The cohomology ring of toric orbifolds with integer coefficients

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January 23rd, 2014

Toric Topology 2014 in Osaka
Osaka City University, Osaka, Japan

Outline

Toric variety

- A toric variety X : A normal complex algebraic variety with algebraic $(\mathbb{C}^*)^n$-action having a dense orbit.

Theorem (Fundamental theorem for toric varieties)

The category of toric varieties is equivalent to the category of fans.

$$X_\Sigma \longleftrightarrow \Sigma_X$$

Construction of toric varieties

- $\Sigma = (K, \Lambda)$: a simplicial fan with $\Sigma^{(1)} = \{\lambda_1, \ldots, \lambda_m\} \subset \mathbb{Z}^n$,
  1. K : underlying simplicial complex,

$$\mathcal{Z}_\Sigma(\mathbb{C}, \mathbb{C}^*) = \bigcup_{\sigma \in K} (\mathbb{C}^{|I|} \times (\mathbb{C}^*)^{m-|I|}),$$

     with standard action of $(C^*)^m$

  2. $\Lambda = [\lambda_1 | \ldots | \lambda_m] : \mathbb{Z}^m \to \mathbb{Z}^n$

$$0 \longrightarrow \ker \Lambda_{\mathbb{C}} \hookrightarrow (\mathbb{C}^*)^m \xrightarrow{\Lambda_{\mathbb{C}}} (\mathbb{C}^*)^n \longrightarrow 0\ .$$

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- $X_\Sigma = \mathcal{Z}_\Sigma / \ker \Lambda_{\mathbb{C}}$ with the residual $(\mathbb{C}^*)^m / \ker \Lambda_{\mathbb{C}}$-action.

Cohomology ring of a toric variety

Correspondence...

- X_Σ : compact iff Σ : complete,
- X_Σ : a smooth variety iff Σ : smooth,
- X_Σ : an orbifold[1] or simplicial iff Σ : simplicial.

[1] has only finite quotient singularities

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$H^*(X_\Sigma; -)$	$\mathbb{Q}$-coefficients	$\mathbb{Z}$-coefficients
smooth variety	$H^*(X_\Sigma;\mathbb{Q}) \cong \mathbb{Q}[\Sigma]/\mathcal{J}_\Sigma$	$H^*(X_\Sigma;\mathbb{Z}) \cong \mathbb{Z}[\Sigma]/\mathcal{J}_\Sigma$
orbifold		$H^*(X_\Sigma;\mathbb{Z}) \cong \mathbb{Z}[\Sigma]/\mathcal{J}_\Sigma$ (??)

[1] has only finite quotient singularities

Piecewise polynomials

Definition

Let Σ be a fan in N. A function

$$f\colon |\Sigma| = \bigcup_{\sigma\in\Sigma} \sigma \subset N_{\mathbb{R}} \to \mathbb{Z}$$

is called *piecewise polynomial* on Σ, if it coincides with some globally defined polynomial on each cone $\sigma \in \Sigma$

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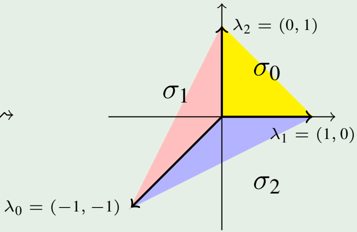
- $PP[\Sigma] := PP[\Sigma; \mathbb{Z}]$: set of piecewise polynomials on Σ.
- $\mathbb{Z}[N]$: the ring of global polynomial functions on N.
- $\{f_\sigma\} := \{f_\sigma \in \mathbb{Z}[N] \mid \sigma \in \Sigma^{(n)}\}$: an element in $PP[\Sigma]$.
- $PP[\Sigma]$ has $\mathbb{Z}[N](\cong H^*(BT))$-algebra structure.

Example

$$\Sigma = \left(\Delta^2, \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}\right) \quad \leftrightsquigarrow$$



$$PP[\Sigma] = \left\{ (f_{\sigma_0}, f_{\sigma_1}, f_{\sigma_2}) \in \mathbb{Z}[x, y]^3 \;\middle|\; \begin{array}{l} f_{\sigma_0} - f_{\sigma_1} \in \langle x \rangle \\ f_{\sigma_1} - f_{\sigma_2} \in \langle y - x \rangle \\ f_{\sigma_2} - f_{\sigma_0} \in \langle y \rangle \end{array} \right\}.$$

Proposition (A.Bahri, M.Franz, and N.Ray, 2009)

- Σ : *a polytopal fan in* N
- X_Σ : *associated compact projective toric variety with* $H^{odd}(X_\Sigma;\mathbb{Z}) = 0$.

