# Equivariant ordinary Chern numbers and Equivariant $K$-theory Chern numbers

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# Main Contents

# Unitary Bordism and Chern numbers

# Unitary bordism

Let $M$ be a smooth manifold. We say $M$ is a **unitary** manifold if the stable tangent bundle

$$TM \oplus \mathbb{R}^k$$

of $M$ admits a fixed complex structure for some $k$.

Let $M_1$ and $M_2$ be two closed unitary manifolds, we say $M_1$ is **unitary (or complex) bordant** to $M_2$ if there is a compact unitary manifold $W$ with boundary

$$\partial W = M_1 \coprod (-M_2),$$

such that the induced unitary structure of $\partial W$ is isomorphic to the unitary structure of $M_1 \coprod (-M_2)$.

In particular, we say $M$ is **null-bordant** if $M$ is just the boundary of $W$

$$\partial W = M,$$

with the induced unitary structure of $W$.

# Ordinary Chern classes of unitary manifolds

Let $M^{2n}$ be a closed unitary manifold with stable tangent bundle:

$$\theta := TM \oplus \mathbb{R}^k.$$

Since $\theta$ is a complex vector bundle, the total Chern class of $M$ is weil-defined:

$$c(M) = c(\theta) = 1 + c_1 + c_2 + \cdots \in H^*(M, \mathbb{Z}).$$

For each partition $\omega = (i_1, i_2, \cdots, i_n)$, the ordinary Chern number $c_\omega(M)$ is defined to be the Poincaré pair:

$$c_\omega(M) =< c_1^{i_1} c_2^{i_2} \cdots c_n^{i_n}, [M] >$$

# Unitary bordism and ordinary Chern numbers

Milnor and Novikov's work on unitary (complex) bordism ring shows that:

**Theorem**

*Let $M$ be a closed unitary manifold, $M$ is null-bordant if and only if all ordinary Chern numbers $c_\omega(M) = 0$.*

Furthermore, one can use (complex) $K$-theory Chern numbers to obtain a similar theorem:

**Theorem**

*Let $M$ be a closed unitary manifold, $M$ is null-bordant if and only if all (complex) $K$- theory Chern numbers vanish.*

# $K$-theory Chern numbers

In $K$ theory, recall the operation of exterior product, for a complex vector bundle $E$ over $M$:

$$\Lambda_t(E) = 1 + \Lambda^1(E)t + \Lambda^2(E)t^2 + \cdots$$

# $\gamma$ operation

The operation $\gamma_t(E) = \Lambda_{\frac{t}{1-t}}(E)$:

$$\gamma_t(E) = 1 + \gamma^1(E)t + \gamma^2(E)t^2 + \cdots + \gamma^m(E)t^m + \cdots$$

In particular, if $E$ is a line bundle, then $\gamma^1(E) = E - 1 \in K(M)$.

# $K$-theory Chern classes

Let $M$ be a closed unitary manifold with stable tangent bundle $\theta$.

We define the total $K$-theory Chern class of $M$ by

$$c^K(M) = 1 + \gamma^1(\theta) + \gamma^2(\theta) + \gamma^3(\theta) + \cdots$$

Let $p : M \longrightarrow pt$ be the constant map. In $K$ theory, the Gysin map

$$p_!^K : K(M) \to K(pt)$$

is well-defined.

The $K$-theory Chern numbers are defined to be:

$$c_\omega^K(M) := p_!^K(\gamma^{i_1}(\theta)\gamma^{i_2}(\theta)\cdots\gamma^{i_m}(\theta)) \in K(pt),$$

where $\omega = (i_1, i_2, \cdots, i_m)$.

# Riemann-Roch relation

$M$ is a closed unitary manifold with stable tangent bundle $\theta$, we have:

$$\begin{array}{ccc} K(M) & \xrightarrow{ch} & H^*(M, \mathbb{Q}) \\ {\scriptstyle p_!^K}\downarrow & & \downarrow{\scriptstyle p_!} \\ K(pt) & \xrightarrow{ch} & H^*(pt, \mathbb{Q}) \end{array}$$

Although this diagram is NOT commutative, we have Riemann-Roch relation: for $x \in K(M)$

$$ch(p_!^K(x)) = p_!(ch(x) \cdot Td(M))$$

where $Td(M)$ is the total Todd class of $M$.

