Embedding theorems for quasitoric manifolds

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Theorem (Mostow-Palais)

Let M be a compact smooth manifold with a smooth action of compact Lie group G.

Then there exists a smooth embedding $M \to \mathbb{R}^N$ equivariant with respect to a linear representation $G \to GL(N, \mathbb{R})$.

Theorem (Kodaira)

Let M be a compact complex manifold with a positive holomorphic linear bundle
(for example, M possesses a rational Kähler form).

Then there exists a complex-analytic embedding $M \to \mathbb{C}P^N$.

Theorem (Gromov-Tishler)

Let M be a compact symplectic manifold with an integral symplectic form ω.

Then there exists a symplectic embedding of M to $\mathbb{C}P^N$ with the standard symplectic form.

The central subject of this talk are theorems on equivariant embeddings of quasitoric manifolds defined by combinatorial data.

One of our tasks is to improve the classical theorems in the case when combinatorial data defines the corresponding structure on the underlying quasitoric manifold.

Notations

$\mathbb{C}^m$ is the standard complex linear m-dimensional space endowed with the canonical basis $e_1 = (1, 0, \dots, 0), \dots, e_m = (0, \dots, 0, 1)$;

$\mathbb{R}^m \subset \mathbb{C}^m$ is the standard linear space generated by $e_1, \dots, e_m$;

$\mathbb{R}^m_{\geqslant 0} \subset \mathbb{R}^m$ is the positive cone i.e. the area formed by all points in $\mathbb{R}^n$ with nonnegative coordinates;

$\mathbb{Z}^m \subset \mathbb{R}^m$ is the standard lattice generated by $e_1, \dots, e_m$;

$\mathbb{T}^m \subset \mathbb{C}^m$ is the standard compact torus $\{(t_1,\dots,t_m)\in\mathbb{C}^m\colon |t_k|=1,\ k\in[1,m]\}$;

the map $exp\colon \mathbb{R}^m \to \mathbb{T}^m$ given by the formula $exp(x_1,\dots,x_m)=(e^{2\pi i x_1},\dots,e^{2\pi i x_m})$ induces the canonical isomorphism of $\mathbb{T}^m$ and $\mathbb{R}^m/\mathbb{Z}^m$.

The standard k-dimensional compact torus is denoted by $\mathbb{T}^k$ and an abstract k-dimensional toric subgroup in the standard torus is denoted by T^k;

$T_I \subset \mathbb{T}^m$ is a toric subgroup corresponding to an index set $I \subset [1,m]$;

the set $I=\{i\}$ defines coordinate torus $T_i \subset \mathbb{T}^m$;

$\rho\colon \mathbb{C}^m \to \mathbb{R}^m$ is the standard moment map given by the formula

$$\rho(z_1, \ldots, z_m) = (|z_1|^2, \ldots, |z_m|^2); \tag{1}$$

$s\colon \mathbb{R}^m_{\geqslant 0} \to \mathbb{C}^m$ is the map given by the formula

$$s(x_1, \ldots, x_m) = (\sqrt{x_1}, \ldots, \sqrt{x_m}), \tag{2}$$

note that $\rho \circ s = id$.

A polytope $P = P^n$ of dimension n with m facets is defined as the set of points in $\mathbb{R}^n$ satisfying m inequalities:

$$P = \{x \in \mathbb{R}^n \colon \langle a_i, x \rangle + b_i \geqslant 0\},\ i = 1 \ldots m. \tag{3}$$

We assume that inequalities are not redundant. The number of i-dimensional faces of P is denoted by $f_i(P)$, so $m = f_{n-1}(P)$. We may rewrite (3) using the matrix form:

$$A_P x + b_P \geqslant 0, \tag{4}$$

where A_P is an $(m \times n)$-matrix and $b_P \in \mathbb{R}^m$.
The matrix A_P and the vector b_P define an affine map

$$i_P \colon \mathbb{R}^n \to \mathbb{R}^m, \quad i_P(x) = A_P x + b_P, \tag{5}$$

and by (4) we have $i_P(P) = i_P(\mathbb{R}^n) \cap \mathbb{R}^m_{\geqslant 0}$.

