Geometric Knot Theory

John M. Sullivan

Institut für Mathematik, TU Berlin
DFG Research Center Matheon

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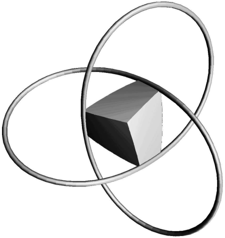
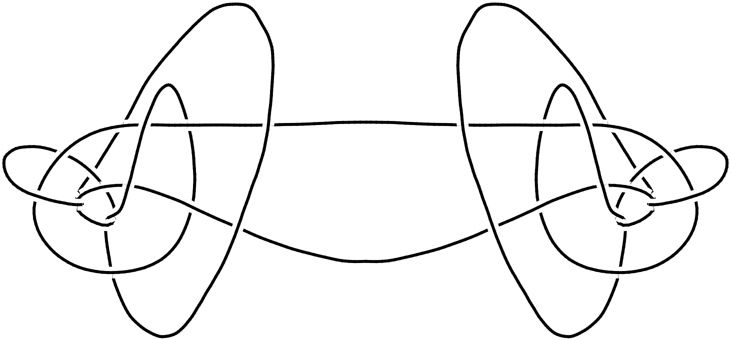


Knots and Links

- Closed curves embedded in space
- Classified topologically up to *isotopy*
- Two knotted curves are equivalent (same knot type)
  if one can be deformed into the other



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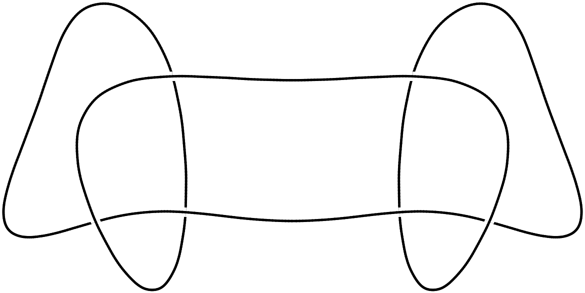


(Topological) Knot Theory

- Classify knot/link types
- Look for easily computed invariants
  to distinguish knots/links
- 3-manifold topology of complement

Geometric Knot Theory

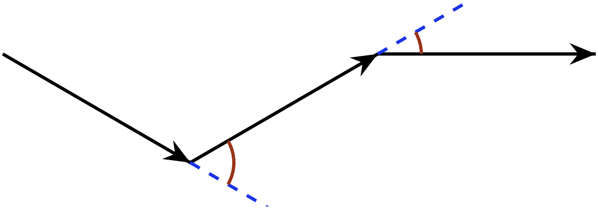
- Geometric properties determined by knot type
  or implied by knottedness
- Seek optimal shape for a given knot
  (optimal geometric form for topological object)
  usually: Minimize geometric energy

Total Curvature

- For smooth curve K

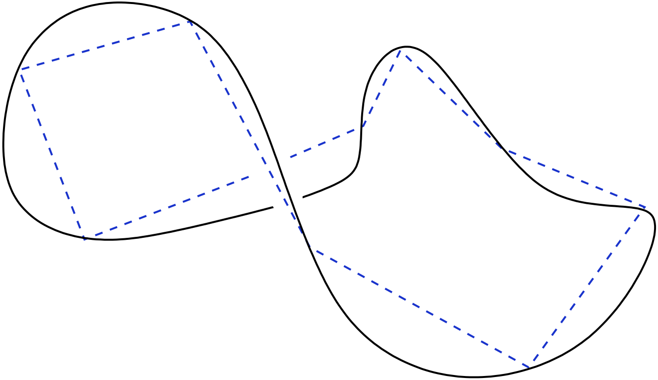
$$\mathsf{TC} = \int_K \kappa \, ds$$

- For polygon P
  TC = sum of turning angles (exterior angles)