$\Rightarrow H^*_T(X_\Sigma;\mathbb{Z}) \cong PP[\Sigma]$ *as* $H^*(BT)$*-algebras.*

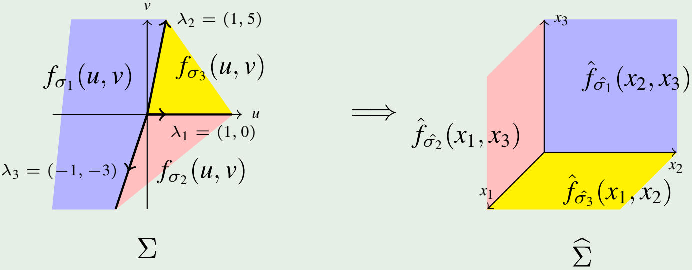
From Piecewise polynomials to Stanley-Reisner ring

- Σ : a polytopal fan in $\mathbb{Z}^n$ with $\Sigma^{(1)} = \{\lambda_1, \ldots, \lambda_m\}$.
- $\Lambda : \mathbb{Z}^m \to \mathbb{Z}^n$, $e_i \mapsto \lambda_i$.
- Define $\widehat{\Sigma} = \{\Lambda^{-1}(\sigma) \mid \sigma \in \Sigma\}$: fan in $\mathbb{Z}^m$.
- $\sigma = cone(\lambda_{i_1}, \ldots, \lambda_{i_n}) \in \Sigma \Rightarrow$
  $\hat{\sigma} := \Lambda^{-1}(\sigma) = cone\{e_{i_1}, \ldots, e_{i_n}\} \in \widehat{\Sigma}$,
- $PP[\widehat{\Sigma}]$: the ring of piecewise polynomials on $\widehat{\Sigma}$, whose elements are of the following form

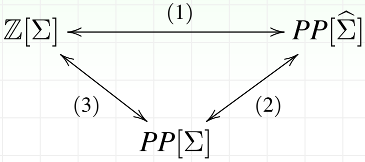
$$\{\hat{f}_{\hat{\sigma}}\} = \{\hat{f}_{\hat{\sigma}}(x_{i_1}, \ldots, x_{i_n}) \mid \hat{\sigma} = cone(e_{i_1}, \ldots, e_{i_n}) \in \widehat{\Sigma}^{(n)}\}$$

Example

$$\Sigma = \left(\Delta^2, \begin{pmatrix} 1 & 1 & -1 \\ 0 & 5 & -3 \end{pmatrix}\right) \Longrightarrow \widehat{\Sigma} = \left(\Delta^2, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right)$$



Given a fan Σ, we have 3 algebraic objects...



RELATION??

Lemma (1)

For a polytopal fan Σ *with* $|\Sigma^{(1)}| = m$*,*

$$\mathbb{Z}[\Sigma] = \mathbb{Z}[x_1, \ldots, x_m]/\mathcal{I}_\Sigma \cong PP[\widehat{\Sigma}], \text{ and}$$
$$\mathbb{Q}[\Sigma] = \mathbb{Q}[x_1, \ldots, x_m]/\mathcal{I}_\Sigma \cong PP[\widehat{\Sigma}; \mathbb{Q}],$$

where $\mathcal{I}_\Sigma$ *is the Stanley-Reisner ideal of* Σ*.*

Lemma (1)

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where $\mathcal{I}_\Sigma$ is the Stanley-Reisner ideal of Σ.

Sketch of proof.

- $\alpha\colon \mathbb{Z}[x_1, \ldots, x_m] \to PP[\widehat{\Sigma}]$, restriction to each cone of $\widehat{\Sigma}$. $\Longrightarrow$ surjective ring homomorphism.
- Sjectivity follows from *inclusion and exclusion principle*.
- $\ker \alpha = \mathcal{I}_\Sigma$.