Theorem (Buchstaber-Panov-Ray)

The set $\mathcal{Z}_P = \rho^{-1}(i_P(P))$, where ρ, P and i_P are given by (1), (3) and (5), is a real-algebraic $(m+n)$-dimensional submanifold in $\mathbb{C}^m$.

The manifold $\mathcal{Z}_P$ is endowed with a smooth action of $\mathbb{T}^m$ induced by the standard action of $\mathbb{T}^m$ in $\mathbb{C}^m$.

Moreover, $\mathcal{Z}_P$ is a complete intersection of $(m-n)$ real quadrics in $\mathbb{C}^m$, which are given by the formulas

$$\sum_{k=1}^{m} c_{j,k}(|z_k|^2 - b_k) = 0,\ j = 1 \ldots (m-n). \tag{6}$$

Moment-angle manifold

Definition

The manifold $\mathcal{Z}_P$ is called a *moment-angle manifold* of the simple polytope P.

The number $s(P)$ is defined as the maximum possible dimension of a toric subgroup in $\mathbb{T}^m$ acting freely on $\mathcal{Z}_P$.

For a given simple polytope $P = P^n$ with m facets the maximum possible value of $s(P)$ is equal to $m - n$.

In this case there exists a subgroup $K \subset \mathbb{T}^m$ isomorphic to $\mathbb{T}^{m-n}$ and acting freely on $\mathcal{Z}_P$.

The subgroup K may be defined via short exact sequence of the form

$$1 \longrightarrow K \longrightarrow \mathbb{T}^m \longrightarrow \mathbb{T}^n \longrightarrow 1. \tag{7}$$

Let us fix an $(m-n)$-dimensional subgroup $K \subset \mathbb{T}^m$ acting freely on $\mathcal{Z}_P$.

Fixing basis in $K \simeq \mathbb{T}^{m-n}$, we obtain an integer $(m \times (m-n))$-matrix C determining a monomorphism $\mathbb{T}^{m-n} \to \mathbb{T}^m$.

The matrix C is determined up to a change of basis in the torus $\mathbb{T}^{m-n}$.

The homomorphism $\ell \colon \mathbb{T}^m \to \mathbb{T}^n$ satisfies the following *independence condition*:

- (I – admissible set) $\Longrightarrow$ $(T_I \cap \ker \ell = 1)$.

The homomorphism ℓ is called a *characteristic function* for the polytope P.

Once the bases in $\mathbb{T}^m$ and $\mathbb{T}^n$ are fixed, we see that the homomorphism $\ell \colon \mathbb{T}^m \to \mathbb{T}^n$ is determined by an integer $(n \times m)$-matrix Λ.

By construction, we have $\Lambda C = 0$.

Quasitoric manifold

Definition

The quotient space $M = \mathcal{Z}_P/K$, endowed with a canonical smooth structure, is called a *quasitoric manifold determined by combinatorial data* (P, Λ).

Monomial functions

Every vector $a = (a_1, \ldots, a_m) \in \mathbb{Z}^m$ determines a real-algebraic monomial function $\varphi_a \colon \mathbb{C}^m \to \mathbb{C}$ given by the formula:

$$\varphi_a(z_1, \ldots, z_m) = \hat{z}_1^{a_1} \cdot \ldots \cdot \hat{z}_m^{a_m},$$

where

- $\hat{z}_i^{a_i} = 1$ if $a_i = 0$,
- $\hat{z}_i^{a_i} = z_i^{a_i}$ if $a_i > 0$,
- $\hat{z}_i^{a_i} = \bar{z}_i^{-a_i}$ if $a_i < 0$.

Let $t = (t_1, \ldots, t_r) \in \mathbb{T}^r$, $a = (a_1, \ldots, a_r) \in \mathbb{Z}^r$ and A be an integer $(s \times r)$-matrix and A_k be the k-th row of the matrix A.

