Intersection Numbers On Moduli Spaces of Curves

Hao Xu

(Talk at The 16th Osaka City University International Academic Symposium 2008 "Riemann Surfaces, Harmonic Maps and Visualization")

Center of Mathematical Sciences, Zhejiang University

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Outline of my talk

- First we introduce the basics of moduli spaces of curves and its intersection theory. We study the integrals of ψ classes in some detail.

- We survey the recursion formulas of various Hodge integrals: Faber's algorithm, intersections of ψ classes, higher Weil-Petersson volumes, Witten's r-spin numbers. We will introduce our work on n-point functions and a proof of the Faber intersection number conjecture. Our work on recursion formulae of higher Weil-Petersson volumes is motivated by the work of Mulase and Safnuk.

- At last, we study the universal relations in Gromov-Witten theory.

Riemann surfaces

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(Apply the Riemann-Hurwitz formula to the morphism from $\mathbb{P}^2$ to $\mathbb{P}^1$, sending (x, y, z) to (x, z).)

Stable curves

A stable curve is a connected and compact curve with nodes

$$\{(x, y) \in \mathbb{C}^2 \mid xy = 0\}$$

and satisfy (i) each genus 0 component has at least three node-branches; (ii) each genus 1 component has at least one node-branch.

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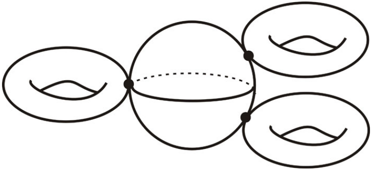
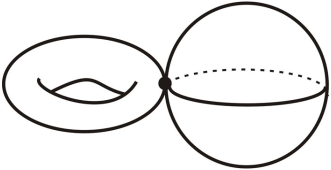


stable curve



unstable curve

Fine moduli spaces of curves deos not exist

Consider contravariant functors

$$Scheme \longrightarrow Sets$$

$$\begin{aligned} Hom(\bullet, S): &\qquad B \longrightarrow Hom(B,S) \ \ \text{(S is a scheme)} \\ F_{\mathcal{M}_g}: &\qquad B \longrightarrow C/B \ \ \text{(family of curves over } B\text{)} \end{aligned}$$

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Fine moduli space of curves over $\mathbb{C}$ fail to exist due to automorphisms of curves (Riemann surfaces).

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Compactification of moduli spaces of curves

Constructions of $\mathcal{M}_g$ and $\overline{\mathcal{M}}_g$

- The quotient of Teichmüller spaces by the action of the mapping class group.
- Geometric invariant theory (Mumford, Gieseker, etc).

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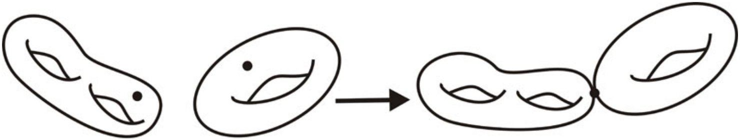
The boundary $\partial\mathcal{M}_g$ indicates that we need to add marked points to curves.

Moduli spaces of curves with marked points

There are two type of boundary morphisms:

$$\overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \longrightarrow \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2}$$



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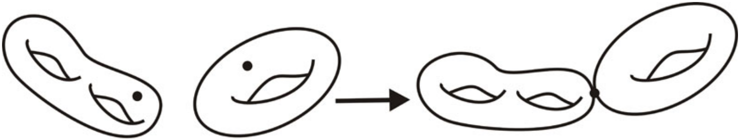

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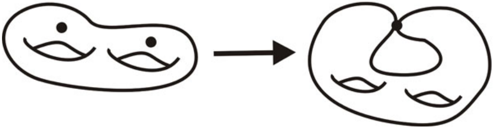
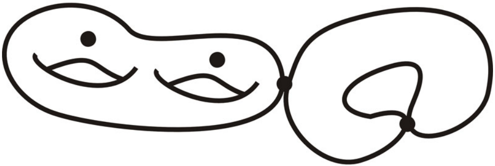


Dual graph of nodal curves

The following is a nodal curve of genus 3 with two components and two marked points.



