Triply Periodic Minimal Surfaces Bounded by Vertical Symmetry Planes

Shoichi Fujimori[1] Matthias Weber[2]

[1]Department of Mathematics
Fukuoka University of Education

[2]Department of Mathematics
Indiana University

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Introduction

A triply periodic minimal surface (TPMS)
$\Longleftrightarrow$
complete, embedded minimal surface in $\mathbb{R}^3$ which is invariant under translations in three independent directions (the three translations generate a lattice Λ in $\mathbb{R}^3$).

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TPMS can be considered as a minimally embedded surface from a compact Riemann surface of genus $g \geq 3$ into $\mathbb{R}^3/\Lambda$.

History

- (19th century) H. A. Schwarz, E. R. Neovius, 5 examples.

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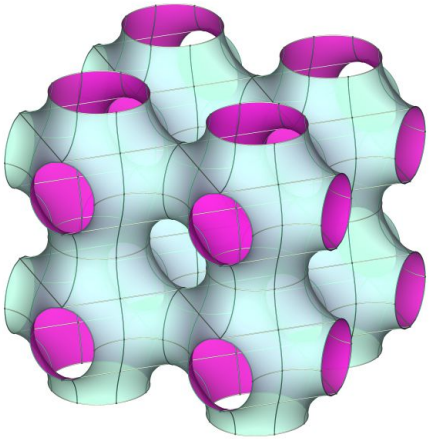
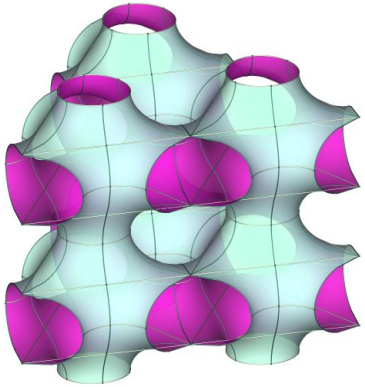
- (1989) H. Karcher, existence of Schoen's examples and further examples.
- (2008) M. Traizet, existence of TPMS for any genus $g \geq 3$.

Examples



Schwarz P



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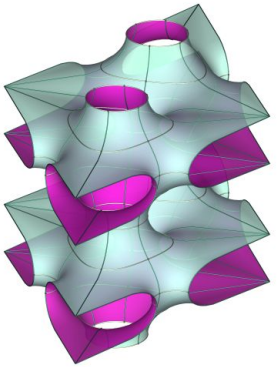
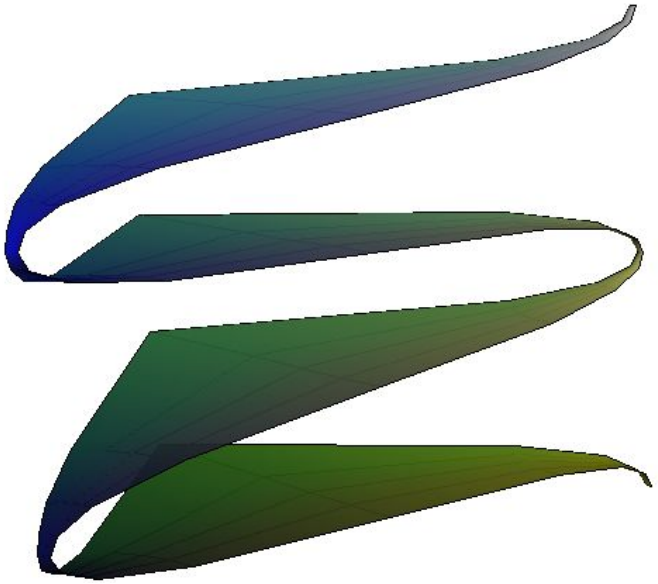


Schoen T'-R

Our construction method

1. Construct a minimal embedding defined on a parallel strip with the following properties:



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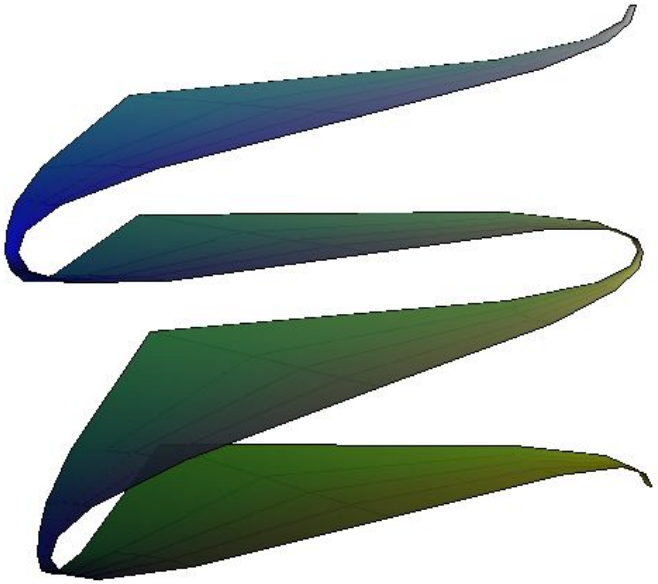
   - bounded by a vertical prism over a certain kind of triangle,
   - the boundary curves lie on the prism,
   - intersects to the prism orthogonally,
   - invariant under vartical translation.