We will use the notation $t^a = t_1^{a_1} \cdot \ldots \cdot t_r^{a_r} \in \mathbb{T}^1$ and $t^A = (t^{A_1}, \ldots, t^{A_s}) \in \mathbb{T}^s$.

It follows directly from the definition that if $a \in \mathbb{Z}^m$ and $t \in \mathbb{T}^m$, then $\varphi_a(tz) = t^a \varphi_a(z)$ for all $z \in \mathbb{C}^m$.

The embedding of the group $T^{m-n} \simeq K \hookrightarrow \mathbb{T}^m$ is given by the matrix C, so any element $t \in K$ has the form $t = \tau^C$ for some (unique) element $\tau \in \mathbb{T}^{m-n}$.

For any $t \in K$ we have

$$\varphi_a(tz) = \tau^{a^T C} \varphi_a(z),$$

where $t = \tau^C$ and $\tau \in \mathbb{T}^{m-n}$.

Let $a, a' \in \mathbb{Z}^m$. Then the following statements are equivalent:

- vectors a and a' generate the same character of the group $K \subset \mathbb{T}^m$,
- $C^T a = C^T a'$,
- $(a - a') \in \operatorname{Im} \Lambda^T$.

Suppose that vectors $a_1, \dots, a_q \in \mathbb{Z}^m$ define the same character

$$\tilde{k}\colon K \subset \mathbb{T}^m \to \mathbb{T}^1.$$

Let $\varphi\colon \mathcal{Z}_P \to \mathbb{C}^q$ be a restriction of the monomial maps $\varphi_{a_1}, \dots, \varphi_{a_q}$ to the moment-angle manifold $\mathcal{Z}_P \subset \mathbb{C}^m$.

- If the character $\tilde{k}$ is trivial, then φ is constant on orbits of the action of K on $\mathcal{Z}_P$. The map φ induces a smooth map $\tilde{\varphi}\colon M \to \mathbb{C}^q$ equivariant with respect to some representation $\mathbb{T}^n \to \mathbb{T}^q$, where action of $\mathbb{T}^q$ on $\mathbb{C}^q$ is supposed to be standard.

- If the maps $\varphi_{a_1}, \dots, \varphi_{a_q}$ don't vanish simultaneously on $\mathcal{Z}_P$, then the map φ induces a smooth map $\tilde{\varphi}_{\mathbb{P}} \colon M \to \mathbb{C}P^{q-1}$ equivariant with respect to some representation $\mathbb{T}^n \to \mathbb{T}^q$, where the action of $\mathbb{T}^q$ on $\mathbb{C}P^{q-1}$ is supposed to be standard.

Maps of the form $\tilde{\varphi} \colon M \to \mathbb{C}^q$ and $\tilde{\varphi}_{\mathbb{P}} \colon M \to \mathbb{C}P^{q-1}$ constructed by a family of vectors $a_1, \dots, a_q$ are called *monomial maps* of the quasitoric manifold M.

Let us fix some character $\tilde{k}\colon K \to \mathbb{T}^1$. Since the matrix C determines the isomorphism of $\mathbb{T}^{m-n}$ and K, the character $\tilde{k}$ is uniquely determined by some vector $\tilde{b} \in \mathbb{Z}^{m-n}$.

Recall that for a given set of indices $I \subset [1, m]$ we denote by C_I the matrix obtained from C by removing rows with indices from I.

- Let $r \subset P$ be an edge of the polytope P. Since the action of $\mathbb{T}^n$ on M is locally standard, the isotropy subgroup $\chi(\pi^{-1}(r))$ of the submanifold $\pi^{-1}(r)$ is an $(n-1)$-dimensional toric subgroup in $\mathbb{T}^n$.

  Denote by $\Phi_r\colon \mathbb{T}^n \to \mathbb{T}^1$ the character with kernel isomorphic to $\chi(\pi^{-1}(r))$, and by μ_r the composition $\mu_r\colon \mathbb{T}^m \xrightarrow{I} \mathbb{T}^n \xrightarrow{\Phi_r} \mathbb{T}^1$.

