Conformal geometry of knots

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Energy of knots, $E_\circ^{(2)}$.

- Generalization of electrostatic energy of charged knots.
- Introduced to produce an “optimal knot” for each knot type (half failed).
- $E_\circ^{(2)}$ is invariant under Möbius transformations.
Table of Contents

- Energy of knots, $E^{(2)}_o$.
  - Generalization of electrostatic energy of charged knots.
  - Introduced to produce an “optimal knot” for each knot type (half failed).
  - $E^{(2)}_o$ is invariant under Möbius transformations.

- Conformal geometry (Joint work with R. Langevin).
  - Infinitesimal cross ratio $\Omega$, which is a conformally invariant complex valued 2-form on $K \times K \setminus \triangle$.
  - $E^{(2)}_o$ can be expressed in terms of $\Omega$.
  - Two interpretations of its real and part; as a symplectic form and as an area element.
Motivation of energy of knots

**Problem** (Fukuhara, Sakuma)

- Define an “energy” $e$ on the space of knots.
- Define a “canonical position” for each knot type, which is an embedding that attains the minimum value of the “energy” within its isotopy class.
- We call it an $e$-minimizer.
Our strategy

A conceptual illustration

Problem
(Fukuhara, Sakuma)

Define an “energy” $e$ on the space of knots.

Define a “canonical position” for each knot type, which is an embedding that attains the minimum value of the “energy” within its isotopy class.
The complement is the set of embeddings, i.e. the space of knots.
Each “cell” corresponds to a knot type, as two points in the space of knots can be connected by a path if and only if two corresponding knots are ambient isotopic.
Take an “energy” 
\( e : \{ \text{knots} \} \rightarrow \mathbb{R} \).

Suppose each “cell” is surrounded by an \( \infty \)-ly high energy wall.

Given a knot.
Our strategy

Deform it along the gradient flow of the “energy” $e$. 

- knot type $[K]$ 
- {non-embedding} 
- {immersion}
Our strategy

If we are lucky the knot might reach an $e$-minimizer, which is a “canonical position” of that knot type.
Our strategy

Required property of our functional

In order to keep the knot type unchanged during the deformation process, crossing changes should be avoided!
Our strategy

Required property of our functional

- Definition.
  - $e$ is self-repulsive
    - def. $\uparrow$
  - $e(K)$ blows up as a knot $K$ degenerates to a singular knot with double points.
  - We say that $e$ is an energy of knots if it is self-repulsive.
Electrostatic energy of charged knots:

\[ E(K) = \int\int_{K \times K} \frac{dxdy}{|x - y|} \]
Electrostatic energy of charged knots:

\[ E(K) = \int \int_{K \times K} \frac{dx \, dy}{|x - y|} = \infty \quad (\forall K) \]

A trick \( \infty - \infty \) produces a finite valued functional \( E^{(1)} \).
Electrostatic energy of charged knots:

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A trick \( \infty - \infty \) produces a finite valued functional \( E^{(1)} \).

But \( E^{(1)}(\tilde{K}) < \infty \) for a singular knot \( \tilde{K} \) with double points.

How to produce a self-repulsive energy?
Electrostatic energy of charged knots:

\[ E(K) = \int \int_{K \times K} \frac{dx \, dy}{|x - y|} = \infty \ (\forall K) \]

A trick \( \infty - \infty \) produces a finite valued functional \( E^{(1)} \).

But \( E^{(1)}(\hat{K}) < \infty \) for a singular knot \( \hat{K} \) with double points.

How to produce a self-repulsive energy? Answer: Increase the power of \( |x - y| \) in the integrand.

\( 2 \leq \text{the power} < 3 \implies \) a well-defined energy.
Definition of $E_2$

Let $d_K(x, y)$ denote the arc-length between $x$ and $y$. 
Definition of $E_0^{(2)}$

Let $d_K(x, y)$ denote the arc-length between $x$ and $y$.

$$E_0^{(2)}(K) = \lim_{\epsilon \to +0} \left( \iint_{\{d_K(x,y) \geq \epsilon\} \subset K \times K} \frac{dx \, dy}{|x - y|^2} - \frac{2}{\epsilon} \right)$$

(We assumed $\text{Length}(K) = 1$ in above.)
Definition of $E^{(2)}_\circ$

Let $d_K(x, y)$ denote the arc-length between $x$ and $y$.

\[ E^{(2)}_\circ(K) = \lim_{\varepsilon \to +0} \left( \iint_{\{d_K(x, y) \geq \varepsilon\}} K \times K \frac{dx \, dy}{|x - y|^2} - \frac{2}{\varepsilon} \right) \]

(We assumed $\text{Length}(K) = 1$ in above.)

\[ E^{(2)}_\circ(K) = -4 + \iint_{K \times K \setminus \Delta} \left( \frac{1}{|x - y|^2} - \frac{1}{d_K(x, y)^2} \right) dx \, dy \]
Definition of $E^{(2)}_o$

Let $d_K(x, y)$ denote the arc-length between $x$ and $y$.

\[
E^{(2)}_o(K) = \lim_{\varepsilon \to 0+} \left( \iint_{\{d_K(x, y) \geq \varepsilon\} \subset K \times K} \frac{dx \, dy}{|x - y|^2} - \frac{2}{\varepsilon} \right)
\]

(We assumed $\text{Length}(K) = 1$ in above.)

\[
E^{(2)}_o(K) = -4 + \iint_{K \times K \setminus \triangle} \left( \frac{1}{|x - y|^2} - \frac{1}{d_K(x, y)^2} \right) dx \, dy
\]

