

Symplectic invariants of spectral curves and their applications to enumerative geometry.

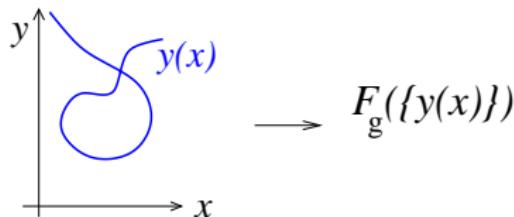
Bertrand Eynard
Institut de Physique Théorique CEA-SACLAY

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Symposium 2008 "Riemann Surfaces, Harmonic Maps and
Visualization", Osaka, dec 2008

$$\text{Diagram showing a decomposition of a Riemann surface element. On the left, a disk labeled } g \text{ with boundary points } z_{n+1}, z_1, z_2, z_n \text{ is shown. It decomposes into two parts: a central part labeled } g-1 \text{ with boundary points } z_{n+1}, \bar{z}, z, z, z_n \text{ and a right part labeled } h \text{ with boundary points } \bar{z}, z, J. \text{ The decomposition is given by:}$$
$$= z_{n+1}^K \frac{z}{z} + z_{n+1}^K \frac{\bar{z}}{z} + z_{n+1}^K \frac{z}{\bar{z}} + z_{n+1}^K \frac{J}{\bar{J}}$$

- Newly discovered invariants:

$\mathcal{S} = \{y(x)\}$ = spectral curve \longrightarrow invariants $F_g(\mathcal{S})$, $g \in \mathbb{N}$.



- F_g = symplectic invariant:

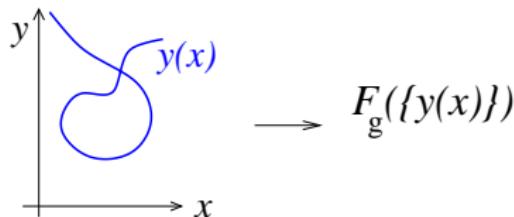
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- Many properties: integrability, modularity, special geometry deformations, ...

- Many applications: random matrices, enumeration of discrete surfaces, Gromov-Witten, Witten-Kontsevich, CFT, ...

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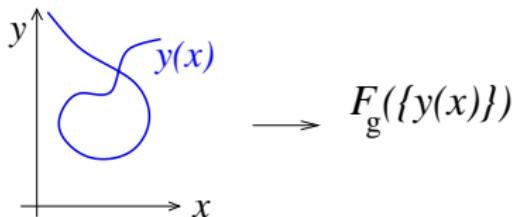
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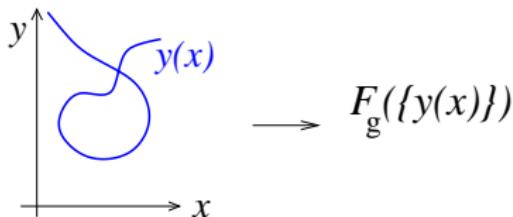
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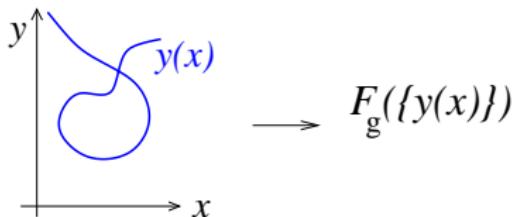
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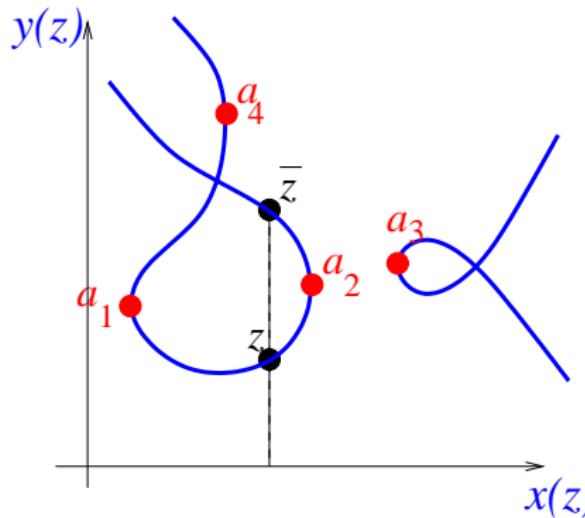
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Outline