- Let $\tilde{k}\colon K \to \mathbb{T}^1$ be a character of the group K and $v = F_I \in P$ be a vertex of P.

  The projection map $p_I\colon K \hookrightarrow \mathbb{T}^m \to T_{[1,m]\setminus I}$ is an isomorphism.

  One may associate with the vertex v a character

$$k_v\colon \mathbb{T}^m \to T_{[1,m]\setminus I} \xrightarrow{p_I^{-1}} K \xrightarrow{\tilde{k}} \mathbb{T}^1.$$

- Let $\tilde{k}\colon K \to \mathbb{T}^1$ be a character of the group K, $r \subset P$ an edge of P and $v \in r$ a vertex lying on the edge r.

  Then the pair (v, r) determines a character $k_{v,r}\colon \mathbb{T}^m \to \mathbb{T}^1$ defined as the sum of the characters $k_v\colon \mathbb{T}^m \to \mathbb{T}^1$ and $\mu_r\colon \mathbb{T}^m \to \mathbb{T}^1$.

The set of pairs of the form

$$\{ \text{ character } \tilde{k}\colon K \to \mathbb{T}^1, \text{ vertex } v \in P\}$$

determines the set of characters $\{k_v | v \in P\} \subset \mathsf{ch}(\mathbb{T}^{\mathsf{m}})$ and the set of triples of the form

$$\{ \text{ character } \tilde{k}\colon K \to \mathbb{T}^1, \text{ edge } r \subset P, \text{ vertex } v \in r\}$$

determines the set $\{k_{v,r} | v \in r \subset P\} \subset \mathsf{ch}(\mathbb{T}^{\mathsf{m}})$, where $\mathsf{ch}(\mathbb{T}^{\mathsf{m}})$ is the set of all characters of $\mathbb{T}^m$.

Therefore, every character $\tilde{k}\colon K \to \mathbb{T}^1$ determines the set $X_{\tilde{k}} = (\{k_v\} \cup \{k_{v,r}\}) \subset \mathsf{ch}(\mathbb{T}^m)$.

We list some of the properties of the set $X_{\tilde{k}}$ and characters k_v and $k_{v,r}$:

- The number of different characters in $X_{\tilde{k}}$ does not exceed $f_0(P)(n+1)$, because there are at most $f_0(P)$ characters $\{k_v\}$ and at most $nf_0(P) = 2f_1(P)$ characters of the form $\{k_{v,r}\}$.

  In practice, $|X_{\tilde{k}}|$ is often much less than this upper bound, since characters k_v and $k_{v,r}$ may be equal for different vertices $v \in P$ and pairs (v, r).

- The restriction of any of characters k_v and $k_{v,r}$ to the subgroup $K \subset \mathbb{T}^m$ is equal to $\tilde{k}\colon K \to \mathbb{T}^1$.
- For any vertex $v \in P$ the character $\tilde{k}_v$ is trivial if and only if the character $\tilde{k}$ is trivial.
- The character μ_r does not depend on $\tilde{k}$ and is nontrivial for every $r \subset P$.
  The restriction of μ_r to the subgroup K is trivial.

- Every character k_v, $v \in P$, is well-defined, but characters $\Phi_r \colon \mathbb{T}^n \to \mathbb{T}^1$ and $\mu_r \colon \mathbb{T}^m \to \mathbb{T}^1$ are defined only up to multiplication by ± 1, so the definition of $k_{v,r}$ is still ambiguous.
  If v_0, v_1 are vertices lying on an edge $r \subset P$, then, as we will show later, the character $k_{v_1} - k_{v_0}$ is a multiple of μ_r. We set $k_{v_0,r} = k_{v_0} + \mu_r$, where μ_r has the same direction as $k_{v_1} - k_{v_0}$. If $k_{v_1} = k_{v_0}$, then we assume that the first nonzero coordinate of m-vector defining $\mu_r \colon \mathbb{T}^m \to \mathbb{T}^1$ is positive.
- If the character $\tilde{k}$ is trivial, the cardinality of $X_{\tilde{k}}$ does not exceed $f_1(P) + 1$, because $k_v \equiv 1$ for all $v \in P$ and if v_0, v_1 are vertices of an edge $r \subset P$, then $k_{v_0,r} = k_{v_1,r} = \mu_r$.