When $x = \gamma^{i_1}(\theta)\gamma^{i_2}(\theta)\cdots\gamma^{i_m}(\theta)$,

$$ch(p_!^K(x)) = p_!(ch(x)\cdot Td(M))$$

is just a combination of ordinary Chern numbers.
So, we see when ordinary Chern numbers vanish:

$$p_!(ch(x)\cdot Td(M)) = ch(p_!^K(x)) = 0.$$

Finally, we see all the $K$-theory Chern numbers

$$p_!^K(x) = 0,$$

since $ch : K(pt) \hookrightarrow H^*(pt, \mathbb{Q})$ is injective.

By this relation, one can deduce that

All **ordinary Chern nunmbers** of $M$ vanish if and only if all $K$-**theory Chern numbers** of $M$ vanish.

Therefore, the following statements are equivalent:
(1). $M$ is **unitary null-bordant**.
(2). All **ordinary Chern numbers vanish**.
(3). All $K$-**theory Chern numbers vanish**.

We try to consider the equivariant case. Naturally, we will ask:

**Question**

*When $G = T^n$, is it true that a closed unitary $G$ manifold is 'equivariant null-bordant' if only if all 'equivariant Chern numbers' vanish.*

# Unitary $G$-manifold

Let $M$ be a closed smooth $G$-manifold, we say $M$ is a unitary $G$-manifold if $M$ admits a complex $G$ structure on

$$\theta = TM \oplus \mathbb{R}^k$$

for some $k$.

# Geometric unitary $G$ bordism

One can copy the definition of (non-equivariant) unitary bordism to define geometric unitary $G$ bordism.

Let $M_1$ and $M_2$ be two unitary $G$-manifolds. $M_1$ is geometric unitary $G$ bordant to $M_2$ if there exists a unitary $G$-manifold $W$ with

$$\partial W = M_1 \coprod (-M_2).$$

$M$ is geometric null-bordant if $\partial W = M$.

# Geometric unitary bordism ring

We denote the geometric unitary $G$-bordism ring by:

$$\Omega_*^{U,G} = \{\text{closed unitary } G \text{ manifold}\}/\text{geometric bordism}$$

with the operations $\coprod$ and $\times$.

# Homotopy theoretic unitary $G$-bordism

tom Dieck introduced the theory of homotopy theoretic unitary $G$-bordism. In modern point of view, it is an important example of equivariant stable homotopy theory.

In non-equivariant case, we see

$$\Omega_*^U \cong \varinjlim [S^{n+m}, MU(m)].$$

The homotopy theoretic unitary $G$-bordism ring is defined to be:

$$MU_*^G = \varinjlim_W [S^W, T(\xi^G_{|W|-n})]^G$$

where $[-.-]^G$ is the group of homotopy classes of pointed $G$ equivariant maps and the direct limit is with respect to a direct set of uniatry $G$ representations, i.e. $W < U$ if $U = W \oplus W^{\perp}$.

# Pontrjagin-Thom map

In non-equivariant bordism theory, one has the Pontrjagin-Thom map:

$$\Phi : \Omega_*^U \longrightarrow \pi_*(MU).$$

In equivariant case, one also has the **Pontrjagin-Thom map** from **geometric** to **homotopy theoretic** unitary $G$ bordism ring:

$$\Phi : \Omega_*^{U,G} \longrightarrow MU_*^G.$$

When $G$ is a compact abelian Lie group, one has

### Theorem

*The Pontrjagin-Thom map is injective*

$$\Phi : \Omega_*^{U,G} \longrightarrow MU_*^{U,G}.$$

*More precisely, it is a split monomorphism of $MU_*$ module.*

# Boardman map and Characteristic number map

tom Dieck constructed a natural transformation

$$MU_*^G \longrightarrow K_G^*[[a_1, a_2, \cdots]]$$

Composed with the Pontrjagin-Thom map, one obtains the characteristic number map:

$$\Psi : \Omega_*^{U,G} \longrightarrow K_G^*[[a_1, a_2, \cdots]].$$

tom Dieck proved that

### Theorem (tom Dieck)

*When $G$ is compact abelian, the characteristic number map*

$$\Psi : \Omega_*^{U,G} \longrightarrow K_G^*[[a_1, a_2, \cdots]]$$

*is injective.*

Let $M$ be a unitary $G$-manifold, the image $\Psi(M)$ is all kinds of combinations of equivariant $K$-theory Chern numbers. Hence, one has

### Corollary

*$M$ is geometric unitary $G$ null-bordant if and only if all equivariant $K$-theory Chern numbers vanish.*

# Questions

In many situations, equivariant ordinary cohomology ($H_G^*(X) = H^*(EG \times_G X)$) seems easier to compute.