- Algebraic definition:
spectral curves and their **symplectic invariants**
- Some applications
 - Topological expansion of **Matrix models**
 - **Discrete surfaces**
 - **Intersection numbers** and double scaling limits of large discrete surfaces, Kontsevich integral
 - **Volumes of moduli spaces**
 - **Partitions**, crystal growth and TASEP
 - **Gromov-Witten invariants**
- Conclusion

Spectral curve

- Spectral curve: $\mathcal{S} = (\mathcal{C}, x(z), y(z))$ $(x, y : \mathcal{C} \rightarrow \mathbb{C})$



- Branchpoints a_i = points with vertical tangent $dx(a_i) = 0$
Assumption: regular spectral curve ($\leftrightarrow \forall i, a_i$ simple).
- Conjugated point near a_i , $\exists!$ point \bar{z} such that $x(\bar{z}) = x(z)$.
- \mathcal{C} = Riemann surf., genus $\bar{g} \rightarrow \theta$ -function, Alg. Geometry...

Topological Recursion

$$B(z_1, z_2) = d_{z_1} d_{z_2} \ln \theta_*(z_1 - z_2) = \text{Bergman kernel}$$

$$K(z_1, z) = \frac{1}{2(y(z) - y(\bar{z})) dx(z)} d_{z_1} \ln \frac{\theta_*(z_1 - \bar{z})}{\theta_*(z_1 - z)}$$

n-forms: $W_n^{(g)}(z_1, \dots, z_n)$ = "n-point function, order g":
 $W_2^{(0)}(z_1, z_2) = B(z_1, z_2)$

Topological recursion:

$$W_{n+1}^{(g)}(z_1, \dots, z_{n+1}) = \sum_i \underset{z \rightarrow a_i}{\text{Res}} K(z_{n+1}, z) \left[$$

$$W_{n+2}^{(g-1)}(z, \bar{z}, z_1, \dots, z_n) + \sum_h' \sum_{J \subset [1, \dots, n]} W_{1+|J|}^{(h)}(z, \mathbf{z}_J) W_{1+n-|J|}^{(g-h)}(\bar{z}, \mathbf{z}_{\bar{J}}) \right]$$

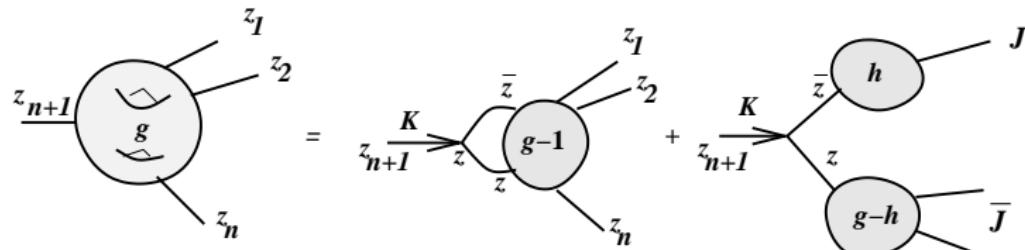
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Diagrammatic representation:

$$K(z_1, z) = z_1 \longrightarrow z, \quad W_2^{(0)}(z_1, z_2) = B(z_1, z_2) = z_1 \longrightarrow z_2$$



Example 3-point function

$$W_3^{(0)}(z_1, z_2, z_3) = \sum_i \text{Res}_{z \rightarrow a_i} K(z_1, z)(B(z, z_2)B(\bar{z}, z_3) + B(z, z_3)B(\bar{z}, z_2))$$