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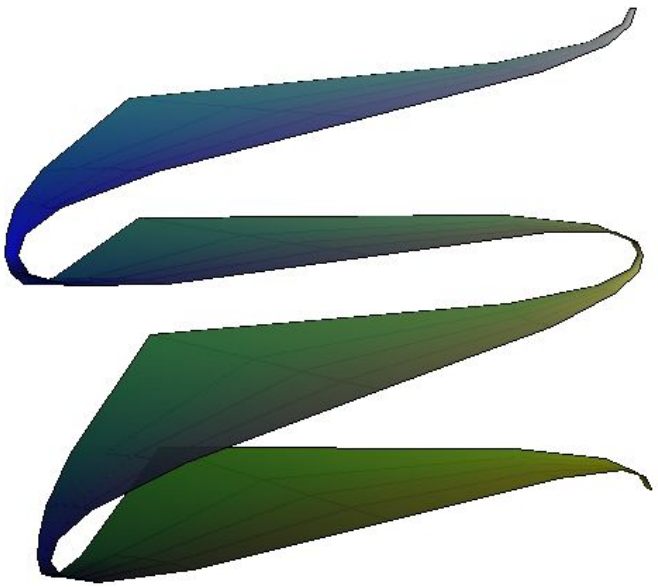
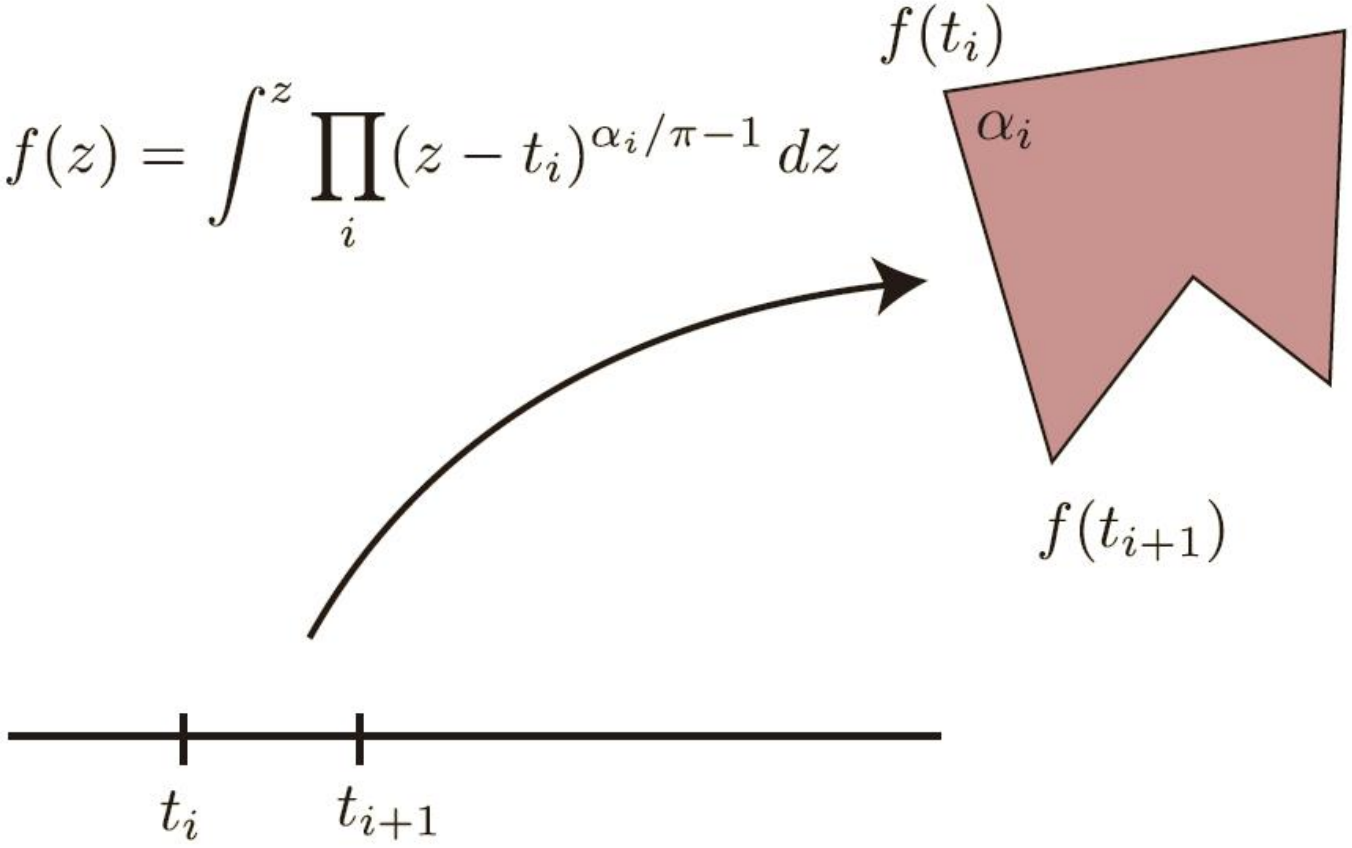
2. Repeat reflection with respect to a vertical plane of the prism.