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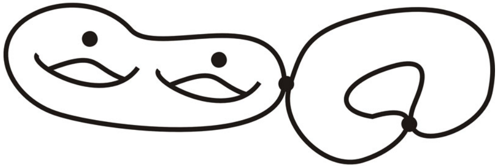

Its associated dual graph is:

Mumford defined the Chow ring $A^*(\overline{\mathcal{M}}_{g,n})$ on moduli spaces of curves in 1983. He emphasized the tautological subring $R^*(\overline{\mathcal{M}}_{g,n}) \subset A^*(\overline{\mathcal{M}}_{g,n})$, which contains all geometrically natural classes.

Intersection Theory on Moduli Spaces of Curves

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Some tautological classes on $\overline{\mathcal{M}}_{g,n}$

- ψ_i the first Chern class of the line bundle whose fiber over each pointed stable curve is the cotangent line at the *i*th marked point.
- $\lambda_i = c_i(\mathbb{E})$ the *i*th Chern class of the Hodge bundle $\mathbb{E}$ (with fibre $H^0(C, \omega_C)$).
- κ classes originally defined by Miller-Morita-Mumford on $\overline{\mathcal{M}}_g$ and generalized to $\overline{\mathcal{M}}_{g,n}$ by Arbarello-Cornalba.

Hodge integrals

We call the following integrals the Hodge integrals on moduli spaces of curves

$$\langle \tau_{d_1} \cdots \tau_{d_n} \kappa_{a_1} \cdots \kappa_{a_m} \mid \lambda_1^{k_1} \cdots \lambda_g^{k_g} \rangle \triangleq \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \kappa_{a_1} \cdots \kappa_{a_m} \lambda_1^{k_1} \cdots \lambda_g^{k_g}.$$

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The following remarkable ELSV formula of Ekedahl, Lando, Shapiro, and Vainshtein relates Hodge integrals to single Hurwitz numbers.

Let μ be a partition, $n = l(\mu)$ and $r = 2g - 2 + |\mu| + n$. Then

$$H_{g,\mu} = r! \prod_{i=1}^{n} \left(\frac{\mu_i^{\mu_i}}{\mu_i!} \right) \int_{\overline{\mathcal{M}}_{g,n}} \frac{1 - \lambda_1 + \cdots + (-1)^g \lambda_g}{(1 - \mu_1 \psi_1) \cdots (1 - \mu_n \psi_n)},$$

Hurwitz numbers

Let $\pi : X_g \to \mathbb{P}^1$ be a ramified cover over the sphere $\mathbb{P}^1$ (or meromorphic function on X_g) whose only degenerate ramification point is over ∞ with ramification type μ. So we have $\deg \pi = |\mu|$. By the Riemann-Hurwitz formula, the number of simple ramification points of π over $\mathbb{P}^1$ is:

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The single Hurwitz number $H_{g,\mu}$ is a weighted count of the distinct covers π of genus g with ramification μ over ∞ and simple ramification over r (arbitrary chosen) fixed points on $\mathbb{P}^1$. Each such cover is weighted by $1/|\mathrm{Aut}(\pi)|$.

Faber's algorithm

Faber's algorithm reduces the calculation of general Hodge integrals to those with pure ψ classes, based on Mumford's Chern character formula

$$\mathrm{ch}_{2m-1}(\mathbb{E}) = \frac{B_{2m}}{(2m)!}\left[\kappa_{2m-1} - \sum_{i=1}^{n}\psi_i^{2m-1} + \frac{1}{2}\sum_{\xi\in\Delta} l_{\xi*}\left(\sum_{i=0}^{2m-2}\psi_1^i(-\psi_2)^{2m-2-i}\right)\right]$$

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Let $\pi_{n+1} : \overline{\mathcal{M}}_{g,n+1} \longrightarrow \overline{\mathcal{M}}_{g,n}$. From push-forward formula

$$\pi_{n+1*}\psi_{n+1}^{a_1+1}\psi_1^{d_1}\cdots\psi_n^{d_n} = \kappa_{a_1}\psi_1^{d_1}\cdots\psi_n^{d_n},$$

we have

$$\int_{\overline{\mathcal{M}}_{g,n}} \kappa_{a_1}\psi_1^{d_1}\cdots\psi_n^{d_n} = \int_{\overline{\mathcal{M}}_{g,n+1}} \psi_{n+1}^{a_1+1}\psi_1^{d_1}\cdots\psi_n^{d_n}.$$

Intersection of pure ψ classes

The following integrals are called intersection indices or descendant integrals.