Example: rational curve $z_i \in \mathcal{C} = \mathbb{P}^1$:

$$B(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2}$$

$$W_3^{(0)}(z_1, z_2, z_3) = \sum_i \frac{dz_1 dz_2 dz_3}{x''(a_i)y'(a_i) (z_1 - a_i)^2(z_2 - a_i)^2(z_3 - a_i)^2}$$

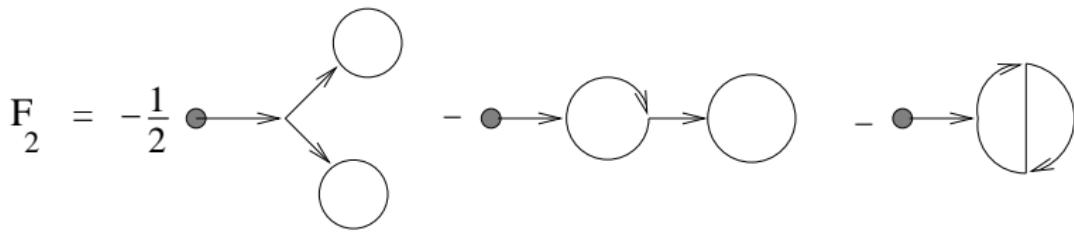
Definition: F_g

Definition of the Symplectic Invariant F_g , $g \geq 2$:

$$F_g = W_0^{(g)} = \frac{1}{2 - 2g} \sum_i \text{Res}_{z \rightarrow a_i} W_1^{(g)}(z) \Phi(z)$$

where $z \bullet = \Phi(z) = \int_*^z y dx$

Example F_2 :



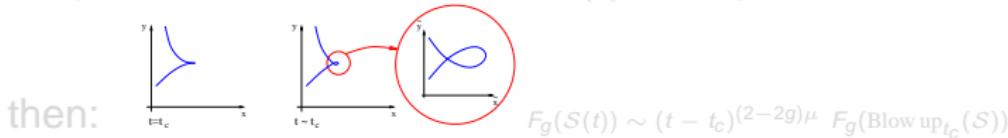
Similar definitions for F_0 and F_1 .

Main properties

- **Symmetry:** $W_n^{(g)}(z_1, z_2, \dots, z_n)$ is symmetric in n variables.
- **Homogeneity:** $y \rightarrow \lambda y \Rightarrow F_g \rightarrow \lambda^{2-2g} F_g$.
- **Symplectic invariance:** $dx \wedge dy = d\tilde{x} \wedge d\tilde{y} \Rightarrow F_g(\mathcal{S}) = F_g(\tilde{\mathcal{S}})$.
- **Deformations** of the spectral curve $y(x) \rightarrow y(x) + \delta y(x) \Rightarrow$

$$\delta W_n^{(g)}(z_1, \dots, z_n) = \int_{(\delta y dx)^*} W_{n+1}^{(g)}(z_1, \dots, z_n, z)$$

- **Singular limits, universality:** If $\mathcal{S}(t)$ is singular at $t = t_c$,



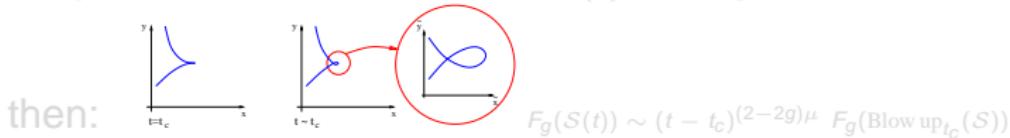
- **Integrability:** Tau-function $\tau = \exp \left(\sum_g N^{2-2g} F_g(\mathcal{S}) \right) \Theta$.
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- And many more ...