The Classical Schwarz-Christoffel formula

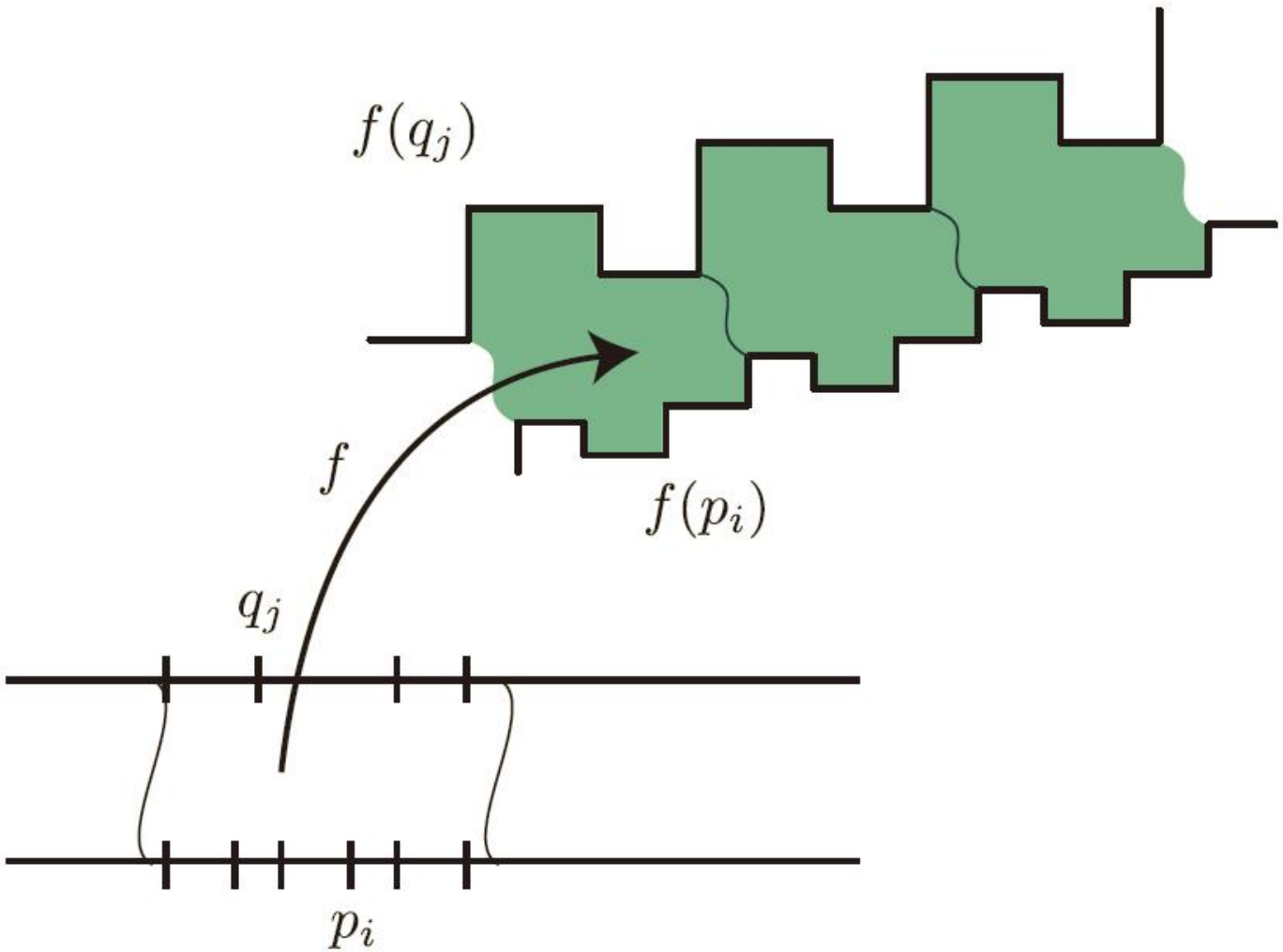
It gives the biholomorphic map from the upper half plane into a polygon.

$$f(z) = \int^z \prod_i (z - t_i)^{\alpha_i/\pi - 1}\, dz$$



An equivariant Schwarz-Christoffel formula

It gives the biholomorphic map from the parallel strip Z into a periodic polygon P.



An equivariant Schwarz-Christoffel formula

A periodic polygon P is a simply connected domain in the plane which is bounded by two infinite piecewise linear curves, and is invariant under a Euclidean translation $V(z) = z + v$ for some $v \neq 0$, and the quotient $P/\langle V \rangle$ is conformally an annulus.

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Let P be a periodic polygon, and define the strip $Z = \{z \in \mathbb{C}\,;\, 0 < \mathrm{Im} z < \tau/2\}$. Then $\exists 1\ \tau \in i\mathbb{R}^+$ s.t. $Z/\langle z \mapsto z+1 \rangle$ and $P/\langle V \rangle$ are conformally equivalent. Moreover, we obtain a biholomorphic map $f : Z \to P$ which is equivariant w.r.t. both translations:

$$f(z+1) = V(f(z)).$$

Proposition

Let P be a periodic polygon and Z the associated parallel strip. Then, up to scaling, rotating, and translating,

$$f(z) = \int^z \prod_{i=1}^{m} \vartheta(z - p_i)^{a_i} \cdot \prod_{j=1}^{n} \vartheta(z - q_j)^{b_j}$$

is the biholomorphic map from Z to P. Conversely, for any choices of $p_i \in \mathbb{R}$ $(1 \leq i \leq m)$, $q_j \in \mathbb{R} + \tau/2$ $(1 \leq j \leq n)$ and $-1 < a_i, b_j < 1$ satisfying the angle condition $\sum_{i=1}^{m} a_i = 0 = \sum_{j=1}^{n} b_j$, f maps Z to a periodic polygon.

Jacobi ϑ-function

Definition

$$\vartheta(z) = \vartheta(z,\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i (n+\frac{1}{2})^2 \tau + 2\pi i (n+\frac{1}{2})(z-\frac{1}{2})}$$

Fact

$$\begin{aligned} \vartheta(-z) &= -\vartheta(z), \\ \vartheta(z+1) &= -\vartheta(z), \\ \vartheta(z+\tau) &= -e^{-\pi i \tau - 2\pi i z}\vartheta(z). \end{aligned}$$

Proposition (Weierstrass representation)

$$Z \ni z \mapsto \mathrm{Re} \int^z \left(\frac{1}{G} - G,\ \frac{i}{G} + iG,\ 2 \right) dz$$

with

$$G(z) = \prod_{i=1}^{m} \vartheta(z - p_i)^{a_i} \cdot \prod_{j=1}^{n} \vartheta(z - q_j)^{b_j}$$

gives a simply connected minimal surface with two boundary components lying in a finite number of vertical symmetry planes. These planes meet at angles πa_i at the image of p_i and πb_j at the image of q_j. Furthermore, the surface is invariant under the vertical translation $x_3 \mapsto x_3 + 1$.

Two corners in one component

- $m = 2$ and $n = 0$.