$E^{(2)}_o(\text{circle } \bigcirc) = 0$. Drop off $-4$ for a long knot $\widetilde{K}$
Properties of $E_o^{(2)}$

**Theorem** (Freedman-He-Wang) $E_o^{(2)}$ is *conformally invariant*, i.e. if $T$ is a Möbius transformation of $\mathbb{R}^3 \cup \{\infty\}$ then $E_o^{(2)}(T(K)) = E_o^{(2)}(K) \ \forall K$. 
Properties of $E^{(2)}_o$

**Theorem** (Freedman-He-Wang) $E^{(2)}_o$ is **conformally invariant**, i.e. if $T$ is a Möbius transformation of $\mathbb{R}^3 \cup \{\infty\}$ then $E^{(2)}_o(T(K)) = E^{(2)}_o(K) \ \forall K$.

**Theorem** (Freedman-He-Wang)

$\exists$ an $E^{(2)}_o$-minimizer for $\forall$ **prime** knot type.
Properties of $E_{o}^{(2)}$

**Theorem** (Freedman-He-Wang) $E_{o}^{(2)}$ is **conformally invariant**, i.e. if $T$ is a Möbius transformation of $\mathbb{R}^3 \cup \{\infty\}$ then $E_{o}^{(2)}(T(K)) = E_{o}^{(2)}(K)$ $\forall K$.

**Theorem** (Freedman-He-Wang) $\exists$ an $E_{o}^{(2)}$-minimizer for $\forall$ *prime* knot type.

**Conjecture** (Kusner-Sullivan) $\nexists$ $E_{o}^{(2)}$-minimizers for any *composite* knot types. Numerical experiments imply:

(why?)
How composite knots behave

Use Möbius transformations to “open” knots.

“Pull-tight”
Properties of $E^{(2)}_{\circ}$

**Theorem** (Freedman-He-Wang) $E^{(2)}_{\circ}$ is **conformally invariant**, i.e. if $T$ is a Möbius transformation of $\mathbb{R}^3 \cup \{\infty\}$ then $E^{(2)}_{\circ}(T(K)) = E^{(2)}_{\circ}(K)$ $\forall K$.

**Theorem** (Freedman-He-Wang) $\exists$ an $E^{(2)}_{\circ}$-minimizer for $\forall$ **prime** knot type. (why?)

**Conjecture** (Kusner-Sullivan) $\not\exists$ $E^{(2)}_{\circ}$-minimizers for any **composite** knot types. Numerical experiments imply:

![Diagram](image-url)
Prime knots Case

Why do prime knots have $E_{\circ}^{(2)}$-minimizers? (How can they avoid pull-tight?)

But if the knot is prime:

$E_{\circ}^{(2)}$ is continuous w.r.t. $C^2$-topology.
Theorem (He) $E^{(2)}_o$-minimizers are smooth.
Remarks I

Theorem (He) $E_\circ^{(2)}$-minimizers are smooth.

Remarks There are uncountably many $E_\circ^{(2)}$-minimizers for each non-trivial prime knot type.

\[ \begin{align*}
K & : \text{an } E_\circ^{(2)} \text{-minimizer of a knot type } [K] \\
T & : \text{a Möbius transformation} \\
\Rightarrow & \quad \begin{cases} 
T(K) \text{ or its mirror image } T(K)^* \in [K], \text{ and} \\
E_\circ^{(2)}(T(K)) = E_\circ^{(2)}(T(K)^*) = E_\circ^{(2)}(K).
\end{cases}
\Rightarrow T(K) \text{ or } T(K)^* \text{ is an } E_\circ^{(2)} \text{-minimizer of } [K].
\end{align*} \]

Open Problem

\[ \# (\{\text{Minimizers of } [K]\}/\text{Möbius group}) = \text{?} \]
Open Problem

\[ \exists E^{(2)}_\circ \text{-critical unknots besides round circles?} \]
Remarks II

Open Problem

∃ $E_o^{(2)}$-critical unknots besides round circles?

If NO $\implies$ Hatcher’s results:

\[
\{\text{unknots in } S^3\} \xrightarrow{\text{deform. retract}} \{\text{great circles in } S^3\}
\]
Remarks II

Open Problem

\( \exists E_{o}^{(2)} \)-critical unknots besides round circles?

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\]

\( E_{o}^{(2)} \) can untie the following “unknots”.
- Ochiai’s unknot (Huang, Kauffman, and Grzeszczuk ’97).
- “Freedman’s unknot” (Kusner and Sullivan ’94).
Unknots

Figure 1: Ochiai’s unknot

Figure 2: Freedman’s unknot
Remarks II

**Open Problem**

\( \exists E^{(2)}_o \)-minimizing unknots which are not the circles?

If NO \( \iff \) Hatcher’s results:

\[ \{ \text{unknots in } S^3 \} \xrightarrow{\text{deform. retract}} \{ \text{great circles in } S^3 \} \]

**Conjecture** (Kusner-Sullivan) \( \exists \) unstable critical points in the \((p, q)\) torus knot type if both \(p\) and \(q\) are greater than 2.
Remarks II

Open Problem

\[ \exists \ E^{(2)}_o \text{-minimizing unknots which are not the circles?} \]

If NO \( \iff \) Hatcher’s results:

\[ \{ \text{unknots in } S^3 \} \overset{\sim}{\longrightarrow} \{ \text{great circles in } S^3 \} \]

deform. retract

Conjecture (Kusner-Sullivan) \( \exists \) unstable critical points in the \((p, q)\) torus knot type if both \( p \) and \( q \) are greater than 2.