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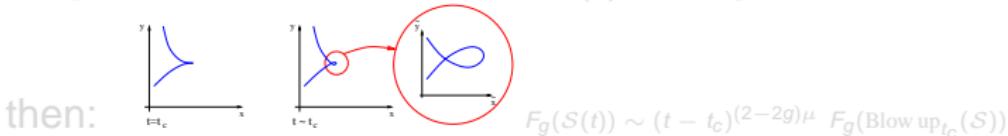
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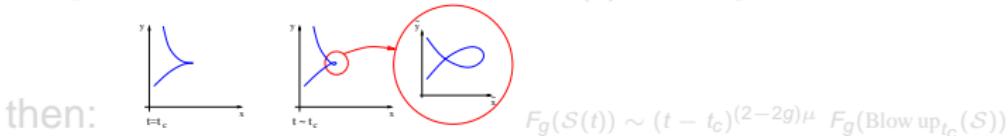
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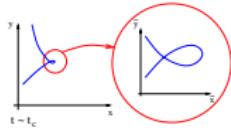
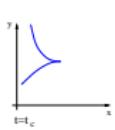
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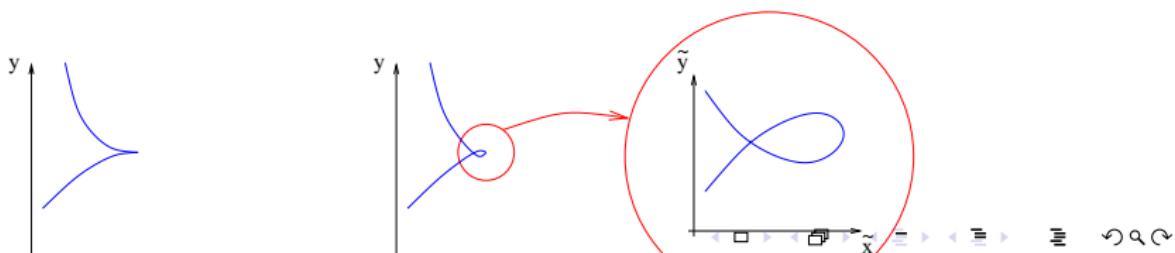
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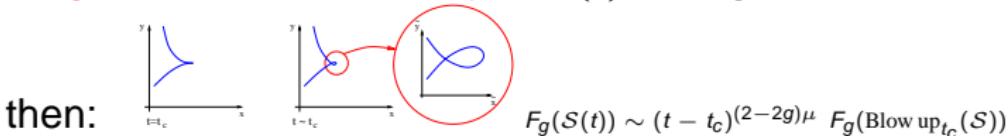


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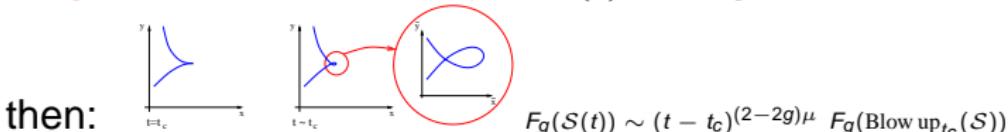
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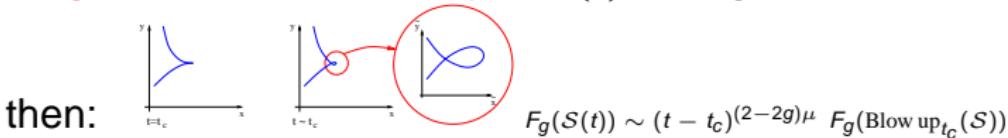
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Hirota equations, Determinantal formulae,
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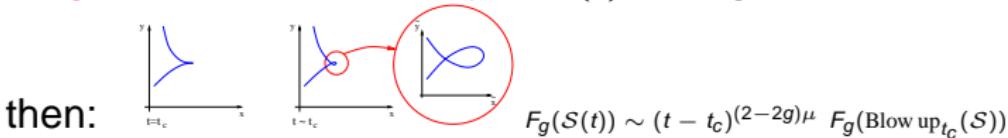
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Application: Matrix models