Vectors w_r that define characters Φ_r, $r \subset P$, form an integer $(n \times q)$-matrix W. Consider the corresponding linear representation Φ of the torus $\mathbb{T}^n$ in linear space $\mathbb{C}^q$.

The moment map $\pi\colon M \to P \subset \mathbb{R}^n$ is equivariant with respect to the trivial torus action on $\mathbb{R}^n$.

Theorem

The moment map $\pi\colon M \to P$ can be extended to a real-algebraic embedding $\pi \times \tilde{\varphi}\colon M \to \mathbb{R}^n \times \mathbb{C}^q$ equivariant with respect to the representation $\Phi\colon \mathbb{T}^n \to \mathbb{T}^q$.

Here the number q does not exceed the number of edges of P, as follows from the construction of the representation Φ.

Theorem

The set of characters $X_{\tilde{k}}$, where $\tilde{k}\colon K \to \mathbb{T}^1$ is an arbitrary character, determines a monomial map $\tilde{\varphi}_{\mathbb{P},\tilde{k}}\colon M \to \mathbb{C}P^{q-1}$ that can be extended to an embedding $\pi \times \tilde{\varphi}_{\mathbb{P},\tilde{k}}\colon M \to P \times \mathbb{C}P^{q-1}$.

The induced cohomology pullback $\tilde{\varphi}^*_{\mathbb{P},\tilde{k}}\colon H^2(\mathbb{C}P^{q-1},\mathbb{Z}) \to H^2(M,\mathbb{Z})$ coincides with the classifying map $H^2(\mathbb{C}P^{\infty},\mathbb{Z}) \xrightarrow{\simeq} H^2(\mathbb{C}P^{q-1},\mathbb{Z}) \to H^2(M,\mathbb{Z})$ of the bundle $\xi_{\tilde{k}}$.

If the character $\tilde{k}$ is trivial, then the image of the map $\tilde{\varphi}_{\mathbb{P},\tilde{k}}$ lies entirely in some affine chart $\mathbb{C}^{q-1} \subset \mathbb{C}P^{q-1}$.

Suppose that combinatorial data (P, Λ) of a quasitoric manifold M satisfies following conditions:

- $b_P \in \mathbb{Z}^m$,
- we have $A_P^T = B\Lambda D$, where B is an invertible integral $(n \times n)$-matrix and D is a diagonal $(m \times m)$-matrix with all nonzero elements equal to ± 1.

Then the vector $b_P \in \mathbb{Z}^m$ determines a character $\mathbb{T}^m \to \mathbb{T}^1$ whose restriction to the subgroup $K \subset \mathbb{T}^m$ gives a character $\tilde{k}_P \colon K \to \mathbb{T}^1$.

Theorem

The projective map $\tilde{\varphi}_{\mathbb{P}, \tilde{k}_P} \colon M \to \mathbb{C}P^{q-1}$ constructed from the set $X_{\tilde{k}_P}$ is a smooth embedding.

Suppose that M is a complete nonsingular toric variety determined by a normal fan of some simple lattice polytope P. Consider monomials determined by vectors $b_v = i_P(v) \in \mathbb{Z}^m$, $v \in P$, and $a_{v,r}$, $v \in r \subset P$, where $a_{v,r}$ is a closest point to $i_P(v)$ on the edge $i_P(r)$.

Denote by X_P the set of all such monomials.