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n},$$

where $d_1 + \cdots + d_n = 3g - 3 + n$.

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The string equation

$$\langle \tau_0 \prod_{i=1}^{n} \tau_{k_i} \rangle_g = \sum_{j=1}^{n} \langle \tau_{k_j - 1} \prod_{i \neq j} \tau_{k_i} \rangle_g$$

The dilaton equation

$$\langle \tau_1 \prod_{i=1}^{n} \tau_{k_i} \rangle_g = (2g - 2 + n) \langle \prod_{i=1}^{n} \tau_{k_i} \rangle_g$$

KdV hierarchy

The KdV hierarchy is the following hierarchy of differential equations for $n \geq 1$,

$$\frac{\partial U}{\partial t_n} = \frac{\partial}{\partial t_0} R_{n+1},$$

where R_n are Gelfand-Dikii differential polynomials in $U, \partial U/\partial t_0, \partial^2 U/\partial t_0^2, \ldots,$ defined recursively by

$$R_1 = U, \qquad \frac{\partial R_{n+1}}{\partial t_0} = \frac{1}{2n+1}\left(\frac{\partial U}{\partial t_0} R_n + 2U \frac{\partial R_n}{\partial t_0} + \frac{1}{4}\frac{\partial^3}{\partial t_0^3} R_n\right).$$

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It is easy to compute R_n recursively

$$R_2 = \frac{1}{2}U^2 + \frac{1}{12}\frac{\partial^2 U}{\partial t_0^2},$$

$$R_3 = \frac{1}{6}U^3 + \frac{U}{12}\frac{\partial^3 U}{\partial t_0^3} + \frac{1}{24}(\frac{\partial U}{\partial t_0})^2 + \frac{1}{240}\frac{\partial^4 U}{\partial t_0^4},$$

$$\vdots$$

Witten's conjecture

The Witten-Kontsevich theorem states that the generating function for ψ class intersection numbers

$$F(t_0, t_1, \ldots) = \sum_g \sum_{\mathbf{n}} \langle \prod_{i=0}^{\infty} \tau_i^{n_i} \rangle_g \prod_{i=0}^{\infty} \frac{t_i^{n_i}}{n_i!}$$

is a τ-function for the KdV hierarchy, i.e. $U \triangleq \partial^2 F / \partial t_0^2$ obeys all equations in the KdV hierarchy.

DVV formula

The following equivalent formulation of Witten's conjecture is due to Dijkgraaf-E.Verlinde-H.Verlinde.

$$\begin{aligned}\langle\tau_{k+1}\prod_{j=1}^{n}\tau_{d_j}\rangle_g &= \frac{1}{(2k+3)!!}\Bigg[\sum_{j=1}^{n}\frac{(2k+2d_j+1)!!}{(2d_j-1)!!}\langle\tau_{d_1}\cdots\tau_{d_j+k}\cdots\tau_{d_n}\rangle_g \\ &\quad+\frac{1}{2}\sum_{r+s=k-1}(2r+1)!!(2s+1)!!\langle\tau_r\tau_s\tau_{d_1}\cdots\tau_{d_n}\rangle_{g-1} \\ &\quad+\frac{1}{2}\sum_{r+s=k-1}(2r+1)!!(2s+1)!!\sum_{\underline{n}=I\coprod J}\langle\tau_r\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\tau_s\prod_{i\in J}\tau_{d_i}\rangle_{g-g'}\Bigg]\end{aligned}$$

where $\underline{n}=\{1,2,\ldots,n\}$.