Consider a $N \times N$ matrix integral with a topological expansion:

$$Z = \int_{H_N(\gamma)} dM e^{-N \text{Tr } V(M)} \sim \exp \left(\sum_g N^{2-2g} \mathcal{F}_{g_{\text{M.M}}} \right)$$

Theorem 1: Schwinger-Dyson (=Loop equations) imply

$$\mathcal{F}_{g_{\text{M.M}}} = F_g(\mathcal{S}):$$

Sp. curve: $\mathcal{S} = \begin{cases} y = \frac{1}{2} \left(V'(x) - \sqrt{V'(x)^2 - 4P(x)} \right) \\ = " \lim_{N \rightarrow \infty} \frac{1}{N} < \text{Tr} \frac{1}{x-M} >" \\ \frac{1}{2i\pi} \oint_{\mathcal{A}_i} y dx = \epsilon_i = \text{fill.fractions} \rightarrow \gamma \end{cases}$

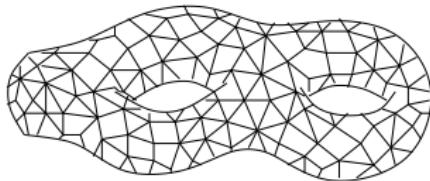
Generalizations: 2-matrix model, chain of matrices, +ext. field.

In all those cases, $Z = \exp \left(\sum_g N^{2-2g} \mathcal{F}_{g_{\text{M.M}}} \right)$,

$\exists \mathcal{S}$ such that $\mathcal{F}_{g_{\text{M.M}}} = F_g(\mathcal{S})$.

Application: discrete surfaces

\mathbb{M}_g = set of **discrete surfaces (= maps)** obtained by gluing n_3 triangles, n_4 quadrangles, ..., and of **genus g** and with v vertices.



$$\mathcal{F}_{g \text{discr. surf.}} = \sum_{\Sigma \in \mathbb{M}_g} t_3^{n_3(\Sigma)} t_4^{n_4(\Sigma)} \dots t_d^{n_d(\Sigma)} \frac{t^{v(\Sigma)}}{\#\text{Aut}(\Sigma)}$$

Theorem 2: Tutte's equations $\Rightarrow \mathcal{F}_{g \text{discr. surf.}} = F_g(\mathcal{S})$:

$$\mathcal{S} = \begin{cases} x(z) = \gamma(z + z^{-1}) + \alpha \\ y(z) = \sum_{k=1}^d u_k(z^k - z^{-k}) \\ \mathcal{C} = \mathbb{P}^1 \\ x(z) - \sum_k t_{k+1} x(z)^k = \sum_{k=1}^{d-1} u_k(z^k + z^{-k}), \quad u_1 = \frac{t}{\gamma} \end{cases}$$

Kontsevich integral and Double scaling limit

Kontsevich integral:

$$Z_{\text{Kontsevich}}(\Lambda) = \int dM e^{-N \text{Tr} \frac{M^3}{3} - M \Lambda^2} = e^{\sum N^{2-2g} \mathcal{F}_g}, \quad t_k = \frac{1}{N} \text{Tr} \Lambda^{-k}$$

$$\mathcal{F}_g = \sum_n \sum_{d_1, \dots, d_n} \prod_i \frac{t_{2i+1}^{d_i}}{d_i!} < \prod \tau_i^{d_i} >_g$$

Theorem 3: Loop equations imply $\mathcal{F}_g = F_g(S_{\text{Kontsevich}})$:

$$S_{\text{Kontsevich}} = \begin{cases} x(z) = z + \frac{1}{2N} \text{Tr} \frac{1}{\Lambda(z-\Lambda)} = z - \frac{1}{2} \sum_{k=0}^{\infty} t_{k+2} z^k \\ y(z) = z^2 + t_1 \end{cases}$$

Theorem 3bis: Double scaling limit of discrete surfaces:

$$\mathcal{F}_{\text{Discr. Surf. } g} \sim (t - t_c)^{(2-2g)\frac{p+2}{p+1}} \mathcal{F}_{\text{DSL}}(p, 2)_g, \text{ with}$$

$$\mathcal{F}_{\text{DSL}}(p, 2)_g = F_g(S_{\text{DSL}}(p, 2)) = \text{"Liouville + (p, 2)"}$$

