Geodesics on Sub-Riemannian manifolds

by

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Def. (M, D, g) : Sub-Riemann. m.f.d.

$\overset{\text{def}}{\Leftrightarrow}$ (i) M : C^∞-m.f.d.

(ii) D : Smooth subbundle of TM

(iii) g : Riemann. fibre. metric on D

Suppose that D is "bracket generating"

i.e., for any local basis $X_1, \dots, X_r$ of D on $U \subset M$

the collection of all vector fields $\{X_i, [X_i, X_j], [X_i, [X_j, X_k]], \dots\}$

generated by Lie brackets of X_i spans the whole

tangent bundle TU.

$\underline{D}$: the sheaf of germs of section of D

Define the sheaves $\{D^k\}_{k\geq 1}$ by setting

$D^1 = \underline{D}$ and

$D^{k+1} = D^k + [D^1, D^k]$ $(k \geq 1)$

$D^1 = D^2 \Rightarrow D$ is completely integrable

$\bigcup D^k = \underline{TM} \Rightarrow D$ is bracket generating

(non holonomic)

* D is also called a "distribution"

Examples of nonholonomic distributions

- Contact distribution

$\mathbb{R}^3\,(x, y, z)$

$D^2 = \{ dz - \frac{1}{2}(x\,dy - y\,dx) = 0 \}$

rank two

growth $(2, 3)$

$\exists\, f_1, f_2 \in \Gamma(D^2)$

s.t. $\{ f_1, f_2, [f_1, f_2] \}$

spans $T\mathbb{R}^3$

· Cartan disttribution

$\mathbb{R}^5 \ (x^1, x^2, x^3, x^4, x^5)$

$$X_1 = \frac{\partial}{\partial x^1} - \frac{1}{2}x^2\frac{\partial}{\partial x^3} - \left(x^3 - \frac{1}{2}x^1x^2\right)\frac{\partial}{\partial x^4}$$

$$X_2 = \frac{\partial}{\partial x^2} + \frac{1}{2}x^1\frac{\partial}{\partial x^3} - \left(x^3 + \frac{1}{2}x^1x^2\right)\frac{\partial}{\partial x^5}$$

$$X_3 = \frac{\partial}{\partial x^3}, \quad X_4 = \frac{\partial}{\partial x^4}, \quad X_5 = \frac{\partial}{\partial x^5}$$

$$[X_1, X_2] = X_3, \quad [X_1, X_3] = X_4, \quad [X_2, X_3] = X_5$$

the others are trivial

$\Gamma(D^2) = \langle X_1, X_2 \rangle$ rank two growth $(2, 3, 5)$

$\dim \mathrm{Aut}(\mathbb{R}^5, D^2) \leq 14$, $\max \mathrm{Aut}(\mathbb{R}^5, D^2) \simeq G_2$

[E. Cartan]

Riemannian geometry

A minimizer between two points is a geodesic.

Every geodesic is local minimizer.

Given by $\ddot{x}^i + \sum \Gamma^i_{jk} \dot{x}^j \dot{x}^k = 0$

Sub-Riemannian geometry

* Given by the projection of the integral curve of D.

There are two types geodesics:

(i) "normal type" depends on (D, g)

(ii) "abnormal type" depends only on D

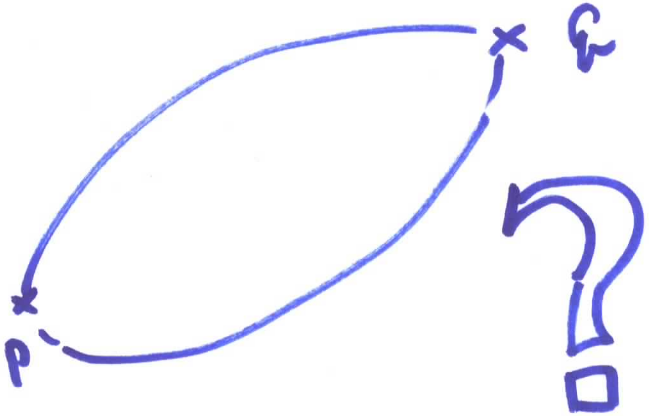
[1990's, I. Kupka, R. Montgomery]

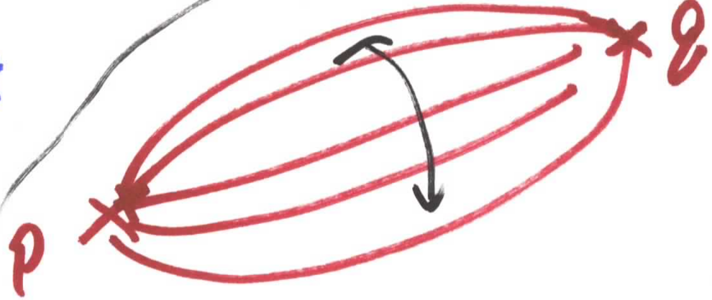
Remark A minimizer on Sub-Riemannian manifold is either normal or abnormal.

