Normalization of differential equations associated to orbifold quantum cohomology

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Contents

Gromov-Witten Invariants and Problem

- Target : a closed symplectic manifold (or orbifold) X.
- Counting certain pseudo-hol. sphere in X (intuitively).
- Gromov-Witten invariant:

$$\Psi^X_{0,k,A}(x_1,\ldots,x_k) := \int_{[\overline{\mathcal{M}}_{0,k}(X,A)]^{\mathrm{vir}}} \mathbf{ev}_1^* x_1 \cup \cdots \cup \mathbf{ev}_k^* x_k$$

(genus-0, k-pointed, $A \in \mathrm{H}_2(X;\mathbb{Z})$)

Problem

How do we compute GW invariants systematically?

Quantum Cohomology

- Quantum cohomology : "$\circ$" is a product on

$$\Gamma^{\mathrm{hol}}\Big(\underbrace{\mathrm{H}^{\mathrm{even}}(X;\mathbb{C})}_{\text{fibre}} \times \mathrm{H}^2(X;\mathbb{C}) \longrightarrow \underbrace{\mathrm{H}^2(X;\mathbb{C})}_{\text{parameter space}}\Big)$$

- Quantum product "$\circ$":

$$\int_X (x \circ y) \cup z = \sum_{A \in \mathrm{Eff}_X} \underbrace{\Psi^X_{0,3,A}(x,y,z)}_{\text{GW inv.}} e^{\langle \tau, A\rangle}$$

for all z. ($\tau \in H^2(X;\mathbb{C})$: the parameter)

- Eff_X : The effective cone (semigroup)

$$\Big\{ A \in \mathrm{H}_2(X;\mathbb{Z}) \,\Big|\, \exists \text{ pseudo-holomorphic curve in class } A \Big\}$$

Quantum Parameters (1/2)

Technical assumptions (please ignore):

- monotonicity : $[\omega] = \lambda c_1(TX)$ for some $\lambda > 0$.
- $A_1, \ldots, A_r$: a basis of $\mathrm{H}_2(X; \mathbb{Z})$ such that
  $\forall A \in \mathrm{Eff}_X$, $A = d_1 A_1 + \cdots + d_r A_r$ for some nonnegative $d_1, \ldots, d_r$.
- $p_1, \ldots, p_r$: the dual basis to $A_1, \ldots, A_r$

- $p_1, \ldots, p_r$: a "good" basis of $\mathrm{H}^2(X; \mathbb{C})$.
- $\tau = t_1 p_1 + \cdots + t_r p_r$. ($t_1, \ldots, t_r$: the coord. sys.)
- $q_i := e^{t_i}$: the quantum parameter
- "$\circ$" is also a product on

$$\mathrm{QH}^*(X) := \mathrm{H}^{\mathrm{even}}(X; \mathbb{C}) \otimes \mathbb{C}[q_1, \ldots, q_r].$$

Quantum Parameters (2/2)

Example: $X = \mathbb{C}\mathrm{P}^1$

- p : the Poincare dual to $[\mathbb{C}\mathrm{P}^1]$
- $\mathrm{H}^{\mathrm{even}}(\mathbb{C}\mathrm{P}^2;\mathbb{C}) \cong \mathbb{C}[p]/(p^2)$:

$$1 \cup 1 = 1, \quad 1 \cup p = p, \quad p \cup p = 0.$$

- $\mathrm{QH}^*(\mathbb{C}\mathrm{P}^2) \cong \mathbb{C}[p]/(p^2 - q) \quad (q = q_1)$

$$1 \circ 1 = 1, \quad 1 \circ p = p, \quad p \circ p = q.$$

- $x \circ y\Big|_{q=0} = x \cup y$

D-module Structure on Quantum Cohomology

- $\hbar$: a parameter of $\mathbb{C}^\times$
- $$\mathcal{D}^\hbar := \left\{ \sum_\alpha f_\alpha (\hbar\partial)^\alpha \;\middle|\; f_\alpha \in \mathbb{C}[\hbar, q_1, \ldots, q_r] \right\}$$
  $$\left(\partial_i := \frac{\partial}{\partial t_i} = q_i \frac{\partial}{\partial q_i},\ (\hbar\partial)^\alpha = (\hbar\partial_1)^{\alpha_1} \cdots (\hbar\partial_r)^{\alpha_r} \right)$$
- $\mathcal{D}^\hbar \curvearrowright \mathrm{QH}^*(X)$; $\quad \hbar\partial_i \cdot \xi := \hbar \dfrac{\partial \xi}{\partial t_i} + p_i \circ \xi$
  ($\xi \in \mathrm{QH}^*(X)$: cohomology-valued function in q)

Quantum Differential Equation (1/2)

Assumption (Important!)

$\mathrm{H}^*(X;\mathbb{C})$ is generated by $\mathrm{H}^2(X;\mathbb{C})$.

- $1 \in \mathrm{H}^0(X;\mathbb{C})$ is a cyclic generator: $\mathcal{D}^\hbar \cdot 1 = \mathrm{QH}^*(X)$.
- $\mathrm{QH}^*(X) \cong \mathcal{D}^\hbar / \mathcal{J}$ as $\mathcal{D}^\hbar$-module.