Unfortunately, \( \not\exists \) theoretically determined \( E^{(2)}_o \) minimum values except for 0 for the trivial knot type.
Numerical experiments by Kusner and Sullivan

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How to produce energy minimizers

**Problem**
Find an energy for which \( \forall \) knot type has an energy minimizer.
How to produce energy minimizers

**Problem** Find an energy for which $\forall$ knot type has an energy minimizer.

Two solutions.

$$E_o^{(2)}(K) = -4 + \iint_{K \times K} \left( \frac{1}{d_{R^3}(x, y)^2} - \frac{1}{d_K(x, y)^2} \right) dx dy$$
How to produce energy minimizers

Problem
Find an energy for which ∀ knot type has an energy minimizer.

Two solutions.

\[ E_{o}^{(2)}(K) = -4 + \int \int_{K \times K} \left( \frac{1}{d^{3}(x, y)^{2}} - \frac{1}{d_{K}(x, y)^{2}} \right) dx dy \]

Make the power greater than 2
Problem Find an energy for which $\forall$ knot type has an energy minimizer.

Two solutions.

$$E_0^{(2)}(K) = -4 + \int_{K \times K} \left( \frac{1}{d_{R^3}(x, y)^2} - \frac{1}{d_K(x, y)^2} \right) dx dy$$

- Make the power greater than 2
- Change the metric of the ambient space

In each case, no Möbius invariance any more.
Increase the power

\[ E^{(2)}_o(K) = -4 + \iint_{K \times K} \left( \frac{1}{|x - y|^2} - \frac{1}{d_K(x, y)^2} \right) dx \, dy \]
Increase the power

\[ E_o^{(2)}(K) = -4 + \iint_{K \times K} \left( \frac{1}{|x - y|^2} - \frac{1}{d_K(x, y)^2} \right) dx \, dy \]

Let \( K \) be a knot with total length 1. Put

\[ E^{(\alpha)}(K) = \iint_{K \times K} \left( \frac{1}{|x - y|^\alpha} - \frac{1}{d_K(x, y)^\alpha} \right) dx \, dy. \]

It is well-defined if \( \alpha < 3 \).
Increase the power

\[ E_0^{(2)}(K) = -4 + \iint_{K \times K} \left( \frac{1}{|x - y|^2} - \frac{1}{d_K(x, y)^2} \right) \, dx \, dy \]

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It is well-defined if \( \alpha < 3 \).

**Theorem** \( \exists E^{(\alpha)} \)-minimizers \( \forall \) knot type if \( \alpha > 2 \).
Knots in a Riemannian mfd.

\[ E_0^{(2)}(K) = -4 + \iint_{\mathcal{K} \times \mathcal{K}} \left( \frac{1}{d_{R^3}(x, y)^2} - \frac{1}{d_K(x, y)^2} \right) dx \, dy \]
Let $\mathcal{M}$ be a Riemannian manifold. Define
\[ d_M(x, y) := \inf \{ \text{Length of path joining } x \text{ and } y \} . \]

\[ E_M^{(\alpha)}(K) = \iint_{K \times K} \left( \frac{1}{d_M(x, y)^\alpha} - \frac{1}{d_K(x, y)^\alpha} \right) dx \, dy. \]

**Theorem** Let $\mathcal{M}$ be a *compact* manifold.\[ \exists \ E_M^{(\alpha)} \text{-minimizers } \forall \ \text{knot type if } \alpha > 2. \]
Knots in a Riemannian mfd.

\[ E^{(2)}_o(K) = -4 + \int \int_{K \times K} \left( \frac{1}{d_{R^3}(x, y)^2} - \frac{1}{d_K(x, y)^2} \right) dx dy \]

Let \( M \) be a Riemannian manifold. Define \( d_M(x, y) := \inf \{ \text{Length of path joining } x \text{ and } y \} \).

\[ E^{(\alpha)}_M(K) = \int \int_{K \times K} \left( \frac{1}{d_M(x, y)^\alpha} - \frac{1}{d_K(x, y)^\alpha} \right) dx dy. \]

**Theorem** Let \( M \) be a compact manifold.
\[ \exists E^{(\alpha)}_M \text{-minimizers } \forall \text{ knot type if } \alpha > 2. \]

**Conjecture**
- Theorem holds for \( M = H^3 \).
- Theorem holds even for \( \alpha = 2 \) if \( M = S^3 \).
Related topics

Geometric knot theory. Complexity of embeddings.

(Distances). Thickness, rope length, global radius of curvature, distortion, “self-distance”, etc.

(Integral geometry). Average crossing number, measure of non-trivial spheres, etc.
Related topics

- Geometric knot theory. Complexity of embeddings.
  - (Distances). Thickness, rope length, global radius of curvature, distortion, “self-distance”, etc.
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- Random knotting and probabilities of knot types.
Related topics

- Geometric knot theory. Complexity of embeddings.
  - (Distances). Thickness, rope length, global radius of curvature, distortion, “self-distance”, etc.
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- Random knotting and probabilities of knot types.