Thm. (Chow) Let M be a connected m.f.d. and D a non-holonomic distribution on M. then there exists for any two points $p, q \in M$ a piecewise smooth integral curve by which p and q can be joined.

In sub-Riemannian geometry, the space $C_D(p,q)$ of all integral curves of D joining p and q may have singularities.

so which makes difficult to apply the method of variation to the case.





In the Riemann. case $C(p,q)$ has no singularity, and is a smooth infinite dimensional manifold.

Def (M, D, g) : Sub-Riemann. mfd.

For $p \in M$, $v \in D_p$, we define the length $\|v\|_g$ of v by $\boxed{\|v\|_g = g_p(v, v)^{\frac{1}{2}}}$

If $\gamma : [a, b] \to M$ is an integral curve of D then we define the length of γ by

$$\boxed{\|\gamma\|_g = \int_a^b \|\dot{\gamma}(t)\|_g \, dt}$$

* If γ is not an integral curve, we agree to define

$$\|\gamma\|_g = +\infty .$$

We define a function $d_g : M \times M \to \mathbb{R} \cup \{\infty\}$ by

$$\boxed{d_g(p, q) = \inf\{\|\gamma\|_g : \partial\gamma = (p, q)\}.}$$

where $\partial\gamma = (\gamma(a), \gamma(b))$.

If M is connected and D is nonholonomic, then $d_g : M \times M \to \mathbb{R}$ is a metric func. on M.

The distance d_g is called sub-Riemann. distance or Carnot-Caratheodory metric.

If an integral curve $\gamma : [a, b] \to M$ of D satisfies

$$d_g(\gamma(a), \gamma(b)) = \|\gamma\|_g$$

γ is called a minimizer.

Def (Normal extremal)

$(p, \lambda) \in T^*M$

$\|\lambda\|_g$: the norm of $\lambda|_{D_p}$

$E : T^*M \to \mathbb{R}$ given by $E(x, \lambda) = -\frac{1}{2}\|\lambda\|_g^2$ is

the energy func of the sub-Riemann. str. (D, g).

A normal biextremal of (D, g) is a curve $\Gamma : I \to T^*M$

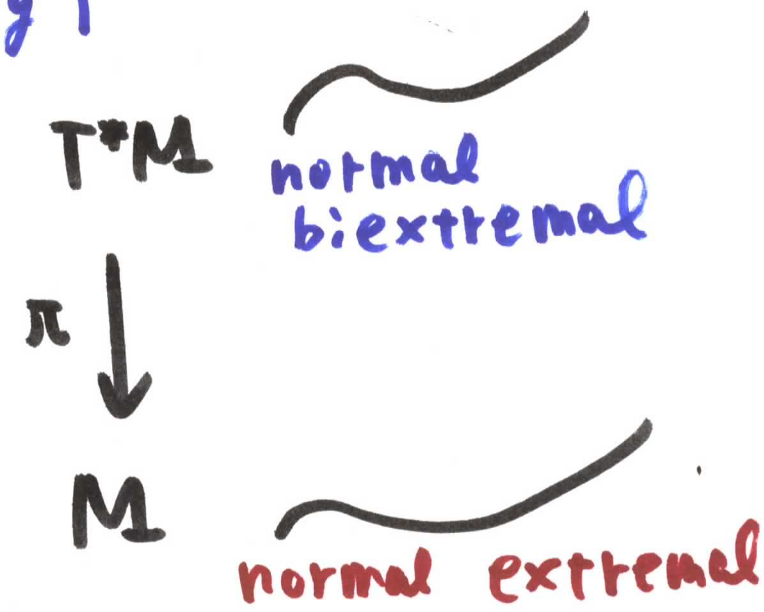
such that (i) $\dot{\Gamma}(t) = \vec{E}_{\Gamma(t)}$

(ii) E does not vanish along Γ

A normal extremal is a curve in M

which is projection of

a normal biextremal.



Def (abnormal extremal) $\{X_1, \cdots, X_r\}$: local basis of $U \subset M$

we define $H_{X_i} : T^*M \to \mathbb{R}$ by $H_{X_i} = \langle \lambda, X_i(p) \rangle$.

$\quad\quad\quad\quad\quad\quad\quad\quad \rotatebox{90}{$\in$}$

$\quad\quad\quad\quad\quad\quad (p, \lambda) \quad\quad\quad\quad\quad\quad (i = 1, \cdots, r)$

An abnormal biextremal of D is a curve $\Gamma : I \to T^*M \setminus \{0\}$

such that (i) $H_{X_i}(\Gamma(t)) = 0$

(ii) $\dot{\Gamma}(t) \in \langle (\vec{H}_{X_1})_{\Gamma(t)}, \cdots, (\vec{H}_{X_r})_{(\Gamma(t))} \rangle$

An abnormal extremal of D is a curve in M which is projection of a normal biextremal.

depend only on D

Let D be the Cartan distribution on $\mathbb{R}^5$ defined by

$\Gamma(D) = \langle X_1, X_2 \rangle$.