□

Lemma (2)

$$PP[\Sigma;\mathbb{Q}] \cong PP[\widehat{\Sigma};\mathbb{Q}].$$

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Sketch of proof.

$$PP[\widehat{\Sigma};\mathbb{Q}] \underset{\gamma}{\overset{\beta}{\rightleftarrows}} PP[\Sigma;\mathbb{Q}]$$

$$\beta(\{\hat{f}_{\hat{\sigma}}\}) = \{\hat{f}_{\hat{\sigma}}(\Lambda_\sigma^{-1}\cdot(u_1,\ldots,u_n)^T) \mid \sigma \in \Sigma^{(n)}\},$$
$$\gamma(\{f_\sigma\}) = \{f_\sigma(\Lambda_\sigma\cdot \mathbf{x}_\sigma^T) \mid \hat{\sigma} \in \widehat{\Sigma}^{(n)}\}.$$

$\Longrightarrow \beta\circ\gamma = id_{PP[\Sigma;\mathbb{Q}]}$ and $\gamma\circ\beta = id_{PP[\widehat{\Sigma}]}$. □

Corollary

$\mathbb{Q}[\Sigma] \cong PP[\Sigma; \mathbb{Q}]$.

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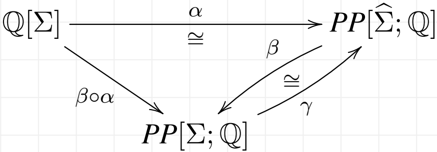
Finally, in $\mathbb{Q}$-coefficients,



Integrality condition

In order to work with $\mathbb{Z}$-coefficients, we need the following definitions.

Definition

We say a piecewise polynomial

$$\{\hat{f}_{\hat{\sigma}}\} = \{\hat{f}_{\hat{\sigma}}(x_{i_1}, \ldots, x_{i_n}) \mid \hat{\sigma} = cone(e_{i_1}, \ldots, e_{i_n}) \in \widehat{\Sigma}^{(n)}\} \in PP[\widehat{\Sigma}]$$

satisfies *integrality condition associated to Σ*, if

$$\beta(\{\hat{f}_{\hat{\sigma}}\}) = \{\hat{f}_{\hat{\sigma}}(\Lambda_{\sigma}^{-1} \cdot (u_1, \ldots, u_n)^T)\}$$

has integer coefficients, for all $\hat{\sigma} \in \widehat{\Sigma}^{(n)}$.

Definition

We say an element $[h(x_1, \ldots, x_m)] \in \mathbb{Z}[\Sigma]$ satisfies *integrality condition associated to Σ*, if

$$\beta \circ \alpha([h]) = \{h(i_\sigma(\Lambda_\sigma^{-1}(u_1, \ldots, u_n)^T))\}$$

has integer coefficients, for all $\sigma \in \Sigma^{(n)}$

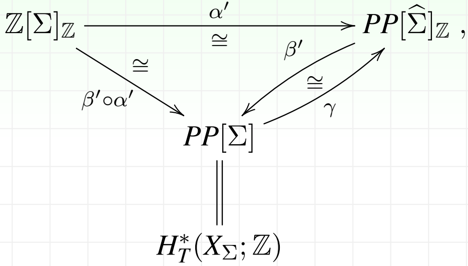
Proposition

The image of $\gamma : PP[\Sigma] \to PP[\widehat{\Sigma}]$ *(resp.* $\alpha^{-1} \circ \gamma : PP[\Sigma] \to \mathbb{Z}[\Sigma]$*) is the collection of elements in* $PP[\Sigma]$ *(resp.* $\mathbb{Z}[\Sigma]$*) which satisfy the integrality condition associated to* Σ*.*

Notations...

- $PP[\widehat{\Sigma}]_{\mathbb{Z}} = im(\gamma\colon PP[\Sigma] \to PP[\widehat{\Sigma}])$
- $\mathbb{Z}[\Sigma]_{\mathbb{Z}} = im(\alpha^{-1} \circ \gamma\colon PP[\Sigma] \to \mathbb{Z}[\Sigma])$

Finally, we have the following commutative diagram :



where α' and β' is the restrictions to their subrings satisfying integrality conditions.

Theorem

- Σ : *a polytopal fan in* N
- X_Σ : *associated compact projective toric variety with* $H^{odd}(X_\Sigma;\mathbb{Z}) = 0$.