- Topology of the spaces of knots.
  - Low dimensional topology of the knot complements and algebraic techniques.
Related topics

- Geometric knot theory. Complexity of embeddings.
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  - (Integral geometry). Average crossing number, measure of non-trivial spheres, etc.
- Random knotting and probabilities of knot types.
- Topology of the spaces of knots.
  - Low dimensional topology of the knot complements and algebraic techniques.
- Conformal geometry and the set of spheres.
  - Complex valued 2-form on $K \times K \setminus \triangle$.
  - Conformal integral geometry.
Part II. Conformal geometry

Joint work with Rémi Langevin (Bourgogne)
Part II. Conformal geometry

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Joint work with Rémi Langevin (Bourgogne)
The Minkowski space $\mathbb{R}_1^5$ with the Lorentz metric:
\[ \langle x, y \rangle = -x_0 y_0 + x_1 y_1 + \cdots + x_4 y_4 \]

\[ V = \{ x \in \mathbb{R}_1^5 | \langle x, x \rangle = 0 \} : \text{light cone} \]

\[ \Lambda = \{ x \in \mathbb{R}_1^5 | \langle x, x \rangle = 1 \} : \text{de Sitter space} \]

The 3-sphere is realized in $\mathbb{R}_1^5$ as $\{ 0 \in \text{lines} \subset V \}$

\[ S_1^3 = V \cap \{ x_0 = 1 \} \]

The Lorentz group $O(4, 1)$ acts on $S^3$. 

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de Sitter sp. as the set of spheres

\[ \mathcal{S}(2, 3) = \{ \Sigma \mid \text{an oriented 2-sphere in } S^3 \} \]

\[ \updownarrow 1:1 \]

Hyperbolic hypersurface of 1 sheet \( \Lambda \)
de Sitter sp. as the set of spheres

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Hyperbolic hypersurface of 1 sheet \( \Lambda \)
de Sitter sp. as the set of spheres

\[ \mathcal{S}(2, 3) = \{ \Sigma | \text{an oriented 2-sphere in } S^3 \} \owns \Sigma_p \]

\[ \upharpoonright 1:1 \]

Hyperbolic hypersurface of 1 sheet \( \Lambda \owns p \)
Willmore Conjecture

$\iota : T^2 \to \mathbb{R}^3$ : a smooth embedding.

$\kappa_1, \kappa_2$ : principal curvatures. Willmore functional $W$ is

$$W(\iota) = \int_{T^2} \left( \frac{\kappa_1 + \kappa_2}{2} \right)^2 \, dv = \int_{T^2} \left( \frac{\kappa_1 - \kappa_2}{2} \right)^2 \, dv$$

$W$ is invariant under Möbius transformations.

Conjecture $W(\iota) \geq 2\pi^2$. "$=$" $\iff \iota(T^2)$ is a torus of revolution $T_{\sqrt{2}, 1}$ mod. Möbius transformations.
Willmore Conjecture

\[ \iota : T^2 \to \mathbb{R}^3 : \text{a smooth embedding.} \]
\[ \kappa_1, \kappa_2 : \text{principal curvatures. Willmore functional } W \text{ is} \]
\[ W(\iota) = \int_{T^2} \left( \frac{\kappa_1 + \kappa_2}{2} \right)^2 dv = \int_{T^2} \left( \frac{\kappa_1 - \kappa_2}{2} \right)^2 dv \]

\( W \) is invariant under Möbius transformations.

**Conjecture** \( W(\iota) \geq 2\pi^2. \quad \text{"=} \iff \iota(T^2) \text{ is a torus of revolution } T_{\sqrt{2},1} \text{ mod. Möbius transformations.} \)

**Theorem** (Bryant) \( W(\iota) = \text{Area } (\varphi_\iota(T^2)), \) where
\[ \varphi_\iota : T^2 \to S(2,3) \text{ is the mean sphere map:} \]
\[ \varphi_\iota : T^2 \ni p \mapsto \Sigma_{\frac{\kappa_1(p)}{\kappa_1(p)+\kappa_2(p)}}^2(p) \in S(2,3), \]
tangent to \( \iota(T^2) \) at \( p \) with curvature \( \frac{\kappa_1(p)+\kappa_2(p)}{2} \).
Infinitesimal cross ratio

Geometric definition. Let $x, x + dx, y, y + dy \in K$. 
Infinitesimal cross ratio

Geometric definition. Let \( x, x + dx, y, y + dy \in K \).

\[
\Sigma = \Sigma(x, x + dx, y, y + dy) : 2\text{-sphere through } x, x + dx, y, y + dy.
\]
Infinitesimal cross ratio

Geometric definition. Let \( x, x + dx, y, y + dy \in K \).

\[
\Sigma = \Sigma(x, x + dx, y, y + dy)
\]

a stereographic projection \( \mathbb{C} \cup \{\infty\} \)

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Infinitesimal cross ratio

Geometric definition. Let \( x, x + dx, y, y + dy \in K \).

Through a stereographic projection
\[
\Sigma(x, x+dx, y, y+dy) \xrightarrow{\sim} \mathbb{C} \cup \{\infty\},
\]
x, \( x + dx \), y, y + dy can be identified with \( \tilde{x}, \tilde{x} + \tilde{dx}, \tilde{y}, \tilde{y} + \tilde{dy} \in \mathbb{C} \).
Infinitesimal cross ratio

Geometric definition. Let \( x, x + dx, y, y + dy \in K \).

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\Sigma(x, x+dx, y, y+dy) \xrightarrow{\cong} \mathbb{C} \cup \{\infty\},
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\( x, x + dx, y, y + dy \) can be identified with \( \tilde{x}, \tilde{x} + \tilde{dx}, \tilde{y}, \tilde{y} + \tilde{dy} \in \mathbb{C} \).