Choosing a sub-Riemannian metric g on D

so that $\{X_1(p), X_2(p)\}$ forms an orthonormal basis of D_p.

We consider the sub-Riemannian manifold $(\mathbb{R}^5, D, g)$

Let us determine the normal extremal and

the abnormal extremal of $(\mathbb{R}^5, D, g)$.

Extremals on the Cartan distribution

$(x^1, x^2, x^3, x^4, x^5, p_1, p_2, p_3, p_4, p_5) \in T^*\mathbb{R}^5$

A normal biextremal of (D, g) satisfies:

(1) $\dot{x}^1 = -A$

(2) $\dot{x}^2 = -B$

(3) $\dot{x}^3 = \frac{1}{2}x^2 A - \frac{1}{2}x^1 B$

(4) $\dot{x}^4 = (x^3 - \frac{1}{2}x^1 x^2) A$

(5) $\dot{x}^5 = (x^3 + \frac{1}{2}x^1 x^2) B$

(6) $\dot{p}_1 = \frac{1}{2}x^2 p_4 A + \frac{1}{2}(p_3 - \frac{1}{2}x^2 p_5) B$

(7) $\dot{p}_2 = (-\frac{1}{2}p_3 + \frac{1}{2}x^1 p_4) A - \frac{1}{2}x^1 p_5 B$

(8) $\dot{p}_3 = -p_4 A - p_5 B$

(9) $\dot{p}_4 = 0$

(10) $\dot{p}_5 = 0$

where $A = p_1 - \frac{1}{2}x^2 p_3 - (x^3 - \frac{1}{2}x^1 x^2) p_4$

$B = p_2 + \frac{1}{2}x^1 p_3 - (x^3 + \frac{1}{2}x^1 x^2) p_5$

Then we have :

(i) x^1 and x^2 run along the line

$$P_4 x^1 + P_5 x^2 + c' = 0$$

or (ii) $P_4 x^1 + P_5 x^2 + c'$ moves periodically between $-k$ and k, where k is constant.

These equation for (x^1, x^2) Can be

$$\begin{pmatrix} \ddot{x}^1 \\ \ddot{x}^2 \end{pmatrix} = P_3 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \dot{x}^1 \\ \dot{x}^2 \end{pmatrix}$$

where $P_3 = P_4 x^1 + P_5 x^2 + \dot{c}$

Since the acceleration vector $\begin{pmatrix} \ddot{x}^1 \\ \ddot{x}^2 \end{pmatrix}$ is obtained by the rotation of $\frac{\pi}{2}$ of the velocity vector $\begin{pmatrix} \dot{x}^1 \\ \dot{x}^2 \end{pmatrix}$, this equation represents the equation of motion of an electron moving in a plane under a magnetic field whose direction is perpendicular to the plane and whose magnitude is given by $P_3 = P_4 x^1 + P_5 x^2 + \dot{c}$.

Abnormal biextremal $\Gamma(t) = (x(t), p(t))$ satisfies:

(1) $\dot{x}^1 = a^1$

(2) $\dot{x}^2 = a^2$

(3) $\dot{x}^3 = -\frac{1}{2}a^1x^2 + \frac{1}{2}a^2x^1$

(4) $\dot{x}^4 = -a^1(x^3 - \frac{1}{2}x^1x^2)$

(5) $\dot{x}^5 = -a^2(x^3 + \frac{1}{2}x^1x^2)$

(6) $\dot{p}_1 = -\frac{1}{2}a^1x^2p_4 - a^2(\frac{1}{2}p_3 - \frac{1}{2}x^2p_5)$

(7) $\dot{p}_2 = a^1(-\frac{1}{2}x^2p_4 + \frac{1}{2}p_3) + \frac{1}{2}a^2x^1p_5$

(8) $\dot{p}_3 = a^1p_4 + a^2p_5$

(9) $\dot{p}_4 = 0$

(10) $\dot{p}_5 = 0$

* a^1, a^2 are some func

We have $\begin{pmatrix} \dot{x}^1 \\ \dot{x}^2 \end{pmatrix} = \begin{pmatrix} a^1 \\ a^2 \end{pmatrix} = \varphi \begin{pmatrix} p_5 \\ -p_4 \end{pmatrix}$, where φ is a function along $\Gamma(t)$.

If we set $\psi = \int_{\alpha}^{t} \varphi(s)ds$, we have $\begin{pmatrix} x^1(t) \\ x^2(t) \end{pmatrix} = \psi(t)\begin{pmatrix} p_5 \\ -p_4 \end{pmatrix} + \begin{pmatrix} q^1 \\ q^2 \end{pmatrix}$

* $q^1 = x^1(\alpha)$, $q^2 = x^2(\alpha)$

Thus the lines in (x^1, x^2)-space give rise to the abnormal extremal.