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Kontsevich integral and Double scaling limit

Kontsevich integral:

$$Z_{\text{Kontsevich}}(\Lambda) = \int dM e^{-N \text{Tr} \frac{M^3}{3} - M \Lambda^2} = e^{\sum N^{2-2g} \mathcal{F}_g}, \quad t_k = \frac{1}{N} \text{Tr} \Lambda^{-k}$$

$$\mathcal{F}_g = \sum_n \sum_{d_1, \dots, d_n} \prod_i \frac{t_{2i+1}^{d_i}}{d_i!} < \prod \tau_i^{d_i} >_g$$

Theorem 3: Loop equations imply $\mathcal{F}_g = F_g(S_{\text{Kontsevich}})$:

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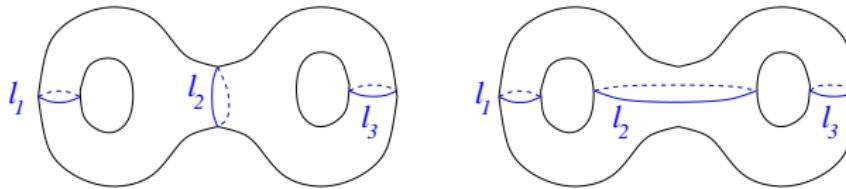
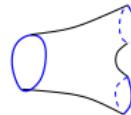
Weil-Petersson volumes

\mathcal{M}_g = moduli-space of Riemann surfaces of genus $g \geq 2$.

→ decompose into $2g - 2$ "pairs of pants"

→ $3g - 3$ geodesic lengths l_i = flat coordinates on \mathcal{M}_g .

$$\text{Vol}(\mathcal{M}_g) = \int_{\overline{\mathcal{M}}_g} \prod_i l_i dl_i.$$

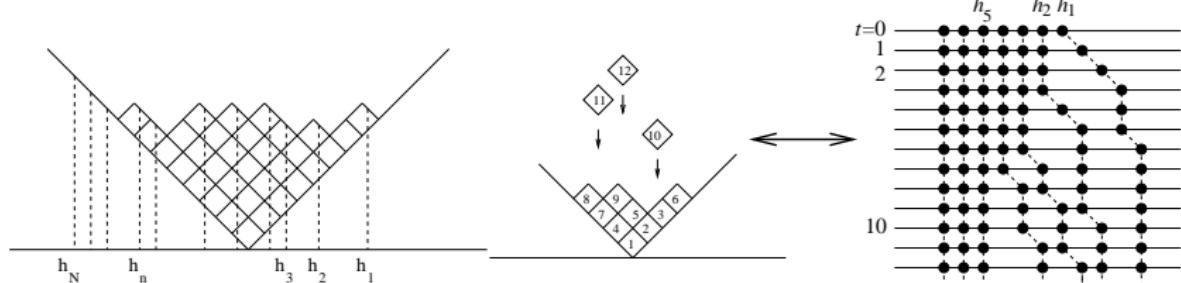


Theorem 4: $\text{Vol}(\mathcal{M}_g) = F_g(\mathcal{S})$:

Spectral curve: $\mathcal{S} = \begin{cases} x(z) = z^2 \\ y(z) = \frac{1}{2\pi} \sin(2\pi z) \end{cases}$

Remark: recursions for $W_n^{(g)}$ \Leftrightarrow Mirzakhani's recursions.