Definition. Let the infinitesimal cross ratio of a knot, \( \Omega(x, y) \), be the cross ratio

\[
\frac{(\tilde{x} + \tilde{dx}) - \tilde{x}}{(\tilde{x} + \tilde{dx}) - (\tilde{y} + \tilde{dy})} : \frac{\tilde{y} - \tilde{x}}{\tilde{y} - (\tilde{y} + \tilde{dy})} \sim \frac{\tilde{dx} \tilde{dy}}{(\tilde{x} - \tilde{y})^2}.
\]
Infinitesimal cross ratio

Geometric definition. Let \( x, x + dx, y, y + dy \in K \).

Through a stereographic projection \( \Sigma(x, x+dx, y, y+dy) \overset{\approx}{\rightarrow} \mathbb{C} \cup \{\infty\} \),
\( x, x + dx, y, y + dy \) can be identified with \( \tilde{x}, \tilde{x} + \tilde{dx}, \tilde{y}, \tilde{y} + \tilde{dy} \in \mathbb{C} \).

**Definition.** Let the *infinitesimal cross ratio* of a knot, \( \Omega(x, y) \), be the cross ratio
\[
\frac{(\tilde{x} + \tilde{dx}) - \tilde{x}}{(\tilde{x} + \tilde{dx}) - (\tilde{y} + \tilde{dy})} : \frac{\tilde{y} - \tilde{x}}{\tilde{y} - (\tilde{y} + \tilde{dy})} \sim \frac{\tilde{dx}\tilde{dy}}{(\tilde{x} - \tilde{y})^2}.
\]

**Remark** The four complex numbers are not uniquely determined. But the cross ratio is well-defined. We need the orientation of \( \Sigma \).

Conformal angle and $\Omega$

**Definition.** (Doyle and Schramm)
Definition. (Doyle and Schramm)

Let $C(x, x, y)$ be a circle tangent to $K$ at $x$ though $y$. 
Definition. (Doyle and Schramm)

Let $C(x, x, y)$ be a circle tangent to $K$ at $x$ though $y$. Let $C(y, y, x)$ be a circle tangent to $K$ at $y$ through $x$. 
Conformal angle and $\Omega$

**Definition.** (Doyle and Schramm)

Let $C(x, x, y)$ be a circle tangent to $K$ at $x$ though $y$.

Let $\theta_K(x, y)$ be the angle between $C(x, x, y)$ and $C(y, y, x)$.

Call it the *conformal angle* between $x$ and $y$. 
Conformal angle and $\Omega$

**Definition.** (Doyle and Schramm)

Let $C(x, x, y)$ be a circle tangent to $K$ at $x$ though $y$.

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The absolute value of the infinitesimal cross ration $\Omega$ is equal to

$$\frac{dx dy}{|x - y|^2}.$$

The argument of $\Omega$ is equal to $\theta_K(x, y)$.
**Definition.** (Doyle and Schramm)

Let \( C(x, x, y) \) be a circle tangent to \( K \) at \( x \) though \( y \).

Let \( \theta_K(x, y) \) be the angle between \( C(x, x, y) \) and \( C(y, y, x) \).

Call it the conformal angle between \( x \) and \( y \).

The absolute value of the infinitesimal cross ration \( \Omega \) is equal to

\[
\frac{dx
dy}{|x - y|^2}
\]

The argument of \( \Omega \) is equal to \( \theta_K(x, y) \).

**Proposition.**

\[
\Omega(x, y) = e^{i\theta_K(x, y)} \frac{dx
dy}{|x - y|^2}.
\]
Proposition

(Doyle and Schramm’s *cosine formula*)

\[
E_{(2)}(K) = \int \int_{K \times K \setminus \triangle} \frac{1 - \cos \theta_K(x, y)}{|x - y|^2} \, dx \, dy.
\]
Inf. X-ratio and $E_{o}^{(2)}$

**Proposition** (Doyle and Schramm’s *cosine formula*)

$$E_{o}^{(2)}(K) = \int \int_{K \times K \setminus \triangle} \frac{1 - \cos \theta_{K}(x, y)}{|x - y|^2} \, dx \, dy.$$  

**Proof** Assume $\text{Length}(K) = 1$.

$$E_{o}^{(2)}(K) = \lim_{\varepsilon \to +0} \left( \int \int \{ d_{K}(x, y) \geq \varepsilon \} \subset K \times K \frac{dx \, dy}{|x - y|^2} - \frac{2}{\varepsilon} \right)$$
Inf. X-ratio and $E^{(2)}_\circ$

**Proposition** (Doyle and Schramm’s *cosine formula*)

$$E^{(2)}_\circ(K) = \int\int_{K \times K \setminus \triangle} \frac{1 - \cos \theta_K(x, y)}{|x - y|^2} \, dx \, dy.$$ 

**Proof** Assume $\text{Length}(K) = 1$.

$$E^{(2)}_\circ(K) = \lim_{\varepsilon \to +0} \left( \int\int_{\{d_K(x,y) \geq \varepsilon\} \subset K \times K} \frac{dx \, dy}{|x - y|^2} - \frac{2}{\varepsilon} \right)$$

$$= \lim_{\varepsilon \to +0} \left\{ \int_K dx \int_{d_K(x,y) \geq \varepsilon} \frac{dy}{|x - y|^2} - \frac{2}{\varepsilon} \right\}$$

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Proof of cosine formula

\[ E_0^{(2)}(K) = \lim_{\varepsilon \to +0} \left\{ \int_K dx \int dK(x,y) \frac{dy}{|x - y|^2} - \frac{2}{\varepsilon} \right\} \]
**Proposition** (Doyle and Schramm’s cosine formula)

\[
E^{(2)}_o(K) = \iint_{K \times K \setminus \triangle} \frac{1 - \cos \theta_K(x, y)}{|x - y|^2} \, dx \, dy.
\]

This is another proof of the conformal invariance of \( E^{(2)}_o \)
Inf. X-ratio and $E^{(2)}_o$

**Proposition** (Doyle and Schramm’s cosine formula)

$$E^{(2)}_o(K) = \int\int_{K \times K \setminus \triangle} \frac{1 - \cos \theta_K(x, y)}{|x - y|^2} \, dx \, dy.$$ 

Recall $\Omega(x, y) = e^{i\theta_K(x, y)} \frac{dxdy}{|x - y|^2}$. 