Therefore, we want to know is it true that the following statements are equivalent:
(1). $M$ is **geometric unitary $G$ null-bordant**.
(2). All **equivariant ordinary Chern numbers** vanish.
(3). All **equivaraint $K$-theory Chern numbers** vanish.
This is the motivation that we consider the relation between equivariant ordinary Chern numbers and $K$-theory Chern numbers.

# Equivariant Chern numbers

# Equivariant Gysin map

In order to introduce (different) equivariant Chern numbers, we need to introduce equivariant Gysin map first.

Let $G$ be a compact Lie group and $h_G()$ be an equivariant multiplication cohomology theory. For example: $H^*_G()$, $K^*_G()$.

Let $M$ and $N$ be two closed unitary $G$ manifolds and $f : M \to N$ be a $G$ map, we can define **equivariant Gysin map**:

$$f_! : h_G(M) \longrightarrow h_G(N).$$

# 3 steps

We need 3 steps:

Step 1. We embed $M$ equivariantly into a $G$ vector space $V$
$e : M \hookrightarrow V$.

Step 2. Consider the embedding $f \times e : M \longrightarrow N \times V$ with normal bundle $\eta$ and we obtain three homomorphisms.

$$\phi_1 : h_G(M) \longrightarrow \widetilde{h_G}(D(\eta)/S(\eta)), (Thom\ isomorphism)$$

$$\phi_2 : \widetilde{h_G}(D(\eta)/S(\eta)) \longrightarrow \widetilde{h_G}(N{\times}D(V)/N{\times}S(V)), (collapsing\ map)$$

$$\phi_3 : \widetilde{h_G}(N \times D(V)/N \times S(V)) \longrightarrow h_G(N), (Thom\ isomorphism)$$

Step 3: The equivariant Gysin map is:

$$f_! := \phi_3 \circ \phi_2 \circ \phi_1 : h_G(M) \longrightarrow h_G(N).$$

In particular, when $N$ is a point, we obtain the equivariant Gysin homomorphism:

$$p_! : h_G(M) \longrightarrow h_G(pt).$$

# Equivariant ordinary Chern classes

Let $M$ be a unitary $G$ manifold with stable tangent bundle $\theta$ and one obtains a unitary $G$ vector bundle $\theta \times_G EG$ over $EG \times_G M$ such that:

$$\begin{array}{ccc} \theta \times EG & \longrightarrow & \theta \times_G EG \\ \downarrow & & \downarrow \\ M \times EG & \xrightarrow{\pi} & M \times_G EG \end{array}$$

The total equivariant ordinary Chern class is defined by:

$$\begin{aligned} c_G(M) &= c(\theta \times_G EG) \\ &= 1 + c_1(\theta \times_G EG) + c_2(\theta \times_G EG) + \cdots \\ &\in H^{**}(M \times_G EG) = H^{**}_G(M) \end{aligned}$$

where $H^{**}(M \times_G EG) := \prod H^i(M \times_G EG)$.

For example, $H^*(BS^1) = \mathbb{Z}[x]$ and $H^{**}(BS^1) = \mathbb{Z}[[x]]$.

# Equivariant ordinary Chern numbers

We denote

$$c_G(M) := 1 + c_1 + c_2 + c_3 + \cdots.$$

Then, for $\omega = (i_1, i_2, \cdots, i_m)$, the corresponding equivariant ordinary Chern number is defined to be

$$c_G^\omega = p_!(c_1^{i_1} c_2^{i_2} \cdots c_m^{i_m}) \in H_G^*(pt).$$

# Equivariant $K$-theory Chern numbers

$M$ is still a unitary $G$ manifold with stable tangent bundle $\theta$. In $K_G(M)$, the operation $\gamma_t(\theta) = \Lambda_{\frac{t}{1-t}}(\theta)$:

$$\gamma_t(\theta) = 1 + \gamma^1(\theta)t + \gamma^2(\theta)t^2 + \cdots + \gamma^m(\theta)t^m + \cdots$$

is still well-defined.

We define the total equivariant $K$-theory Chern class of $M$ by

$$c_G^K(M) = 1 + \gamma^1(\theta) + \gamma^2(\theta) + \gamma^3(\theta) + \cdots$$

and the equivariant $K$-theory Chern number is defined by:

$$c_\omega^K(M) := p_!^K(\gamma^{i_1}(\theta)\gamma^{i_2}(\theta)\cdots\gamma^{i_m}(\theta)) \in K_G(pt),$$

where $\omega = (i_1, i_2, \cdots, i_m)$ and $p_!^K$ is the equivariant $K$ theroy Gysin map.