Virasoro constraints

$$L_k = \begin{cases} \sum_{m=1}^{\infty}(t_m - \delta_{m1})\partial_{m-1} + \frac{1}{2\hbar}t_0^2, & k=-1, \\ \displaystyle\sum_{m=0}^{\infty}(m+\frac{1}{2})(t_m-\delta_{m1})\partial_m + \frac{1}{16}, & k=0, \\ \displaystyle\sum_{m=0}^{\infty}\frac{\Gamma(k+m+\frac{3}{2})}{\Gamma(m+\frac{1}{2})}(t_m-\delta_{m1})\partial_{m+k} & \\ \qquad +\frac{\hbar}{2}\sum_{m=0}^{k-1}(-1)^{m+1}\frac{\Gamma(k-m+\frac{1}{2})}{\Gamma(-m-\frac{1}{2})}\partial_m\partial_{k-m-1}, & k>0. \end{cases}$$

which satisfy the commutation relations

$$[L_k, L_\ell] = (k-\ell)L_{k+\ell}.$$

Equivalent formulation of Witten-Kontsevich theorem

$$L_k(\exp(F)) = 0, \qquad k \geq -1$$

Different Proofs of Witten's conjecture

- Kontsevich's proof uses combinatorial model for the intersection numbers and matrix integral.
- Mirzakhani's proof depends on her recursion formula of Weil-Petersson volumes.
- Via ELSV fomula that relates intersection numbers with Hurwitz numbers (Okounkov-Pandharipande, Kazarian-Lando, Kim-Liu, Chen-Li-Liu).

Compute intersection numbers

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Witten-Kontsevich theorem provides a way to compute all intersection indices on moduli spaces of curves, which baffled mathematicians for many years.

Orbifold Singularities

A neighborhood of $\Sigma \in \overline{\mathcal{M}}_{g,n}$ is of the form $U/\mathrm{Aut}(\Sigma)$, where U is an open subset of $\mathbb{C}^{3g-3+n}$. The denominators of intersection numbers all come from these orbifold quotient singularities.

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$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_0 = \binom{n-3}{d_1, \ldots, d_n}$$

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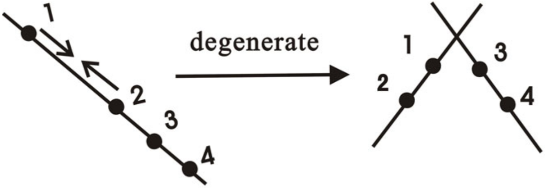

Some examples

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$$\langle \tau_5 \tau_5 \tau_5 \rangle_5 = \frac{13 \cdot 3221}{2^{13} \cdot 3^4 \cdot 5 \cdot 7 \cdot 11}, \quad \langle \tau_{10} \tau_{10} \rangle_7 = \frac{5 \cdot 113 \cdot 839}{2^{23} \cdot 3^9 \cdot 7 \cdot 11 \cdot 13}$$

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It is well-known that the order of automorphism groups of a compact Riemann surface of genus g does not contain prime factors greater then $2g + 1$.

Itzykson-Zuber Conjecture

In the paper "Combinatorics of the Modular Group II The Kontsevich integrals", arXiv: 9201001, page 37, C. Itzykson and J. Zuber posed the following conjecture:

Let $\mathcal{D}_g$ denote the least common multiple of denominators of all intersection indices $\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g$. For $2 \le g' \le g$, the order of any automorphism group of a compact Riemann surface of genus g' divides $\mathcal{D}_g$.

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Itzykson-Zuber's conjecture was proved in our paper "Intersection numbers and automorphisms of stable curves", arXiv: 0608209.

Conjectural multinomial value property

Based on calculations in low genera, we conjecture:

For $\sum_{i=1}^{n} d_i = 3g - 3 + n$ and $d_1 < d_2$, we have

$$\langle \tau_{d_1} \tau_{d_2} \cdots \tau_{d_n} \rangle_g \leq \langle \tau_{d_1+1} \tau_{d_2-1} \cdots \tau_{d_n} \rangle_g .$$

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Conjectural multinomial value property

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On the other hand, we know from Okounkov's work that

$$\sum_{g=0}^{\infty} \sum_{\sum d_j = 3g-3+n} \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g \prod_{j=1}^{n} x_j^{d_j}$$

converges all positive real numbers x_j.