Total Curvature

- For arbitrary curve K [Milnor]
  supremal TC of inscribed polygons

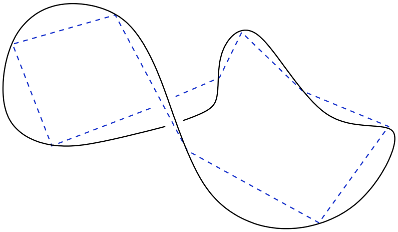


Curves of Finite Total Curvature

- FTC means TC $< \infty$
- Unit tangent vector
  BV function of arclength
- Curvature measure
  $T' = \kappa N\, ds$ as Radon measure
- Countably many corners
  where $T_+ \neq T_-$
  (curvature measure has atom)

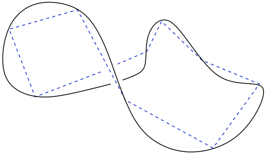
Approximation of FTC curves

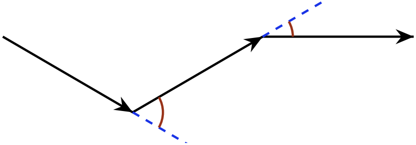
- FTC knot has isotopic inscribed polygon [Milnor]
- Tame knot type
- K, K' each FTC and C^1-close $\implies$ isotopic [DDS]
- FTC is "geometrically tame"



Projection of FTC curves

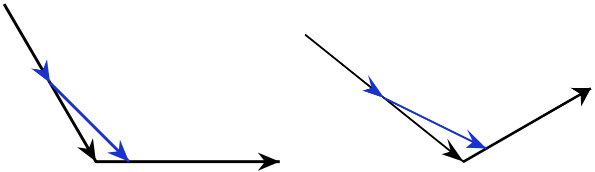
- Fix $k < n$ and an FTC $K \subset \mathbb{R}^n$
- Consider all projections of K to $\mathbb{R}^k$s
- Their average TC equals the TC of K
- Pf: Suffices to prove for polygons
  Suffices to prove for one corner





Projection of FTC curves (Proof)

- Given angle θ, average turning angle of its projections is some function $f_k^n(\theta)$
- By cutting corner into two, f_k^n clearly additive, hence linear $f_k^n(\theta) = c_k^n\,\theta$
- Any projection of a cusp is a cusp, so $f_k^n(\pi) = \pi$ Hence $c_k^n = 1$ as desired



Fenchel's Theorem

$\gamma \subset \mathbb{R}^n$ closed curve $\implies TC(\gamma) \geq 2\pi$

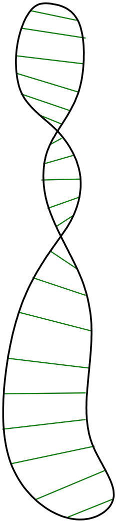
Pf: This is true in $\mathbb{R}^1$, where every angle is 0 or π

Fáry/Milnor Theorem

$K \subset \mathbb{R}^3$ knotted $\implies TC(K) \geq 4\pi$

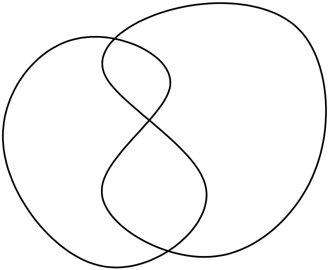
Milnor's Pf:

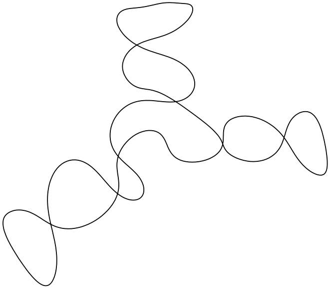
- No projection to $\mathbb{R}^1$
  can just go up & down
- So true in $\mathbb{R}^1$



Fáry/Milnor Theorem: Fáry's Proof

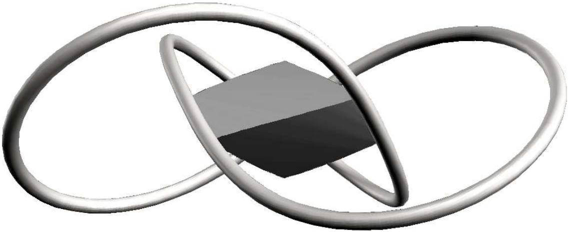
- True for knot diagrams in $\mathbb{R}^2$
- Because some region enclosed twice
  (perhaps not winding number two)





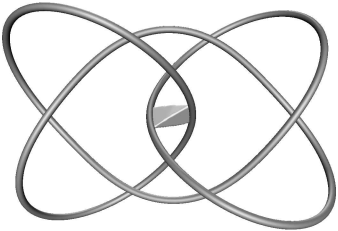
Second Hull: Intiuition

- Fary/Milnor says knot K “wraps around” twice
- Intuition says K “wraps around some point” twice
- Some region (second hull) doubly enclosed by K
- How to make this precise?