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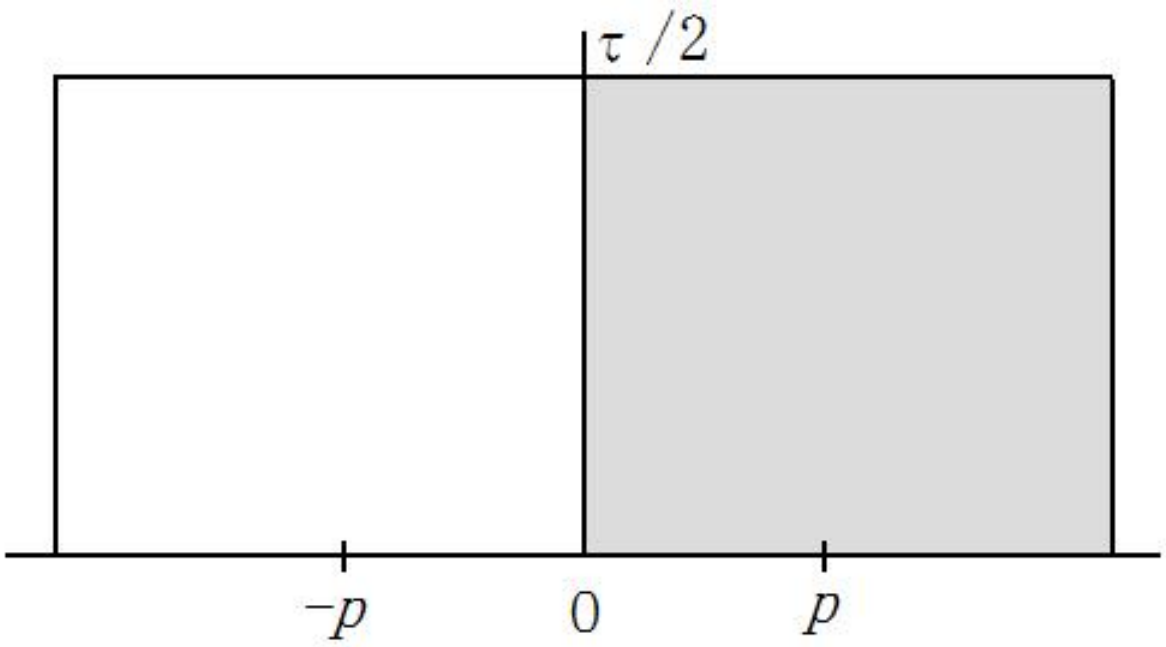
- The surface lies in vertical prism over a triangle.

Angles between symmetry planes

Proposition

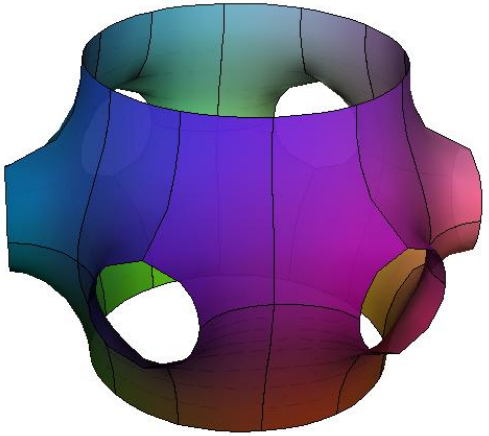
The angle α_0 between the symmetry planes correspond to the intervals $[0, p]$ and $[0, 1/2] + \tau/2$ is

$$\alpha_0 = \pi a(2p - 1).$$

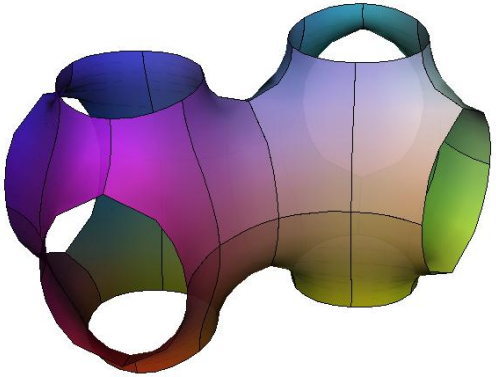


Basic case

Name	(r, s, t)	p	g
Schwarz P	(2,4,4)	1/4	3
Schoen H'-T	(2,6,3)	1/3	4
Schoen H'-T	(2,3,6)	1/6	4
Schwarz H	(3,3,3)	1/4	3
Schoen H"-R	(3,2,6)	1/8	5
Schoen H"-R	(3,6,2)	3/8	5
Schoen S'-S"	(4,4,2)	1/3	4
Schoen S'-S"	(4,2,4)	1/6	4
Schoen T'-R	(6,2,3)	1/5	6
Schoen T'-R	(6,3,2)	3/10	6



(2,3,6)



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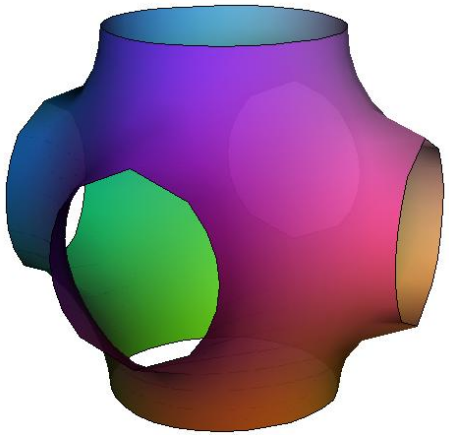