$\Longrightarrow H^*(X_\Sigma) \cong \mathbb{Z}[\Sigma]_\mathbb{Z}/(\mathcal{J}_\Sigma)_\mathbb{Z}$,
where $\mathcal{J}_\Sigma$ *is the ideal generated by linear relations determined by* Σ *and* $(\mathcal{J}_\Sigma)_\mathbb{Z} = \mathcal{J}_\Sigma \cap \mathbb{Z}[\Sigma]_\mathbb{Z}$.

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Sketch of proof.

$H^{odd}(X_\Sigma; \mathbb{Z}) = 0 \Rightarrow$ Serre spectral sequence collapses, which gives the isomorphism

$$\begin{aligned} H^*(X_\Sigma; \mathbb{Z}) &\cong H_T^*(X_\Sigma; \mathbb{Z}) \otimes_{H^*(BT)} \mathbb{Z} \\ &\cong H_T^*(X_\Sigma; \mathbb{Z})/\mathcal{J}_\Sigma \cong \mathbb{Z}[\Sigma]_\mathbb{Z}/\mathcal{J}_\Sigma \end{aligned}$$

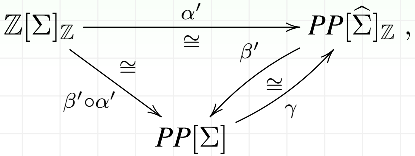
□

Finally, we have...

$H^*(X_\Sigma; -)$	$\mathbb{Q}$-coefficients	$\mathbb{Z}$-coefficients
smooth variety	$H^*(X_\Sigma;\mathbb{Q}) \cong \mathbb{Q}[\Sigma]/\mathcal{J}_\Sigma$	$H^*(X_\Sigma;\mathbb{Z}) \cong \mathbb{Z}[\Sigma]/\mathcal{J}_\Sigma$
orbifold		$H^*(X_\Sigma;\mathbb{Z}) \cong \mathbb{Z}[\Sigma]_\mathbb{Z}/(\mathcal{J}_\Sigma)_\mathbb{Z}$

Comprehension of Integrality condition

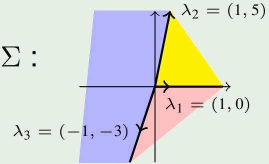
Recall the diagram and maps



- $\beta'(\{\hat{f}_{\hat{\sigma}}\}) = \{\hat{f}_{\hat{\sigma}}(\Lambda_\sigma^{-1} \cdot (u_1, \ldots, u_n)^T) \mid \sigma \in \Sigma\}$,
- $\alpha' \circ \beta'([h]) = \{h(i_\sigma(\Lambda_\sigma^{-1}(u_1, \ldots, u_n)^T)) \mid \sigma \in \Sigma\}$.

Example ($X_\Sigma \cong \mathbb{C}P^2_{(2,3,5)}$)

Σ :



$$\Lambda_{23} = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix} \rightsquigarrow \Lambda_{23}^{-1} = \begin{pmatrix} -\frac{3}{2} & \frac{1}{2} \\ -\frac{5}{2} & \frac{1}{2} \end{pmatrix},$$

$$\Lambda_{13} = \begin{pmatrix} 1 & -1 \\ 0 & -3 \end{pmatrix} \rightsquigarrow \Lambda_{13}^{-1} = \begin{pmatrix} 1 & -\frac{1}{3} \\ 0 & -\frac{1}{3} \end{pmatrix},$$

$$\Lambda_{12} = \begin{pmatrix} 1 & 1 \\ 0 & 5 \end{pmatrix} \rightsquigarrow \Lambda_{12}^{-1} = \begin{pmatrix} 1 & -\frac{1}{5} \\ 0 & \frac{1}{5} \end{pmatrix}.$$

$$\therefore \quad H_T^*(\mathbb{C}P^2_{(2,3,5)};\mathbb{Z}) \cong PP[\Sigma] \cong \mathbb{Z}[\Sigma]_\mathbb{Z} =$$

$$(\mathbb{Z}[x_1,x_2,x_3]/(x_1x_2x_3))_\mathbb{Z} =$$

$$\left\{ [h] \in \mathbb{Z}[\Sigma] \;\middle|\; \begin{array}{l} h\left(0, -\frac{3}{2}u_1 + \frac{1}{2}u_2, -\frac{5}{2}u_1 + \frac{1}{2}u_2\right) \in \mathbb{Z}[u_1,u_2] \\ h\left(u_1 - \frac{1}{3}u_2, 0, -\frac{1}{3}u_2\right) \in \mathbb{Z}[u_1,u_2] \\ h\left(u_1 - \frac{1}{5}u_2, \frac{1}{5}u_2, 0\right) \in \mathbb{Z}[u_1,u_2] \end{array} \right\}.$$

