t^2) = 0$

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The cohomology of the base graph

generators (as an $H^*(BT)$-algebra): s, w

relations: $s^9 - 3w^2 s \equiv 0,\ \ w^3 + 15w^2 s^4 - 9ws^8 \equiv 0 \pmod{H^{\geq 2}(BT)}$