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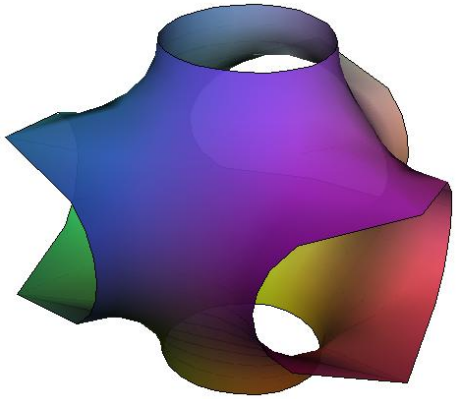
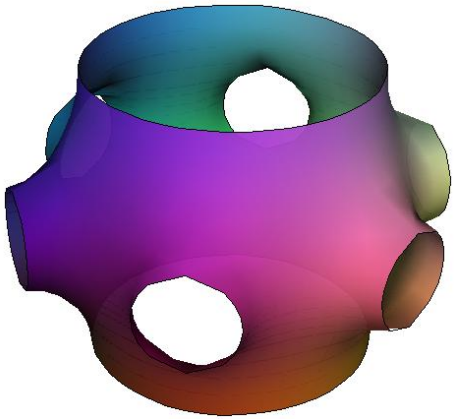


(3,3,3)



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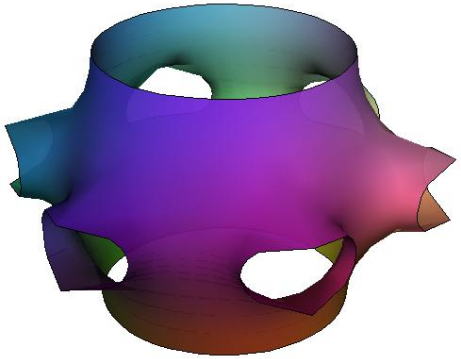

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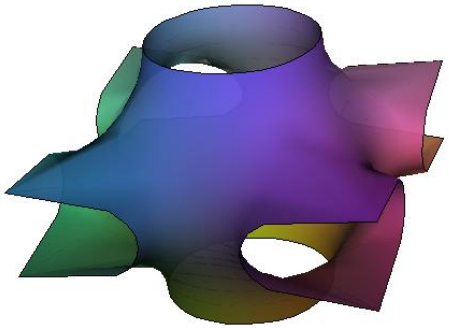
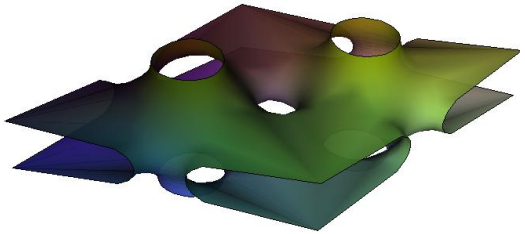
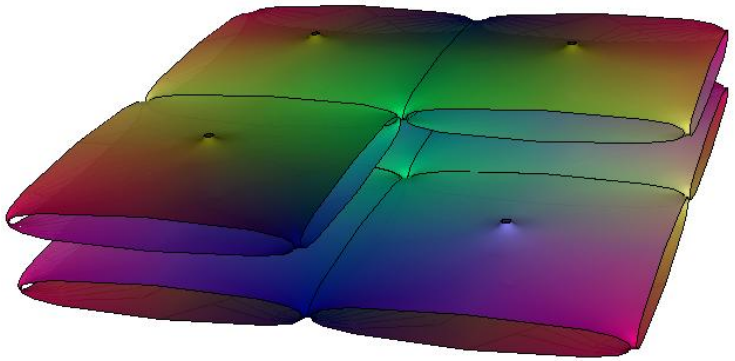


(4,2,4)

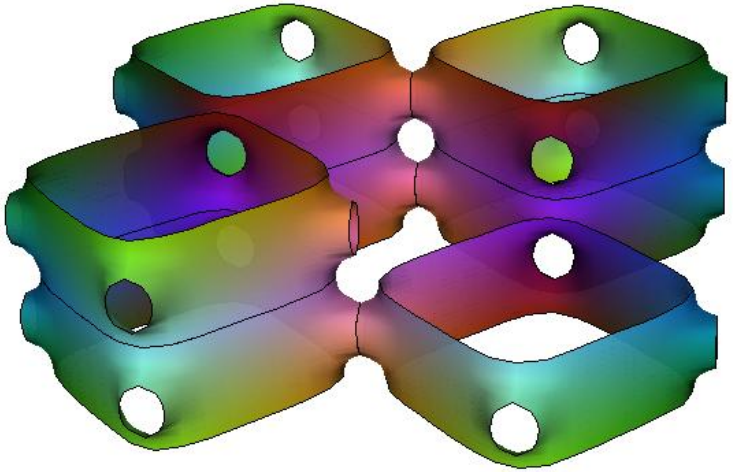


(6,2,3)

Limits



$\tau \to 0$



$\tau \to \infty$

(2,4,4)

Next complicated cases

- Periodic polygons with four corners.

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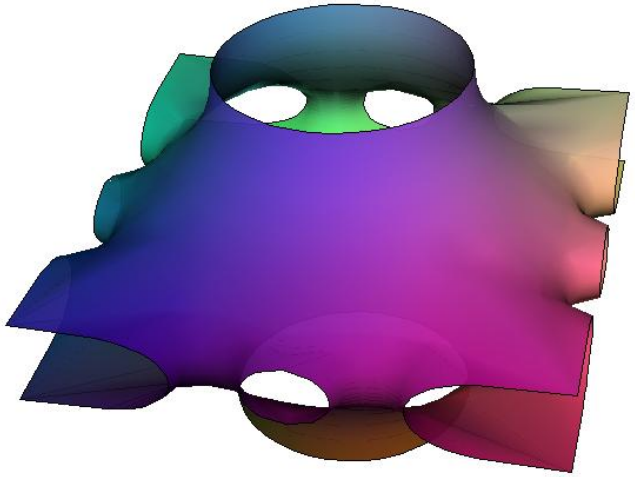
- If all corners lie in one boundary component, the period problem cannot be solved.