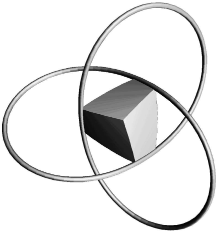
Second hull: Definition

- Characterize convex hull of K as
  set of points p such that
  every plane through p cuts K (at least twice)
- Then n^{th} hull of K is
  set of points p such that
  every plane through p cuts K at least $2n$ times



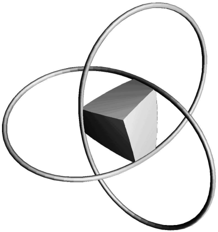
Second hull: Theorem

- Work with Jason Cantarella, Greg Kuperber, Rob Kusner
- *Amer. J. Math 125* (2003) 1335-1348
  arXiv:math.GT/0204106
- Thm: Knotted curve has nonempty second hull



Second hull: Proof

- Proof for prime FTC knot
- *Essential halfspace* contains all of K
  except one unknotted arc
- Intersection of all essential halfspaces is (part of) second hull

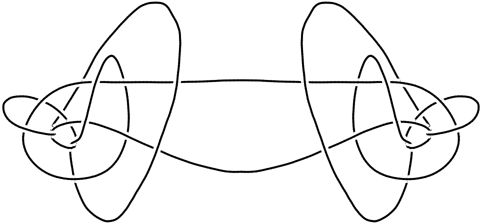


Möbius energy

- Inspired by Coulomb energy (repelling electrical charges)

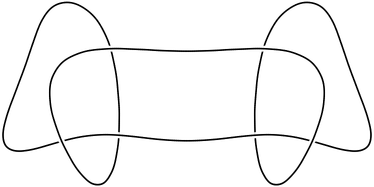
$$\iint_{K\times K} \frac{dx\,dy}{|x-y|^p}$$

- Renormalize to make this finite [O'Hara]
- Scale-invariant for $p = 2$
- Invariant under Möbius transformations [FHW]



Möbius energy

- Minimizers for prime knots [FHW]
- Probably not for composite knots
- Perhaps untangles all unknots
- Simulations (with Kusner) $\longrightarrow$ video

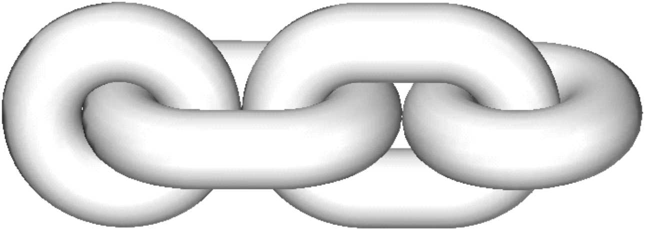


Ropelength: Definitions

- *Ropelength* of L: quotient length / thickness
- *Thickness*: diameter of largest embedded normal tube
  Positive thickness implies $C^{1,1}$
- *Gehring thickness*: minimum distance between components
  Works with Milnor's *link homotopy*

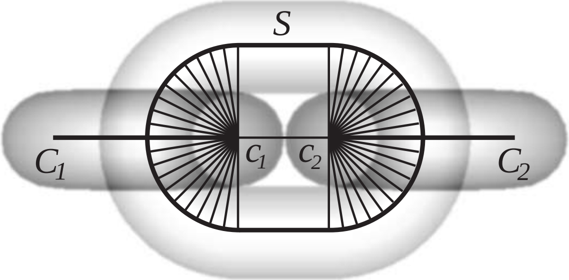
Ropelength: Theory

- Work with Jason Cantarella, Rob Kusner
  arXiv: math.GT/0103224
  Inventiones 150 (2002) pp 257-286
- Minimizers exist for any link type
- Some known from sharp lower bounds
- Need not be C^2