$H^*(\mathbb{C}P^2_{(2,3,5)};\mathbb{Z})$ and Integrality matrix

$$\mathbb{Z}[\Sigma]_{\mathbb{Z}} = \left\{ [h] \in \mathbb{Z}[\Sigma] \;\middle|\; \begin{array}{l} h\left(0, -\frac{3}{2}u_1 + \frac{1}{2}u_2, -\frac{5}{2}u_1 + \frac{1}{2}u_2\right) \in \mathbb{Z}[u_1,u_2] \\ h\left(u_1 - \frac{1}{3}u_2, 0, -\frac{1}{3}u_2\right) \in \mathbb{Z}[u_1,u_2] \\ h\left(u_1 - \frac{1}{5}u_2, \frac{1}{5}u_2, 0\right) \in \mathbb{Z}[u_1,u_2] \end{array} \right\}.$$

"Integrality matrix"

$$\mathcal{G}_\Sigma = \begin{array}{c} \quad x_1 \;\; x_2 \;\; x_3 \\ \begin{pmatrix} 0 & 2 & 2 \\ 3 & 0 & 3 \\ 5 & 5 & 0 \end{pmatrix} \end{array} \quad \begin{array}{l} \leftrightsquigarrow \; cone\{\lambda_2, \lambda_3\} \\ \leftrightsquigarrow \; cone\{\lambda_1, \lambda_3\} \\ \leftrightsquigarrow \; cone\{\lambda_1, \lambda_2\} \end{array}$$

- degree 2 monomials : $15x_1, 10x_2, 6x_3$.
- degree 4 monomials :
  $225x_1^2, 100x_2^2, 36x_3^2, 4x_2x_3, 9x_1x_3, 25x_1x_2$.

$$\Sigma = \left(\Delta^2, \begin{pmatrix} 1 & 1 & -1 \\ 0 & 5 & -3 \end{pmatrix}\right) \longrightarrow \mathcal{J}_\Sigma = (x_1 + x_2 - x_3, 5x_2 - 3x_3)$$

Hence,

- degree 2 monomials : $15x_1 = 10x_2 = 6x_3 =: w_1$
- degree 4 monomials :
  $4x_2x_3 = \frac{1}{15}w_1^2, 9x_1x_3 = \frac{1}{10}w_1^2, 25x_1x_2 = \frac{1}{6}w_1^2.$

Finally, choose degree 4 generator

$$w_2 := 9x_1x_3 - 4x_2x_3 = w_2 = \frac{1}{30}w_1^2.$$

$$\begin{aligned} H^*(\mathbb{C}P^2_{(2,3,5)}; \mathbb{Z}) &\cong \mathbb{Z}[\Sigma]_\mathbb{Z}/(\mathcal{J}_\Sigma)_\mathbb{Z} \\ &\cong \mathbb{Z}[w_1, w_2]/ < w_1^2 - 30w_2, w_1w_2 > \end{aligned}$$

J-construction and canonical extension

- $\Sigma = (K, \Lambda)$: a fan in $\mathbb{Z}^n$ with $\Sigma^{(1)} = \{\lambda_1, \dots, \lambda_m\}$,
  1. K : underlying simplicial complex on $V = \{v_1, \dots, v_m\}$ with minimal non-faces $\{v_{i_1}, \dots, v_{i_\ell}\}$
  2. $\Lambda = [\lambda_1 \mid \cdots \mid \lambda_m]$
- $J = (j_1, \dots, j_m) \in \mathbb{N}^m$