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Inf. X-ratio and $E^{(2)}_\circ$

**Proposition** (Doyle and Schramm’s cosine formula)

$$E^{(2)}_\circ(K) = \iint_{K \times K \setminus \triangle} \frac{1 - \cos \theta_K(x, y)}{|x - y|^2} \, dx \, dy.$$ 

Recall $\Omega(x, y) = e^{i \theta_K(x,y)} \frac{dx \, dy}{|x - y|^2}$.

**Proposition**

$$E^{(2)}_\circ(K) = \iint_{K \times K \setminus \triangle} (|\Omega| - \Re \Omega).$$
Recall every $T^*M$ admits a **canonical symplectic form** (locally $\omega_M = \sum dq_i \wedge dp_i$), which is exact;

$\omega_M = -d \sum p_i dq_i$. 
Recall every $T^*M$ admits a *canonical symplectic form* (locally $\omega_M = \sum dq_i \wedge dp_i$), which is exact;

$$\omega_M = -d \sum p_i dq_i.$$

$\iota : S^n \times S^n \setminus \Delta \xrightarrow{\cong} T^*S^n$ via $\text{id} \times \text{stereo}$. and the dual

$$\bigcup_{p \in S^n} \{p\} \times (S^n \setminus \{p\}) \cong \bigcup_{p \in S^n} \{p\} \times \mathbb{R}^n \cong \bigcup_{p \in S^n} T^*_p S^n$$

(We need to fix the metric of $S^n$.)
Canonical symplectic form of $T^* M$

Recall every $T^* M$ admits a canonical symplectic form (locally $\omega_M = \sum dq_i \wedge dp_i$), which is exact;

$$\omega_M = -d \sum p_i dq_i.$$

\[ \iota : S^n \times S^n \setminus \triangle \overset{\cong}{\longrightarrow} T^* S^n \text{ via } \text{id} \times \text{stereo.} \text{ and the dual} \]
\[ \bigcup_{p \in S^n} \{p\} \times (S^n \setminus \{p\}) \cong \bigcup_{p \in S^n} \{p\} \times \mathbb{R}^n \cong \bigcup_{p \in S^n} T_p S^n \]

We write the pull-back $\iota^* \omega_M$ by the same letter $\omega_{S^n}$.
Recall every $T^* M$ admits a canonical symplectic form (locally $\omega_M = \sum dq_i \wedge dp_i$), which is exact;

$$\omega_M = -d \sum p_i dq_i.$$

Let $\iota: S^n \times S^n \setminus \Delta \to T^* S^n$ via $\text{id} \times \text{stereo.}$ and the dual

$$\bigcup_{p \in S^n} \{p\} \times (S^n \setminus \{p\}) \cong \bigcup_{p \in S^n} \{p\} \times \mathbb{R}^n \cong \bigcup_{p \in S^n} T^*_p S^n$$

We write the pull-back $\iota^* \omega_M$ by the same letter $\omega_{S^n}$.

**Proposition** $\omega_{S^n}$ is invariant under the diagonal action of a Möbius transformation: $(T \times T)^* \omega_{S^n} = \omega_{S^n}$. 

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The cross ratio of $w, w + \Delta w, z, z + \Delta z \in \mathbb{C}$ is:

$$\frac{(w + \Delta w) - w}{(w + \Delta w) - (z + \Delta z)} : \frac{z - w}{z - (z + \Delta z)} \sim \frac{\Delta w \Delta z}{(w - z)^2}.$$
Re $\Omega$ and a can. symplectic form

The cross ratio of $w, w + \Delta w, z, z + \Delta z \in \mathbb{C}$ is:

$$\frac{(w + \Delta w) - w}{(w + \Delta w) - (z + \Delta z)} : \frac{z - w}{z - (z + \Delta z)} \sim \frac{\Delta w \Delta z}{(w - z)^2}.$$ 

Let $\omega_{cr} = \frac{dw \wedge dz}{(w - z)^2}$ be a complex 2-form on $\mathbb{C} \times \mathbb{C} \setminus \Delta$. 
The cross ratio of $w, w + \Delta w, z, z + \Delta z \in \mathbb{C}$ is:

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Let $\omega_{cr} = \frac{dw \wedge dz}{(w - z)^2}$ be a complex 2-form on $\mathbb{C} \times \mathbb{C} \setminus \triangle$.

**Proposition** via $\mathbb{C} \cup \{\infty\} \cong S^2$, $\operatorname{Re} \omega_{cr} = -\frac{1}{2} \omega_{S^2}$. 

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The cross ratio of \( w, w + \Delta w, z, z + \Delta z \in \mathbb{C} \) is:

\[
\frac{(w + \Delta w) - w}{(w + \Delta w) - (z + \Delta z)} : \frac{z - w}{z - (z + \Delta z)} \sim \frac{\Delta w \Delta z}{(w - z)^2}.
\]

Let \( \omega_{cr} = \frac{dw \wedge dz}{(w - z)^2} \) be a complex 2-form on \( \mathbb{C} \times \mathbb{C} \setminus \Delta \).