  ($\mathcal{J}$: the annihilator of 1)
- $\mathcal{J} = (P_1, \ldots, P_k)$
- The quantum differential equation :

$$P_1 u = 0, \cdots, P_k u = 0 \qquad \text{(a system of PDEs)}$$

Quantum Differential Equation (2/2)

There are many spaces s.t. the quantum differential equation is explicitly obtained WITHOUT GW invariants.

- **Flag manifolds (Kim)**
- **Symplectic toric manifolds (Givental)**

Problem

Can we deduce GW invariants from quantum diff. eqn.?

Recovering Gromov-Witten Invariants

M. Guest suggested using Birkhoff factorization.

It is true for the followings:

- Amarzaya-Guest : full flag manifolds
- Iritani : certain toric complete intersections
  (making use of fundamental solution)
- S : Fano complete intersections in $\mathbb{C}P^n$
  monotone symplectic toric manifolds
  (making use of grading of quantum D-module)

Where is the Answer?

- We work on $\mathcal{D}^{\hbar}/\mathcal{J}$ (e.g. $\mathcal{D}^{\hbar}/\big((\hbar\partial)^2 - q\big)$ for $\mathbb{C}\mathrm{P}^1$).
- QH is a space of sections of "$\mathrm{H}^{\mathrm{even}} \times \mathrm{H}^2 \longrightarrow \mathrm{H}^2$".
- $\mathrm{H}^{\mathrm{even}} =$ the space of constant sections

Problem

Find a hidden basis of $\mathrm{H}^{\mathrm{even}}$ behind $\mathcal{D}^{\hbar}/\mathcal{J}$.

Normalized Trivialization

- $\xi_1, \ldots, \xi_N$: frame ($\mathbb{C}[\hbar, q_1, \ldots, q_r]$-basis)
- Define the connection 1-form (w.r.t. the frame):

$$\Omega := \sum_{i=1}^{r} \frac{1}{\hbar} \Omega_i dt_i \qquad \left(\hbar \partial_i \cdot \xi_j = \sum_{k=1}^{N} (\Omega_i)_j^k \xi_k \right).$$

- $\xi_1, \ldots, \xi_N$ is a normalized trivialization

$$\overset{\text{def}}{\Longleftrightarrow} \quad \Omega_1, \ldots, \Omega_r \text{ are } \hbar\text{-indep.}$$

- $\xi_1, \ldots, \xi_N$ is a basis of $\mathrm{H}^{\mathrm{even}}$

$$\Longrightarrow \quad \xi_1, \ldots, \xi_N \text{ is a normalized trivialization.}$$

Birkhoff Factorization

- $\Omega := \sum_{i=1}^{r} \frac{1}{\hbar} \Omega_i dt_i$ w.r.t. a frame $\xi_1, \ldots, \xi_N$.
- $\exists L(\hbar, q)$ s.t. $\Omega = L^{-1} dL$. (L contains $\hbar$ and $\hbar^{-1}$.)
- $L = L_- L_+$: the Birkhoff factorization
  - $L_- = I + \frac{1}{\hbar} A_1 + O\left(\hbar^{-2}\right) \in \Lambda_-^1 \mathrm{GL}_N \mathbb{C}$
  - $L_+ = B_0 + \hbar B_1 + O(\hbar^2) \in \Lambda_+ \mathrm{GL}_N \mathbb{C}$
- $(\hat{\xi}_1, \ldots, \hat{\xi}_N) := (\xi_1, \ldots, \xi_N) L_+^{-1}$: a new frame
- The connection 1-form $\hat{\Omega}$ w.r.t. the new frame is a normalized trivialization.

Uniqueness of Normalized Trivialization

Theorem (Amarzaya-Guest, Iritani, S)

A normalized trivialization is unique in some sense.

$\Longrightarrow$ **Birkhoff factorization gives a frame which we want.**

Key observation:

The change of frames belongs to $\Lambda_{+}\mathrm{GL}_N\mathbb{C}$.

Gromov-Witten invariant for orbifolds

- **Chen and Ruan :**

  GW invariants for closed symplectic orbifolds

- **Coates, Corti, Lee and Tseng :**

  the quantum diff. eqn for weighted projective spaces

- **Extend our method to weighted projective spaces.**

Chen-Ruan cohomology (1/2)

- $\mathrm{QH}^*(\mathcal{X})$ is defined on the Chen-Ruan cohomology.
- $\wedge\mathcal{X} = \mathcal{X} \sqcup (\text{twisted sectors})$: the inertia orbifold

Chen-Ruan cohomology

$$\begin{aligned} \mathrm{H}^*_{\mathrm{CR}}(\mathcal{X}) &:= \mathrm{H}^*(\wedge\mathcal{X};\mathbb{C}) \\ &= \mathrm{H}^*(\mathcal{X};\mathbb{C}) \oplus \underbrace{\mathrm{H}^*(\text{twisted sectors};\mathbb{C})}_{\text{coming from singularity}} \end{aligned}$$

Chen-Ruan cohomology (2/2)

- $\mathrm{H}^*_{\mathrm{CR}}(\mathcal{X}) = \mathrm{H}^*(\mathcal{X};\mathbb{C}) \oplus \mathrm{H}^*(\text{twisted sectors};\mathbb{C})$.
- Chen-Ruan cup product
- Quantum product is a quantization of the CR product.