Four corners, case I



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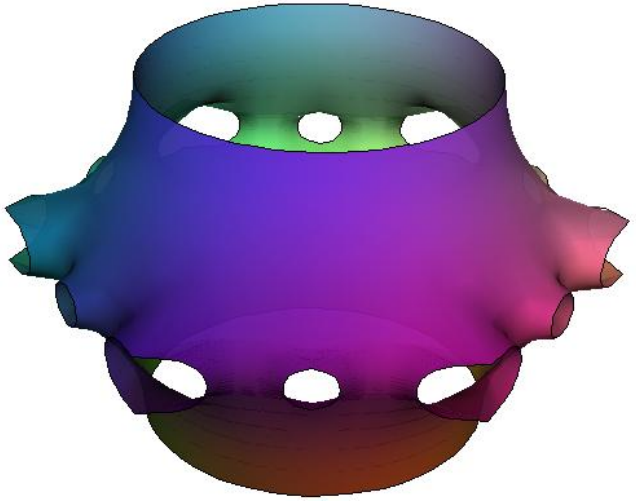

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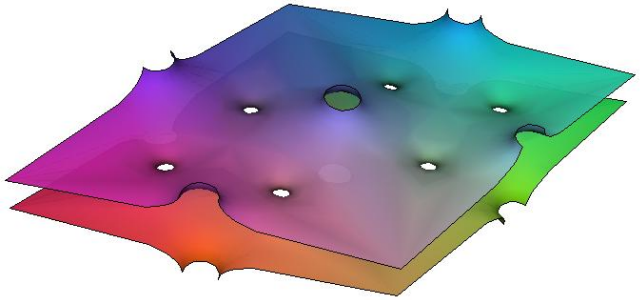

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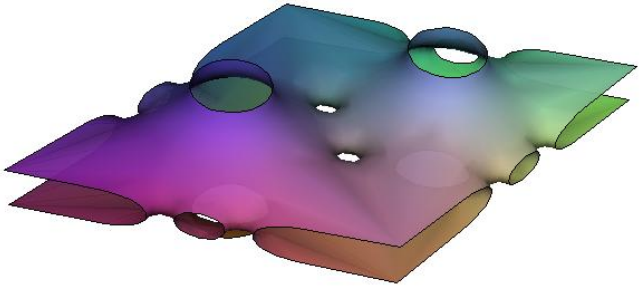

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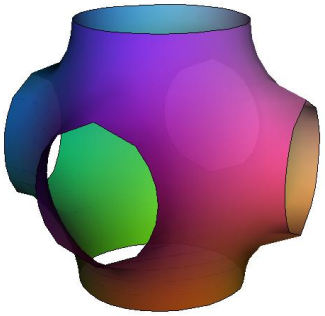
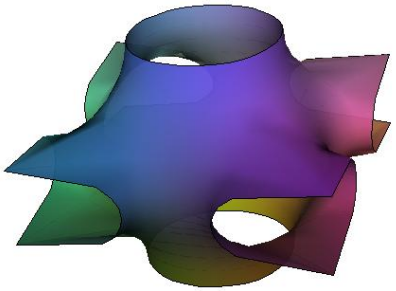
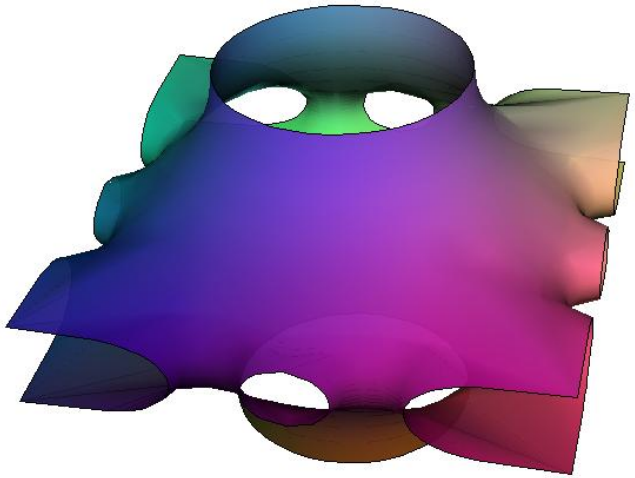


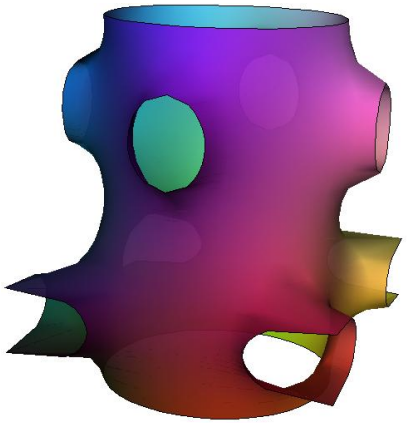


(2,4,4) and (4,2,4) surfaces in the basic case



(2,4,4) surface in this case

Four corners, case II



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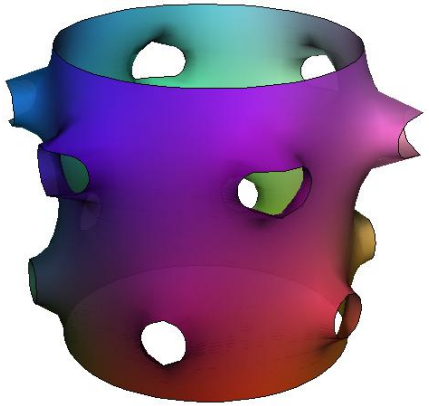

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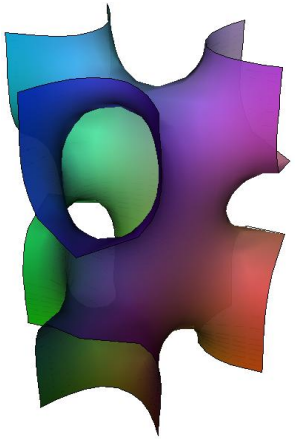