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  2. $\Lambda = \left[\lambda_1 \mid \cdots \mid \lambda_m\right]$
- $J = (j_1, \ldots, j_m) \in \mathbb{N}^m$
- $\Sigma(J) = (K(J), \Lambda(J))$: a fan in $\mathbb{Z}^{n-m+\sum_{i=1}^m j_j}$ with
  1. $K(J)$: simplicial complex on
  $$\{\underbrace{v_1^1, \ldots, v_1^{j_1}}_{j_1}, \ldots, \underbrace{v_i^1, \ldots, v_i^{j_i}}_{j_i}, \ldots, \underbrace{v_m^1, \ldots, v_m^{j_m}}_{j_m}\}$$
  with minimal non-faces of the form $V_{i_1} \cup \cdots \cup V_{i_\ell}$, where $V_i = \{v_i^1, \ldots, v_i^{j_i}\}$.
  2. $\Lambda(J) = ...$

$$\Lambda(J) = \left[\begin{array}{c|c|c|c|ccc}
I_{j_0 -1} & & & & \begin{matrix}-1\\ \vdots \\ -1\end{matrix} & & \\ \hline
& I_{j_1 -1} & & & & \begin{matrix}-1\\ \vdots \\ -1\end{matrix} & \\ \hline
& & \ddots & & & \vdots & \\ \hline
& & & I_{j_n -1} & & & \begin{matrix}-1\\ \vdots \\ -1\end{matrix} \\ \hline
& & & & & \Lambda &
\end{array}\right],$$

simplicial wedge construction

Especially, for $J = (1, \ldots, 1, \underset{\substack{\uparrow \\ i-th}}{2}, 1, \ldots 1)$,

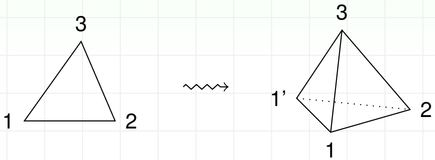
$\Sigma(J) =: \tilde{\Sigma} = (wedge_i(K), \tilde{\Lambda})$, "canonical extension of Σ", where

- $wedge_i(K) := [\{i', i\} * link_K\{i\}] \cup [\{\{i'\}, \{i\}\} * (K \setminus \{i\})]$,
  Call it "simplicial wegde construction".
- $$\tilde{\Lambda} := \Lambda(J) = \left[\begin{array}{c|ccccc} 1 & 0 & \cdots & -1 & \cdots & 0 \\ \hline 0 & & & & & \\ \vdots & \lambda_1 & \cdots & \lambda_i & \cdots & \lambda_m \\ 0 & & & & & \end{array}\right]$$

Examples

$$wedge_i(K) := [\{i', i\} * link_K\{i\}] \cup [\{\{i'\}, \{i\}\} * (K \setminus \{i\})]$$

$\partial\Delta^2$:



: $wedge_1(\partial\Delta^2) = \partial\Delta^3$

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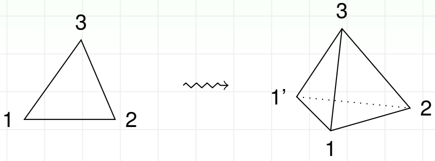
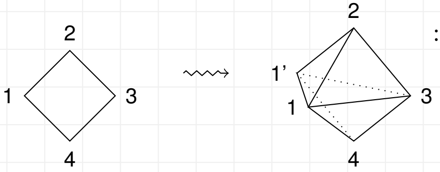
$\partial\Delta^2$:



: $wedge_1(\partial\Delta^2) = \partial\Delta^3$

$\partial\Delta^1 * \partial\Delta^1$:



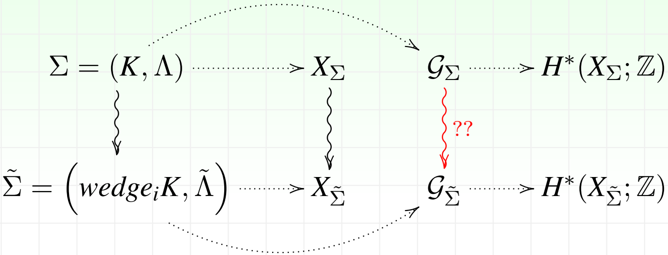
: $wedge_1(\partial\Delta^1 * \partial\Delta^1)$
$= \partial\Delta^2 * \partial\Delta^1$

Example ($J = (1, 2, 1)$)

$$\Sigma = \left(\Delta^2, \begin{bmatrix} 1 & 1 & -1 \\ 0 & 5 & -3 \end{bmatrix}\right), \quad X_\Sigma \cong \mathbb{C}P^2_{(2,3,5)}$$

$$\rightsquigarrow \quad J = (1, 2, 1)$$

$$\tilde{\Sigma} = \left(\Delta^3, \left[\begin{array}{c|ccc} 1 & 0 & -1 & 0 \\ \hline 0 & 1 & 1 & -1 \\ 0 & 0 & 5 & -1 \end{array}\right]\right), \quad X_{\tilde{\Sigma}} \cong \mathbb{C}P^3_{(3,2,3,5)}$$



If we know (??), $H^*(X_{\tilde{\Sigma}}; \mathbb{Z})$ can be easily read off from the original one.