**Proposition** via \( \mathbb{C} \cup \{\infty\} \cong S^2 \), \( \Re \omega_{cr} = -\frac{1}{2} \omega_{S^2} \).

Let \( \iota : K \times K \setminus \Delta \hookrightarrow S^3 \times S^3 \setminus \Delta \) be the inclusion.
\( \Re \Omega \) and a can. symplectic form

The cross ratio of \( w, w + \Delta w, z, z + \Delta z \in \mathbb{C} \) is:

\[
\frac{(w + \Delta w) - w}{(w + \Delta w) - (z + \Delta z)} : \frac{z - w}{z - (z + \Delta z)} \sim \frac{\Delta w \Delta z}{(w - z)^2}.
\]

Let \( \omega_{\text{cr}} = \frac{dw \wedge dz}{(w - z)^2} \) be a complex 2-form on \( \mathbb{C} \times \mathbb{C} \setminus \triangle \).

**Proposition** via \( \mathbb{C} \cup \{\infty\} \cong S^2 \), \( \Re \omega_{\text{cr}} = -\frac{1}{2} \omega_{S^2} \).

Let \( \iota : K \times K \setminus \triangle \hookrightarrow S^3 \times S^3 \setminus \triangle \) be the inclusion.

**Proposition** \( \Re \Omega(x, y) = -\frac{1}{2} \iota^* \omega_{S^3} \)

\( \Re \) inf. X-ratio = pull-back of can. sympl. form of \( T^*S^3 \).
Pseudo-Riem. str. of $S^3 \times S^3 \setminus \triangle$

Put $\mathcal{S}(0, 3) := \{S^0 \subset S^3\} \cong S^3 \times S^3 \setminus \triangle$. 
Put \( S(0, 3) := \{ S^0 \subset S^3 \} \cong S^3 \times S^3 \setminus \triangle \).

**Proposition** \( \exists \) a *pseudo-Riemannian structure* (i.e. \( \forall T_\Sigma S(0, 3) \) has an indefinite metric) of signature \((3, 3)\).
Put $\mathcal{S}(0, 3) := \{S^0 \subset S^3\} \cong S^3 \times S^3 \setminus \triangle$.

**Proposition**  \exists a *pseudo-Riemannian structure* (i.e. \(\forall T_\Sigma S(0, 3)\) has an indefinite metric) of signature \((3, 3)\).

It is invariant under Möbius transformations.
Pseudo-Riem. str. of $S^3 \times S^3 \setminus \triangle$

Put $\mathcal{S}(0, 3) := \{ S^0 \subset S^3 \} \cong S^3 \times S^3 \setminus \triangle$.

**Proposition** \exists a pseudo-Riemannian structure (i.e. \( \forall T_{\Sigma} S(0, 3) \) has an indefinite metric) of signature (3, 3).

It is invariant under Möbius transformations.

Proof: There are three ways.
Pseudo-Riem. str. of $S^3 \times S^3 \setminus \triangle$

Put $\mathcal{S}(0, 3) := \{S^0 \subset S^3\} \cong S^3 \times S^3 \setminus \triangle$.

**Proposition** $\exists$ a *pseudo-Riemannian structure* (i.e. $\forall T_\Sigma \mathcal{S}(0, 3)$ has an indefinite metric) of signature $(3, 3)$.

It is invariant under Möbius transformations.

Proof: There are three ways. The first proof:

- Homogeneous space
  
  $\mathcal{S}(0, 3) \cong SO(4, 1)/SO(3) \times SO(1, 1)$. 

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Pseudo-Riem. str. of $S^3 \times S^3 \setminus \triangle$

Put $\mathcal{S}(0, 3) := \{ S^0 \subset S^3 \} \cong S^3 \times S^3 \setminus \triangle$.

**Proposition** $\exists$ a *pseudo-Riemannian structure* (i.e. $\forall T_\Sigma \mathcal{S}(0, 3)$ has an indefinite metric) of signature $(3, 3)$.

It is invariant under Möbius transformations.

Proof: There are three ways. The second proof:

Plücker coordinates.

Let $\mathbb{R}^5_1 = \mathbb{R}^{4,1}$ be the Minkowski space, and $V$, the light cone. $\mathcal{S}(0, 3) \cong \{ 2$-plane $\Pi \subset \mathbb{R}^5_1 \mid 0 \in \Pi, \Pi \cap V \text{ transversely} \} \subset \bigwedge^2 \mathbb{R}^{5}_1 \cong \mathbb{R}^{10}_6$. 

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Pseudo-Riem. str. of $S^3 \times S^3 \setminus \triangle$

Put $\mathcal{S}(0, 3) := \{S^0 \subset S^3\} \cong S^3 \times S^3 \setminus \triangle$.

**Proposition** ∃ a pseudo-Riemannian structure (i.e. $\forall T_\Sigma \mathcal{S}(0, 3)$ has an indefinite metric) of signature $(3, 3)$.

It is invariant under Möbius transformations.

Proof: There are three ways. The second proof:

Plücker coordinates.

Let $\mathbb{R}^5_1 = \mathbb{R}^{4,1}$ be the Minkowski space, and $V$, the light cone. $\mathcal{S}(0, 3) \cong \{2$-plane $\Pi \subset \mathbb{R}^5_1$ | $0 \in \Pi$, $\Pi \cap V$ transversely$\}$

$\bigwedge^2 \mathbb{R}^5_1 \cong \mathbb{R}^{10}_6$. 