Theorem

The corresponding map of $\mathbb{C}^m$ to $\mathbb{C}^q$, $q = |X_P|$, induces an equivariant embedding $\tilde{\varphi}_{\mathbb{P},\tilde{k}_P} \colon M \to \mathbb{C}P^{q-1}$.

It is well known in algebraic geometry that a complete toric variety is projective if and only if its fan is a normal fan of some simple polytope.

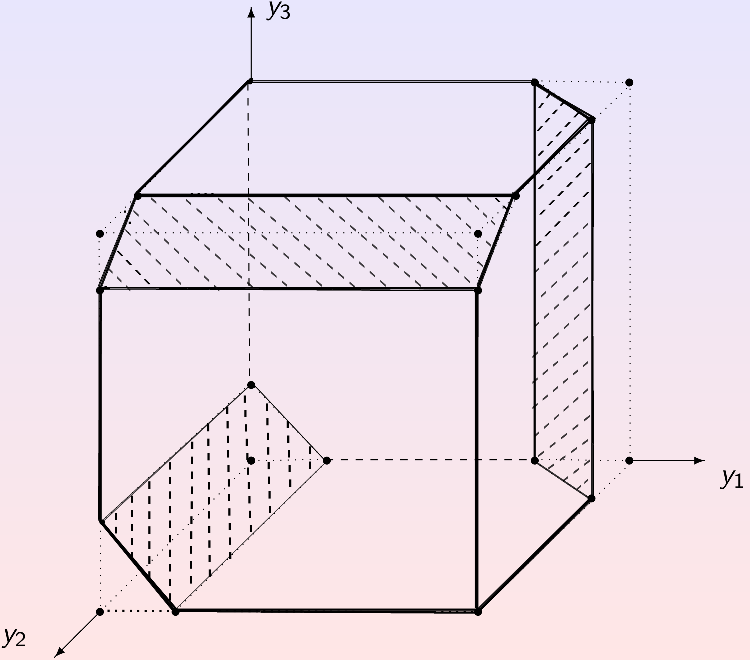
As shown by Yusuke Suyama, there exist complete toric varieties that are not quasitoric manifolds.

Let us consider an example of toric variety of complex dimension three over Stasheff polytope K_5.

The matrix $\Lambda = A_P^T$ has the form

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 & -1 & 1 & 0 \end{pmatrix},$$

and the vector b_P is equal to $(0, 0, 0, 3, 3, 3, 5, -1, 2)^T$.


y_3
y_1
y_2

The manifold $\mathcal{Z}_P \subset \mathbb{C}^9$ is given by the following equations:

$$
\begin{aligned}
|z_1|^2 + |z_4|^2 &= 3, \\
|z_2|^2 + |z_5|^2 &= 3, \\
|z_3|^2 + |z_6|^2 &= 3, \\
|z_1|^2 + |z_3|^2 + |z_7|^2 + |z_9|^2 &= 7, \\
|z_7|^2 + |z_8|^2 + |z_9|^2 &= 6, \\
|z_2|^2 + |z_3|^2 + |z_7|^2 &= 5.
\end{aligned}
$$

The moment-angle manifold $\mathcal{Z}_P \subset \mathbb{C}^9$ has a real dimension 12. It is endowed with the action of the compact torus $\mathbb{T}^9$, with six-dimensional subgroup $K \subset T^9$ acting freely on $\mathcal{Z}_P$.

In the case of Stasheff polytope we have $q = 6$ and $f_1(P) = 21$. These monomial functions embed the quasitoric manifold corresponding to Stasheff polytope to Euclidean space $\mathbb{R}^3 \times \mathbb{C}^6$.