Proposition

Consider $J = (1, \ldots, 1, \underset{\substack{\uparrow \\ i-th}}{2}, 1, \ldots 1)$.

Let $(g_1^\sigma, \ldots, g_i^\sigma, \ldots, g_m^\sigma)$ *be a row vector in* $\mathcal{G}_\Sigma$ *corresponding to* $\sigma \in K_\Sigma$.

- *If* $g_i^\sigma \neq 0$ *(*$\Leftrightarrow i \in \sigma$*), then*

$$(g_1^\sigma, \ldots, g_i^\sigma, \ldots, g_m^\sigma) \rightsquigarrow (\boldsymbol{g_i^\sigma}, g_1^\sigma, \ldots, \boldsymbol{g_i^\sigma}, \ldots, g_m^\sigma) \text{ in } \mathcal{G}_{\tilde{\Sigma}}$$

- *If* $g_i^\sigma = 0$ *(*$\Leftrightarrow i \notin \sigma$*), then*

$$(g_1^\sigma, \ldots, \underset{\substack{\uparrow \\ i-th}}{0}, \ldots, g_m^\sigma) \rightsquigarrow \begin{pmatrix} \mathbf{1} & g_1^\sigma & \ldots & \mathbf{0} & \ldots, g_m^\sigma \\ \mathbf{0} & g_1^\sigma & \ldots & \mathbf{1} & \ldots, g_m^\sigma \end{pmatrix} \text{ in } \mathcal{G}_{\tilde{\Sigma}}$$

Example

Example

Recall, for $\Sigma = \left(\Delta^2, \begin{bmatrix} 1 & 1 & -1 \\ 0 & 5 & -3 \end{bmatrix}\right)$, and $J = (1,2,1)$,

$$X_\Sigma \cong \mathbb{C}P^2_{(2,3,5)} \rightsquigarrow X_{\tilde{\Sigma}} \cong \mathbb{C}P^3_{(3,2,3,5)}$$

$$\mathcal{G}_\Sigma = \begin{pmatrix} 0 & 2 & 2 \\ 3 & 0 & 3 \\ 5 & 5 & 0 \end{pmatrix} \rightsquigarrow \mathcal{G}_{\tilde{\Sigma}} = \begin{pmatrix} \mathbf{2} & 0 & \mathbf{2} & 2 \\ \mathbf{1} & 3 & \mathbf{0} & 3 \\ \mathbf{0} & 3 & \mathbf{1} & 3 \\ \mathbf{5} & 5 & \mathbf{5} & 0 \end{pmatrix}$$

Computation

- Degree 2 : $10x_{22}, 15x_1, 10x_{21}, 6x_3$.
  By the ideal $\mathcal{J}_{\Sigma(J)}$, $10x_{22} = 15x_1 = 10x_{21} = 6x_3 =: w_1$.
- Degree 4 : $75x_{21}x_1 = 75x_1x_{22} = \frac{1}{2}w_1^2$, $100x_{21}x_{22} = w_1^2$,
  $12x_{21}x_3 = x_{22}x_3 = \frac{1}{5}x_1^2$, $9x_1x_3 = \frac{1}{10}w_1^2$.
  Choose $9x_1x_3 =: w_2$.
- Degree 6 : $8x_{21}x_{22}x_3 = \frac{1}{75}w_1^3$, $9x_{21}x_1x_3 = 9x_{22}x_1x_3 = \frac{1}{100}w_1^3$,
  $125x_{21}x_1x_{23} = \frac{1}{12}w_1^3$.
  Choose $8x_{21}x_{22}x_3 - x_{21}x_1x_3 = \frac{1}{300}w_1^3 =: w_3$.

$$\therefore H^*(\mathbb{C}P^3_{(3,2,3,5)}) \cong H^*(\mathbb{C}P^3_{(2,3,3,5)})$$

$$\mathbb{Z}[w_1, w_2, w_3]/ < w_1^2 - 10w_2, w_1^3 - 300w_3, w_2^2, w_1w_3 > .$$

Thank you for your attention!