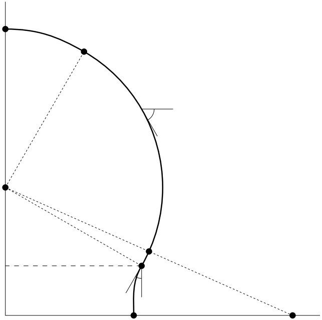
Tight Knots: Theory

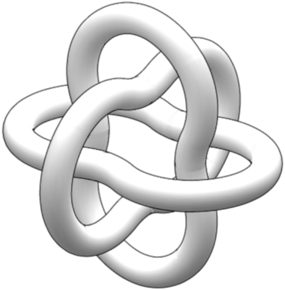
- New work with also Joe Fu, Nancy Wrinkle
  arXiv: math.DG/0402212, math.DG/0409369
- *Balance criterion* for ropelength-critical links



Tight Knots: Example

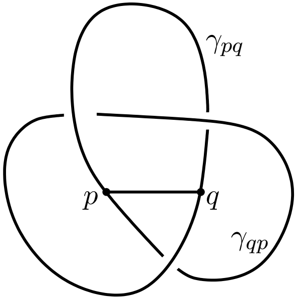
- critical Borromean rings
- piecewise smooth (14 pieces per component)





Distortion

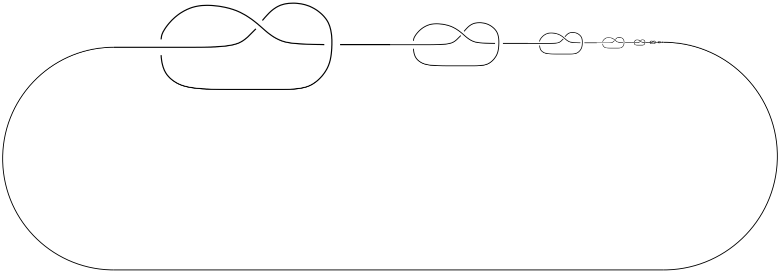
- Given $p, q \in K$,
  subarcs γ_{pq}, γ_{qp}
  of lengths ℓ_{pq}, ℓ_{qp}
- $d(p, q) := \min(\ell_{pq}, \ell_{qp})$
- $\delta(p, q) := d(p, q)/|p - q|$
  arc/chord ratio
- Distortion: $\delta(K) := \sup_{p,q} \delta(p, q)$.



Gromov: $\delta(K) \geq \pi/2$, equality only for round circle
Can every knot be built with $\delta < 100$?

Distortion: Upper bounds

- Trefoil can be built with $\delta < 8.2$
- Open trefoil with $\delta < 11$
- So infinitely many (even wild) knots with $\delta < 11$

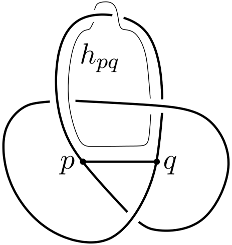


Distortion: Lower bounds

- Work with Elizabeth Denne, arXiv: math.GT/0409438
  K knotted implies $\delta > 4$
- with Denne and Yuanan Diao, arXiv: math.DG/0408026
  Geometry and Topology, to appear
  Ropelength ≥ 15.66 (within 5% for trefoil)
- Proofs use essential secants

Essential arcs

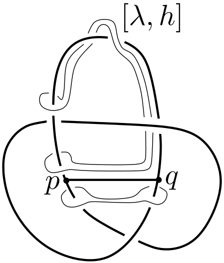
- Given $p, q \in K$, when is γ_{pq} *essential*?
- Construct free homotopy class h_{pq} in $\mathbb{R}^3 \smallsetminus K$
  Parallel to $\gamma_{pq} \cup \overline{qp}$, linking zero with K.
- γ_{pq} essential iff h_{pq} nontrivial
  iff $\gamma_{pq} \cup \overline{qp}$ spanned by no disk in $\mathbb{R}^3 \smallsetminus K$