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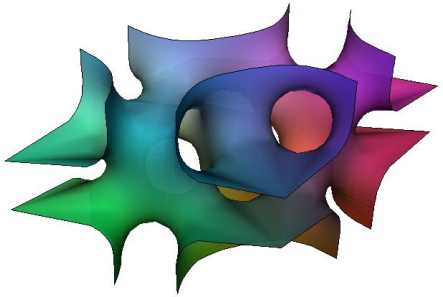

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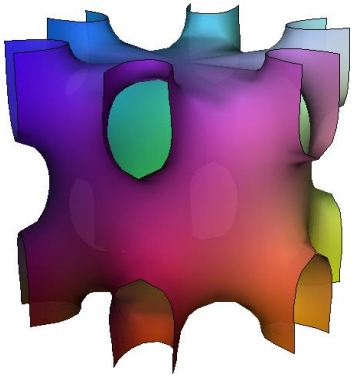
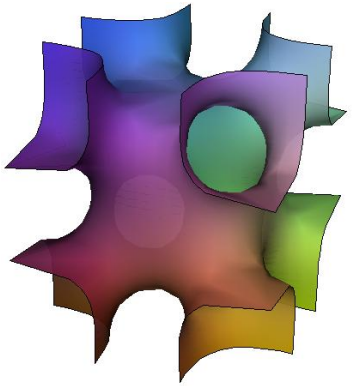


(6,2,3)



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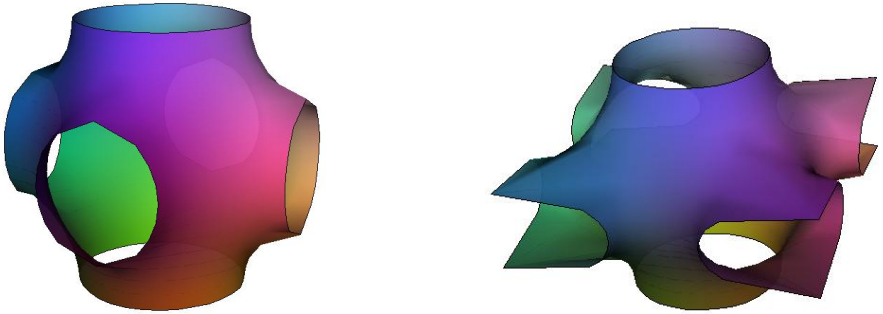
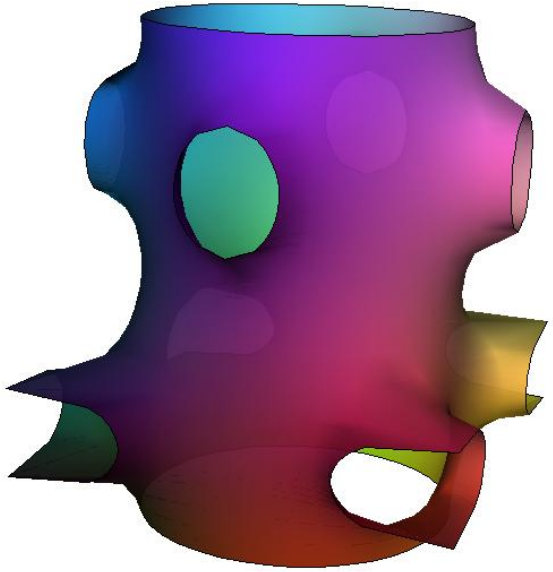


(2,4,4) and (4,2,4) surfaces in the basic case



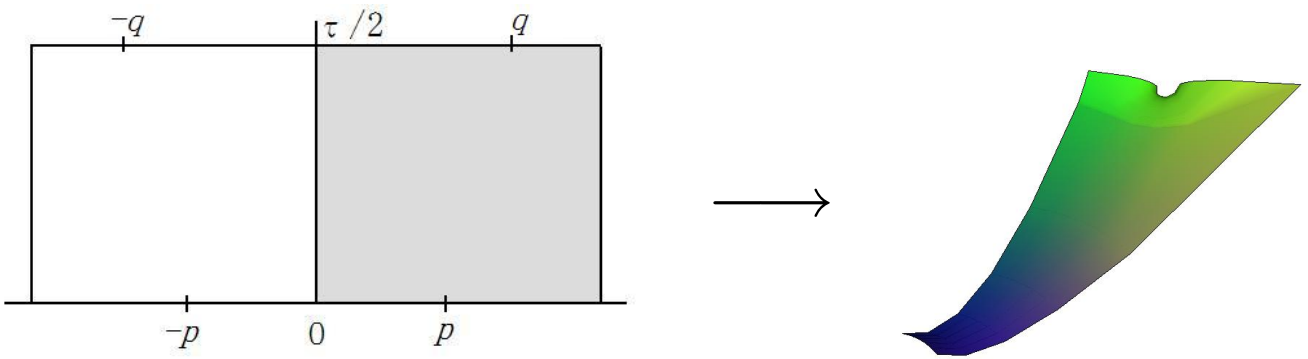
(2,4,4) surface in this case

Existence proof for case I

We asswume the pair of planes corresponds to the intervals $[0, p]$ and $[\tau/2, q]$ to be parallel. Then we have