Application: partitions, Crystal growth, TASEP



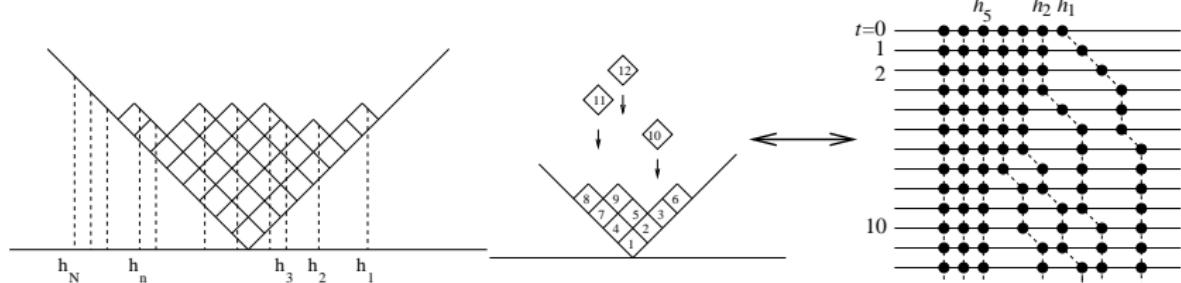
Plancherel Partitions \leftrightarrow Growing crystal \leftrightarrow T.A.S.E.P.

$$Z(Q, t_k) = \sum_{\lambda} Q^{2|\lambda|} \left(\frac{\dim(\lambda)}{|\lambda|!} \right)^2 e^{-Q \sum_k \frac{t_k}{k} Q^{-k} C_k(\lambda)}$$

$$C_k(\lambda) = k^{\text{th}} \text{ Casimir} = \sum_i h_i^k$$

$$\text{Question: } \ln Z = \sum_{g=0}^{\infty} Q^{2-2g} \mathcal{F}_{\text{Plancherel } g}(t_k)$$

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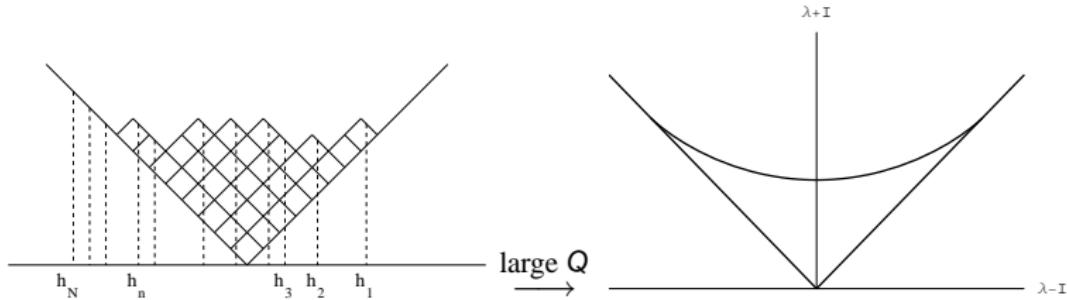
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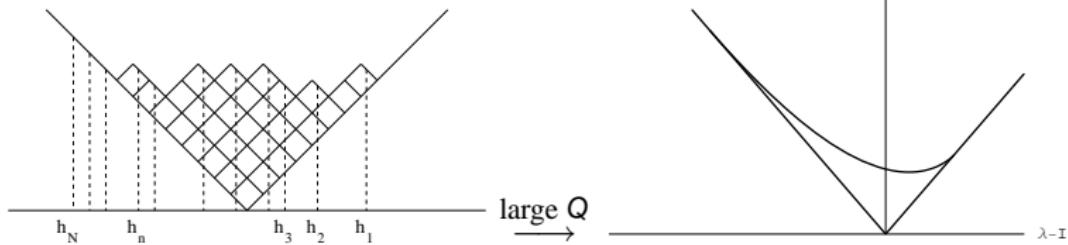
Theorem 5 : $\mathcal{F}_{\text{Plancherel } g}(t_k) = F_g(\mathcal{S}(t_k))$



Application: partitions, Crystal growth, TASEP

Spectral curve: $\mathcal{S}(t_k) = \begin{cases} x(z) = e^{-u_0} (z + z^{-1} - u_1) \\ y(z) = \ln(z) + \sum_k u_k (z^k - z^{-k}) \\ \sum_k t_{k+1} x(z)^k = \sum_k u_k (z^k + z^{-k}) \end{cases}$