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Pseudo-Riem. str. of $S^3 \times S^3 \backslash \triangle$

Put $\mathcal{S}(0, 3) := \{S^0 \subset S^3\} \cong S^3 \times S^3 \backslash \triangle$.

**Proposition** $\exists$ a *pseudo-Riemannian structure* (i.e. $\forall T_\Sigma \mathcal{S}(0, 3)$ has an indefinite metric) of signature $(3, 3)$.

Proof: There are three ways. The third proof:
- Construct pseudoorthonormal basis using pencils.
- **6 pencils** of $S^0 \subset S^1$, 3 spacelike and 3 timelike.

3 spacelike pencils

3 timelike pencils
Re $\Omega$ as area element

Let $\nu$ be a composite

$$\nu : K \times K \setminus \triangle \hookrightarrow S^3 \times S^3 \setminus \triangle \xrightarrow{\cong} \mathcal{S}(0, 3).$$

Then $\nu(K \times K \setminus \triangle)$ is a surface in $\mathcal{S}(0, 3)$. 

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Re \( \Omega \) as area element

Let \( \nu \) be a composite

\[
\nu : K \times K \setminus \triangle \hookrightarrow S^3 \times S^3 \setminus \triangle \xrightarrow{\sim} S(0, 3). \]

Then \( \nu(K \times K \setminus \triangle) \) is a surface in \( S(0, 3) \).

**Theorem.** The real part of the inf. X-ratio is equal to the area element of \( \nu(K \times K \setminus \triangle) \) w.r.t. the semi-Riem. str.:

\[
\sqrt{\begin{vmatrix}
\langle \nu_x, \nu_x \rangle & \langle \nu_x, \nu_y \rangle \\
\langle \nu_y, \nu_x \rangle & \langle \nu_y, \nu_y \rangle 
\end{vmatrix}} \ dx \, dy = 2\sqrt{-1} \ Re \ \Omega.
\]

\[
\langle \nu_x, \nu_x \rangle = \langle \nu_y, \nu_y \rangle = 0
\]
Re Ω as area element

Let \( v \) be a composite

\[
v : K \times K \setminus \triangle \hookrightarrow S^3 \times S^3 \setminus \triangle \xrightarrow{\cong} S(0, 3).
\]

Then \( v(K \times K \setminus \triangle) \) is a surface in \( S(0, 3) \).

**Theorem.** The real part of the inf. X-ratio is equal to the area element of \( v(K \times K \setminus \triangle) \) w.r.t. the semi-Riem. str.:

\[
\sqrt{\begin{vmatrix} \langle v_x, v_x \rangle & \langle v_x, v_y \rangle \\ \langle v_y, v_x \rangle & \langle v_y, v_y \rangle \end{vmatrix}} \ dx dy = 2\sqrt{-1} \ Re \Omega.
\]

\( \langle v_x, v_x \rangle = \langle v_y, v_y \rangle = 0 \)

**Corollary.** Let \( \gamma_1 \cup \gamma_2 \) be a 2-component link. Then the area of \( v(\gamma_1 \times \gamma_2) \subset S(0, 3) \) is equal to 0.
$\Im \Omega$ as a local area element

The imaginary part $\Im \Omega$ of the infinitesimal cross ratio does not have a nice global interpretation.
\( \text{Im } \Omega \) as a local area element

The imaginary part \( \text{Im } \Omega \) of the infinitesimal cross ratio does not have a nice global interpretation.

\( \text{Im } \Omega \) may be singular at \((x, y) \in K \times K \setminus \Delta\) where the conformal angle \( \theta_K(x, y) \) vanishes. We orient spheres \( \Sigma(x, x + dx, y, y + dy) \) so that \( \theta_K(x, y) \geq 0 \).

Recall \( \text{Im } \Omega = \frac{\sin \theta_K(x, y) dx dy}{|x - y|^2} \).
\( \Im \Omega \) as a local area element

- The imaginary part \( \Im \Omega \) of the infinitesimal cross ratio does not have a nice global interpretation.

- \( \Im \Omega \) may be singular at \( (x, y) \in K \times K \setminus \triangle \) where the conformal angle \( \theta_K(x, y) \) vanishes. We orient spheres \( \Sigma(x, x + dx, y, y + dy) \) so that \( \theta_K(x, y) \geq 0 \).

Recall \( \Im \Omega = \frac{\sin \theta_K(x, y) dx dy}{|x - y|^2} \).

- Identify \( S^3 \cong \partial \mathbb{H}^4 \). \( \Im \Omega(x_0, y_0) \) is locally equal to the “transversal area element” at \( (x_0, y_0) \) of geodesics in \( \mathbb{H}^4 \) joining \( x \) and \( y \).
Identify $S^3 \cong \partial \mathbb{H}^4$. $\mathfrak{m} \Omega(x_0, y_0)$ is locally equal to the “transversal area element” at $(x_0, y_0)$ of geodesics in $\mathbb{H}^4$ joining $x$ and $y$.

$l(x, y)$: a geodesic in $\mathbb{H}^4$ joining $x$ and $y$.

$\Pi_0$: a totally geodesic 3-space of $\mathbb{H}^4$ s.t. $\Pi_0 \perp l(x_0, y_0)$.

$S(x, y) = l(x, y) \cap \Pi_0$: a surface in $\Pi_0$.

Then $\mathfrak{m} \Omega(x_0, y_0)$ is equal to the area element of $S(x, y)$ at $(x_0, y_0)$. 
References


- R. Langevin and J. O’Hara, in preparation