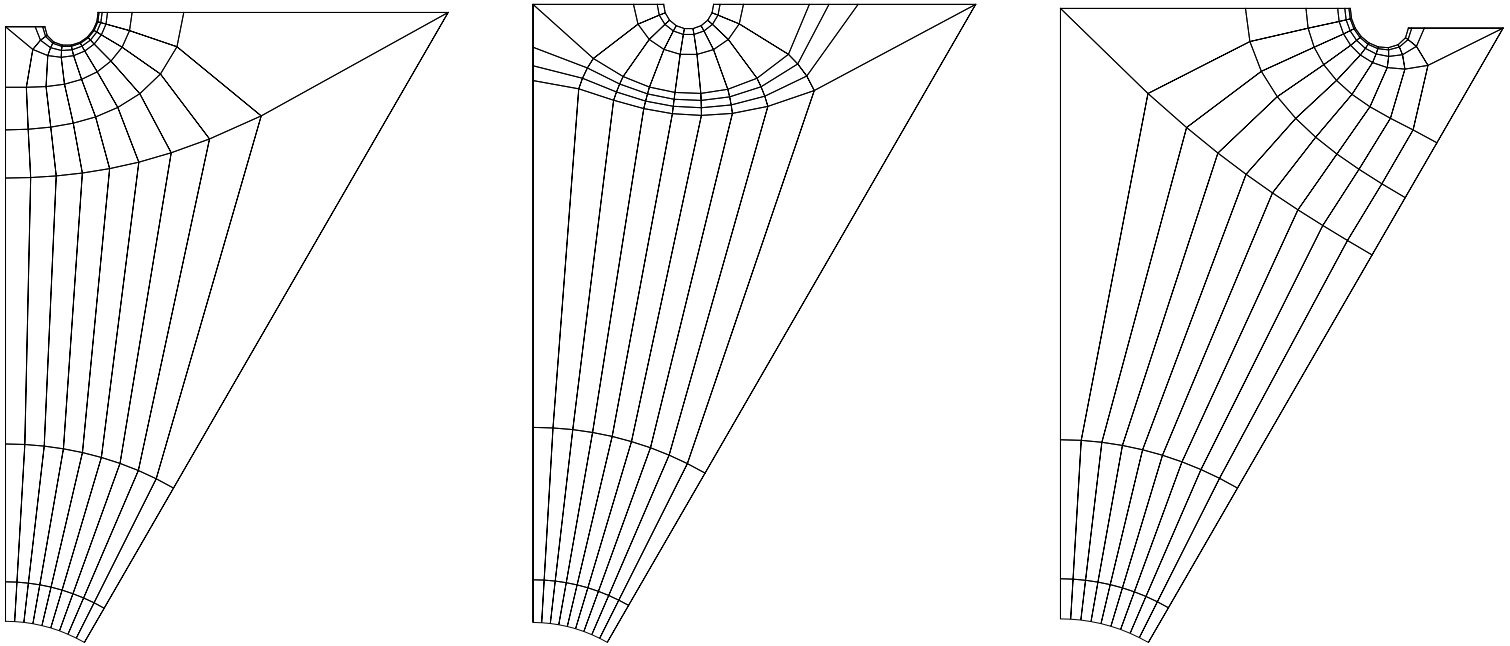
$$a(2p-1) + b(2\mathrm{Re}(q) - 1) = 1.$$

So the period problem reduces to 1-dimensional.



The fundamental piece of the (2,3,6) surface

We consider the projection $F(z)$ of the fundamental piece into the x_1x_2-plane. Then the period problem solved iff the image edges $F([0,p])$ and $F([\tau/2,q])$ are colinear.



$p \to 0$ solved $p \to 1/2$

The limit situations can be analyzed explicitly, so the solution follows from an intermediate value argument.

Next complicated cases

- Periodic polygons with six corners.

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- 2-dimensional period problem.

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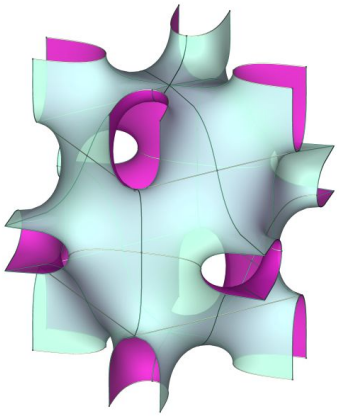
- 2-dimensional period problem.
- In some cases we have established numerically.

The Neovius family



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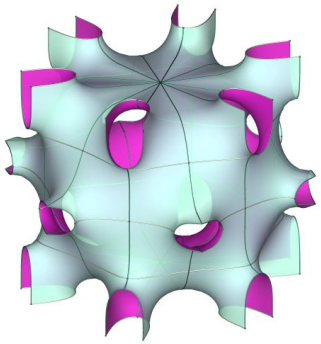
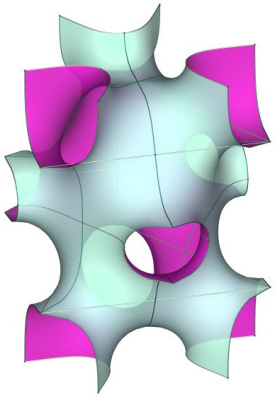
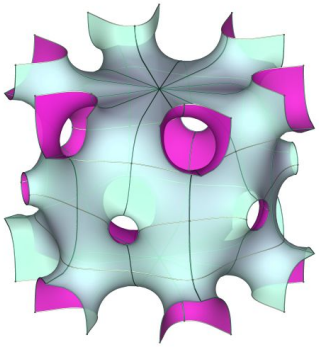


(2,3,6)



(3,3,3)



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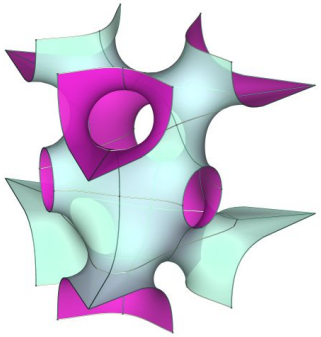

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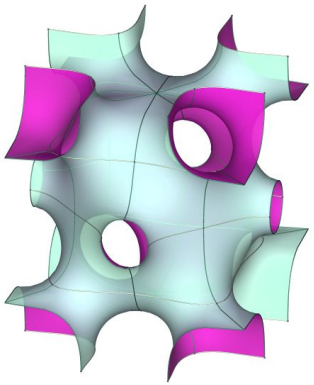

(4,2,4)