- $\mathrm{H}^*(\mathcal{X};\mathbb{C})$ is closed under CR cup product.

Proposition

$\mathrm{H}^2(\mathcal{X};\mathbb{C})$ NEVER generates $\mathrm{H}^*_{\mathrm{CR}}(\mathcal{X})$.

Weighted Projective Spaces

- $S^1 \curvearrowright S^{2n+1}$;

$$t \cdot (z_0, \ldots, z_n) = (t^{-w_0} z_0, \ldots, t^{-w_n} z_n)$$

$$(S^1 \subset \mathbb{C},\ S^{2n+1} \subset \mathbb{C}^{n+1},\ w_0, \ldots, w_n \in \mathbb{Z}_{>0})$$

- $\mathbb{P}(w) = \mathbb{P}(w_0, \ldots, w_n) := S^{2n+1}/S^1$

  : the weighted projective space (quotient orbifold)

Quantum Cohomology of $\mathbb{P}(w)$ (1/2)

Example: $\mathbb{P}(1,3)$

- $\wedge\mathbb{P}(1,3) = \mathbb{P}(1,3) \sqcup \underbrace{\mathbb{P}(3) \sqcup \mathbb{P}(3)}_{\text{twisted sectors}}$
- $\mathrm{H}^*_{\mathrm{CR}}(\mathbb{P}(1,3)) = \mathrm{Span}_{\mathbb{C}}\{\underbrace{1,\ p_1,}_{\text{original}}\ \underbrace{p_2,\ p_3}_{\text{twisted}}\}$

  $\left(p_1 := c_1(\mathcal{O}(1)) \in \mathrm{H}^2(\mathbb{P}(1,3);\mathbb{C})\right)$
- $\mathrm{QH}^*(\mathbb{P}(1,3)) := \mathrm{H}^*_{\mathrm{CR}}(\mathcal{X}) \otimes \mathbb{C}[q^{\frac{1}{3}}]$
- $\exp(\langle t_1 p_1, dA\rangle) = \exp(\frac{1}{3} t_1 d) = q^{\frac{1}{3}d}$

  (A is the generator of Eff)

Quantum Cohomology of $\mathbb{P}(w)$ (2/2)

- Continue

$$\mathrm{QH}^*(\mathbb{P}(1,3)) = \mathrm{Span}_{\mathbb{C}}\{1,\ p_1,\ p_2,\ p_3\} \otimes \mathbb{C}[q^{\frac{1}{3}}].$$

- $$p_1 \circ 1 = p_1,$$

$$p_1 \circ p_1 = 3C_{113} q^{\frac{1}{3}} p_2,$$

$$p_1 \circ p_2 = 3C_{122} q^{\frac{1}{3}} p_3,$$

$$p_1 \circ p_3 = 3C_{131} q^{\frac{1}{3}} 1. \qquad \left(C_{ijk} := \Psi_{0,3,1}^{\mathbb{P}(1,3)}(p_i, p_j, p_k)\right)$$

Proposition

If we permit taking $q^{-\frac{1}{3}}$, then

$\mathrm{H}^2(\mathcal{X};\mathbb{C})$ generates $\mathrm{H}^*_{\mathrm{CR}}(\mathcal{X})$ under quantum product.

Quantum Differential Equation of $\mathbb{P}(w)$

Theorem (Coates-Corti-Lee-Tseng)

In quantum D-module of $\mathbb{P}(w)$, the annihilator of 1 is generated by

$$T = \prod_{i=0}^{n} \prod_{\mu=0}^{w_i - 1} (w_i \hbar \partial - \mu h) - q.$$

Example: $\mathbb{P}(1,3)$

- $T = (\hbar\partial)(3\hbar\partial)(3\hbar\partial - \hbar)(3\hbar\partial - 2\hbar) - q$

Main Result

Theorem (Guest-S)

The Birkhoff factorization gives a basis of $\mathrm{H}^*_{\mathrm{CR}}(\mathbb{P}(w))$.

- **Key observation:**

  A change of frames which we want belongs to $\Lambda_+ \mathrm{GL}_N \mathbb{C}$.

Hypersurface in Weighted Projective Space (WIP)

- Can we extend our method to hypersurfaces in $\mathbb{P}(w)$?.
- A change of frames which we want does NOT belong to $\Lambda_+ \mathrm{GL}_N \mathbb{C}$.
- Thus we need the general Birkhoff factorization

$$L = L_- \gamma L_+ \quad \text{where} \quad \gamma = \mathrm{diag}(\hbar^{a_1}, \ldots, \hbar^{a_N})$$

That's all

Thank you for listening!