Essential secants

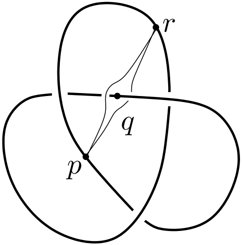
- K unknotted $\implies$ all arcs inessential ($\pi_1 = H_1$)
- γ_{pq} and γ_{qp} inessential $\implies$ K unknotted (Dehn)
- Defn: $\overline{pq}$ essential if both γ_{pq} and γ_{qp} are.
- If $\lambda \in \pi_1$ is meridian loop

  $[\lambda, h_{pq}] = [\lambda, h_{qp}]$

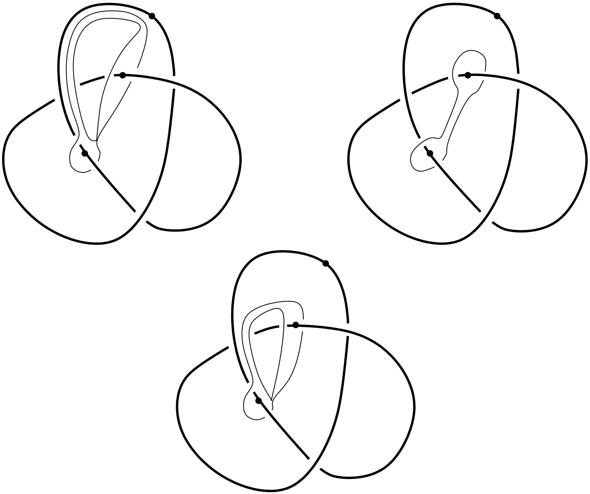


Arcs becoming essential

- Change in h_{pr} to essential happens because $\overline{pr}$ crosses $q \in K$
- Difference is $[\lambda, h_{pq}] = [\lambda, h_{qr}]$
- For γ_{pr} to become essential
  need $\overline{pq}$ and $\overline{qr}$ both essential



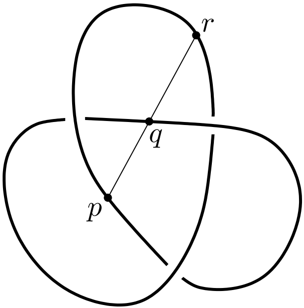
pq and qr are both essential



Distortion: Theorem

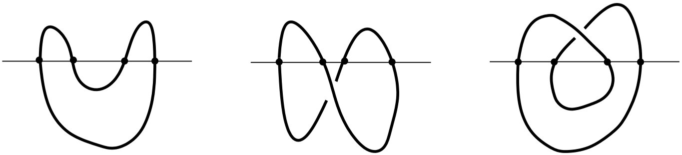
Thm: $\delta \geq \pi$ for any (FTC) knot

- Find shortest essential arc γ_{pr}
- Scale so $\ell_{pr} = \delta = \delta(K)$ (!)
- $\overline{ab}$ essential $\implies \ell_{ab}, \ell_{ba} \geq \delta$
  so $\delta(a,b) \leq \delta \implies |a-b| \geq 1$
- γ_{pr} stays outside $B_1(q)$
  so $\ell_{pr} \geq \pi$.



Quadrisecants

- Quadrisecant: line intersecting knot four times
- Every knot has one (Pannwitz, Kuperberg)
- Three order types: simple, flipped, alternating
- Denne thesis: all knots have alternating quadrisecants
- Alternating $\implies$ midsegment in second hull



Ropelength: Theorem

Thm: Ropelength > 15.66 for any knotted curve

- Denne gives essential alternating quadrisecant $abcd$
- Write lengths as $r := |a-b|$, $s := |b-c|$, $t := |c-d|$
- Scaling to thickness 1, we have $r, s, t \geq 1$
- Define $f(x) := \sqrt{x^2-1} + \arcsin(1/x)$
- $\ell_{ac} \geq f(r) + f(s)$, $\ell_{bd} \geq f(s) + f(t)$,
- $\ell_{da} \geq f(r) + s + f(t)$,
- $\ell_{cb} \geq \pi$ and $\ell_{cb} \geq 2\pi - 2\arcsin s/2$ if $s < 2$.
- Minimize sum separately in r, s, t.