The following monomial functions define a projective embedding of the manifold corresponding to Stasheff polytope K_5:

$$\varphi_{b_1} = z_1 z_4^2 z_5^3 z_6^3 z_7^5 z_9, \qquad \varphi_{b_2} = z_3 z_4^3 z_5^3 z_6^2 z_7^4 z_9^2,$$

$$\varphi_{b_3} = z_1 z_2^3 z_4^2 z_6^3 z_7^2 z_9^4, \qquad \varphi_{b_4} = z_2^3 z_3 z_4^3 z_6^2 z_7 z_9^5,$$

$$\varphi_{b_5} = z_1^2 z_4 z_5^3 z_6^3 z_7^5 z_8, \qquad \varphi_{b_6} = z_1^3 z_2 z_5^2 z_6^3 z_7^4 z_8^2,$$

$$\varphi_{b_7} = z_1^3 z_2 z_3^3 z_5^2 z_7 z_8^5, \qquad \varphi_{b_8} = z_1^2 z_3^3 z_4 z_5^3 z_7^2 z_8^4,$$

$$\varphi_{b_9} = z_1^3 z_2^2 z_3^3 z_5 z_8^5 z_9, \qquad \varphi_{b_{10}} = z_2^2 z_3^3 z_4^3 z_5 z_8^2 z_9^4,$$

$$\varphi_{b_{11}} = z_2^3 z_3^2 z_4^3 z_6 z_8 z_9^5, \qquad \varphi_{b_{12}} = z_1^3 z_2^3 z_3^2 z_6 z_8^4 z_9^2,$$

$$\varphi_{b_{13}} = z_3^3 z_4^3 z_5^3 z_7^2 z_8^2 z_9^2, \qquad \varphi_{b_{14}} = z_1^3 z_2^3 z_6^3 z_7^2 z_8^2 z_9^2.$$

$$\varphi_{a_1} = z_1 z_2 z_4^2 z_5^2 z_6^3 z_7^4 z_9^2, \qquad \varphi_{a_2} = z_1 z_2^2 z_4^2 z_5 z_6^3 z_7^3 z_9^3,$$
$$\varphi_{a_3} = z_1^2 z_2^3 z_4 z_6^3 z_7^2 z_8 z_9^3, \qquad \varphi_{a_4} = z_1^3 z_2^2 z_5 z_6^3 z_7^3 z_8^2 z_9,$$
$$\varphi_{a_5} = z_2 z_3 z_4^3 z_5^2 z_6^2 z_7^3 z_9^3, \qquad \varphi_{a_6} = z_2^2 z_3 z_4^3 z_5 z_6^2 z_7^2 z_9^4,$$
$$\varphi_{a_7} = z_1^3 z_2^3 z_3^1 z_6^2 z_7 z_8^3 z_9^2, \qquad \varphi_{a_8} = z_1^3 z_2 z_3 z_5^2 z_6^2 z_7^3 z_8^3,$$
$$\varphi_{a_9} = z_1^2 z_3 z_4 z_5^3 z_6^2 z_7^4 z_8^2, \qquad \varphi_{a_{10}} = z_1 z_2^3 z_3^2 z_4^2 z_6 z_8^2 z_9^4,$$
$$\varphi_{a_{11}} = z_1^2 z_2^3 z_3^2 z_4 z_6 z_8^3 z_9^3, \qquad \varphi_{a_{12}} = z_1^3 z_2 z_3^2 z_5^2 z_6 z_7^2 z_8^4,$$
$$\varphi_{a_{13}} = z_1^2 z_3^2 z_4 z_5^3 z_6 z_7^3 z_8^3, \qquad \varphi_{a_{14}} = z_3^2 z_4^3 z_5^3 z_6 z_7^3 z_8 z_9^2,$$
$$\varphi_{a_{15}} = z_1 z_3^3 z_4^2 z_5^3 z_7^2 z_8^3 z_9, \qquad \varphi_{a_{16}} = z_2 z_3^3 z_4^3 z_5^2 z_7 z_8^2 z_9^3,$$
$$\varphi_{a_{17}} = z_1 z_2^2 z_3^3 z_4^2 z_5 z_8^3 z_9^3, \qquad \varphi_{a_{18}} = z_1^2 z_2^2 z_3^3 z_4 z_5 z_8^4 z_9^2.$$

We see that the dimension of the projective embedding is much more than the dimension of the affine embedding.

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