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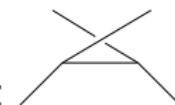


Gromov-Witten

$X = \text{Calabi-Yau}$ 3-fold. $N_{g,d}(X) = \text{Gromov-Witten invariants} =$
number of rational surfaces of genus g and degree d
embedded in X , passing through given points.

$$\mathcal{F}_g(X) = \sum_d e^{-t d} N_{g,d}(X)$$

Topological vertex method: $Z = \exp \left(\sum_g (\ln q)^{2g-2} \mathcal{F}_g \right) = \text{sum of plane partitions.}$



In particular choose Hirzebruch manifolds ($p \in \mathbb{Z}$):
 $X_p = \mathcal{O}(p) \oplus \mathcal{O}(2-p) \rightarrow \mathbb{P}^1$:

$$Z(X_p) = \sum_{\lambda} \left(\frac{\dim_q(\lambda)}{|\lambda|_q!} \right)^2 q^{\frac{p-1}{2} C_2(\lambda)} e^{-t|\lambda|}$$

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Theorem 6 bis: $\mathcal{F}_g = F_g(\mathcal{S})$ with:

$$\mathcal{S} = \begin{cases} x(z) = (1 - \frac{z}{z_0})(1 - \frac{1}{zz_0})/(1 - 1/z_0)^2 \\ y(z) = \frac{1}{x(z)} \ln \left[\frac{1}{z} \frac{(z-z_0)^p}{(z-1/z_0)^{2-p}} \right] \\ e^{-t} = \frac{1}{z_0^2} (1 - 1/z_0)^{p(p-2)} \end{cases}$$

$\mathcal{S} \sim \tilde{X}_p = \{H(e^x, e^y) = 0\} \leftarrow \text{mirror of } X_p$

General conjecture [BKMP]: $GW_g(X) \stackrel{?}{=} F_g(\Sigma_{\text{Mirror}(X)})$

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Main results:

- **Symplectic invariants** are very universal, they appear in many problems of enumerative geometry and integrable systems (more applications: coloured surfaces, CFT, ...).
- They are **easy to compute, easy to compare**.
- They give easily all **universal limits** (Sine, Tracy-Widom, CFT, KdV, KP...).
- they allow to **compute** the **Gromov-Witten invariants** (proved for Hirzebruch X_p , and Seiberg-Witten $SU(n)$ so far).

Further prospects:

- plane partitions, topological vertex, geometric engineering → Gromov-Witten invariants of more general toric CY.
- Boundary symplectic invariants...
- non-orientable surfaces, quantum deformation...

More applications

- Airy, Tracy Widom (1,2), $c = -2$

$$y = \sqrt{x}$$

- Liouville + Minimal model (p, q) , $c = 1 - 6(p - q)^2/pq$

$$x = \text{Pol}_q(z), y = \text{Pol}_p(z).$$

- Liouville + pure gravity (3, 2), $c = 0$

$$x = z^2 - 2, y = z^3 - 3z.$$

- Liouville + Ising (4, 3), $c = 1/2$

$$x = z^3 - 3z, y = z^4 - 4z^2 + 2.$$

- Liouville + Unitary models $(q + 1, q)$, $c = 1 - 6/q(q + 1)$

$$x = T_q(z), y = T_{q+1}(z) \text{ (Tchebychev polynomials).}$$

- Kontsevich integral, times $t_k = \frac{1}{N} \text{Tr } \Lambda^{-k}$

$$x(z) = z^2 + t_1, y(z) = z - \frac{1}{2} \sum_k t_{k+2} z^k,$$

- Weil-Petersson volumes

$$x(z) = z^2, y(z) = \frac{1}{2\pi} \sin(2\pi z)$$

- Seiberg-Witten

$$x(z) = \wp(z), y(z) = \wp'(z).$$