Recall that we want to know the question:

Does equivariant **ordinary Chern numbers** vanish $\Longrightarrow$ equivariant $K$-**theory Chern numbers** vanish $\Longrightarrow$ $M$ **is unitary null-bordant**?

# Special case: isolated fixed points

Let $M$ be a unitary $G$-manifolds with isolated fixed points. Guillemin and etc proved that:

**Theorem**

*$M$ is geometric unitary $G$ null-bordant if and only if all equivariant ordinary Chern numbers vanish.*

The main tool is the localization theorem:

$$S^{-1}H_G^{**}(M) \cong S^{-1}H_G^{**}(M^G) = \oplus_{u_i} S^{-1}H_G^{**}(u_i)$$

$$p_!(x) \mapsto \sum_{u_i} \frac{i_{u_i}^*(x)}{e_x}$$

For general case, we want to get rid of the discussion of fixed points. Hence, we try to find some global formula.

We need two keys:

# Equivariant Chern character

One is equivariant Chern character, which has been defined by many people,

$$ch_G : K_G(M) \to K(EG \times_G M) \to H^{**}(EG \times_G M, \mathbb{Q}) = H_G^{**}(M, \mathbb{Q})$$

$$ch_G : K_G(pt) \to K(BG) \to H^{**}(BG, \mathbb{Q}) = H_G^{**}(pt, \mathbb{Q})$$

# Equivariant Riemann-Roch relation

The second key is the equivariant Riemann-Roch relation. Recall the classical Riemann-Roch relation:

$$\begin{array}{ccc} K(M) & \xrightarrow{ch} & H^*(M, \mathbb{Q}) \\ {\scriptstyle p_!^K}\downarrow & & \downarrow{\scriptstyle p_!} \\ K(pt) & \xrightarrow{ch} & H^*(pt, \mathbb{Q}) \end{array}$$

For$x \in K(M)$, one has

$$ch(p_!^K(x)) = p_!(ch(x) \cdot Td(M))$$

# Equivariant case 1

One has a commutative diagram:

$$\begin{array}{ccc} K_G(M) & \longrightarrow & K^*(M \times_G EG) \\ {\scriptstyle p_!^K}\downarrow & & {\scriptstyle p_!^K}\downarrow \\ K_G(pt) & \longrightarrow & K^*(BG) \end{array}$$

# Equivariant case 2

Then, we consider the diagram:

$$\begin{array}{ccc} K(M \times_G EG) & \xrightarrow{ch} & H^{**}(M \times_G EG, \mathbb{Q}) \\ p_!^K \downarrow & & p_! \downarrow \\ K(BG) & \xrightarrow{ch} & H^{**}(BG) \end{array}$$

Since the Borel construction has a filtration:

$$M \times_G EG(n) \subset M \times_G EG(n+1) \subset \cdots M \times_G EG$$

One can obtain the relation

$$ch_G(p_!(x)) = p_!(ch_G(x) \cdot Td_G(M)),$$

by taking inverse limit, where $Td_G(M) = Td(\theta \times_G EG)$ denotes the total equivariant Todd class.

Therefore,

$$\begin{array}{ccc} K_G(M) & \xrightarrow{ch_G} & H^{**}(M \times_G EG, \mathbb{Q}) \\ {\scriptstyle p_!^K}\big\downarrow & & {\scriptstyle p_!}\big\downarrow \\ K_G(pt) & \xrightarrow{ch_G} & H^{**}(BG). \end{array}$$

Let $x = c_1^{i_1} \cdot \cdot c_N^{i_N} \in K_G(M)$,

$ch_G(p_!(x)) = p_!(ch_G(x) \cdot Td_G(M)) =$ combinations of equivariant ordinary Chern nunmbers.

Hence, equivariant ordinary Chern numbers vanish $\Longrightarrow$ $ch_G(p_!(x)) = 0$.

When $G = T^n$, the equivariant Chern character

$$ch_G : K_G(pt) = \mathbb{Z}[x_i, \frac{1}{x_i}] \longrightarrow H^{**}(BG, \mathbb{Q}) = \mathbb{Q}[[t_i]]$$

$$x_i \mapsto e^{t_i}$$

is injective. It follows that $p_!(x) = 0$.

Thus, when $G$ is a torus, equivariant ordinary Chern numbers vanish $\Longrightarrow$ equivariant $K$-theory Chern numbers vanish $\Longrightarrow$ $M$ is geometric unitary $G$ null-bordant.

# Thank you for your attention!