The n-point function

Definition

The following generating function

$$F(x_1,\ldots,x_n)=\sum_{g=0}^{\infty}\sum_{\sum d_j=3g-3+n}\langle\tau_{d_1}\cdots\tau_{d_n}\rangle_g\prod_{j=1}^{n}x_j^{d_j}$$

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is called the n-point function.

Note that n-point functions encoded all information of intersection numbers on moduli spaces of curves. Note its difference with Witten's "free energy"

$$F(t_0,t_1,\dots)=\sum_{g=0}^{\infty}\sum_{\sum d_j=3g-3+n}\langle\tau_{d_1}\cdots\tau_{d_n}\rangle_g\frac{t_{d_1}\cdots t_{d_n}}{n!}.$$

Less than three point functions

Let $G(x_1,\dots,x_n)=\exp\left(\frac{-\sum_{j=1}^n x_j^3}{24}\right)\cdot F(x_1,\dots,x_n)$.

- Witten's One-point function $G(x)=\frac{1}{x^2}$
- R. Dijkgraaf's two-point function (1993)

$$G(x,y)=\frac{1}{x+y}\sum_{k=0}^{\infty}\frac{k!}{(2k+1)!}\left(\frac{1}{2}xy(x+y)\right)^k.$$

- Don Zagier's three-point function (1996)

$$G(x,y,z)=\sum_{r,s\geq 0}\frac{r!S_r(x,y,z)}{4^r(2r+1)!!}\frac{(\Delta/8)^s}{(r+s+1)!},$$

where S_r and Δ are some homogeneous symmetric polynomials.

Recursive formula of n-point functions

Let $n \geq 2$.

$$G(x_1,\ldots,x_n) = \sum_{r,s\geq 0} \frac{(2r+n-3)!!P_r(x_1,\ldots,x_n)\Delta(x_1,\ldots,x_n)^s}{4^s(2r+2s+n-1)!!}$$

where P_r and Δ are homogeneous symmetric polynomials

$$\Delta = \frac{(\sum_{j=1}^n x_j)^3 - \sum_{j=1}^n x_j^3}{3},$$

$$P_r = \left(\frac{1}{2\sum_{j=1}^n x_j} \sum_{\underline{n}=I\coprod J} (\sum_{i\in I} x_i)^2 (\sum_{i\in J} x_i)^2 G(x_I)G(x_J)\right)_{3r+n-3}$$

$$= \frac{1}{2\sum_{j=1}^n x_j} \sum_{\underline{n}=I\coprod J} (\sum_{i\in I} x_i)^2 (\sum_{i\in J} x_i)^2 \sum_{r'=0}^{r} G_{r'}(x_I)G_{r-r'}(x_J).$$

Coefficients of n-point functions

i) Let $k > 2g-2+n$, $d_j \geq 0$ and $\sum_{j=1}^n d_j = 3g-2+n-k$. Then

$$\mathcal{C}\left(z^k \prod_{j=1}^n x_j^{d_j}, G_g(z, x_1, \ldots, x_n)\right) = 0$$

ii) Let $d_j \geq 0$ and $\sum_{j=1}^n d_j = g$. Then

$$\mathcal{C}\left(z^{2g-2+n} \prod_{j=1}^n x_j^{d_j}, G_g(z, x_1, \ldots, x_n)\right) = \frac{1}{4^g \prod_{j=1}^n (2d_j+1)!!}$$

iii) Let $\sum_{j=1}^n d_j = g+1$, $a = \#\{j \mid d_j = 0\}$ and $b = \#\{j \mid d_j = 1\}$.

$$\mathcal{C}\left(z^{2g-3+n} \prod_{j=1}^n x_j^{d_j}, G_g(z, x_{\underline{n}})\right) = \frac{2g^2 + (2n-1)g + \frac{n^2-n}{2} - 3 + \frac{5a-a^2}{2}}{4^g \prod_{j=1}^n (2d_j+1)!!}$$

New effective recursion formula

The following recursion formula may be used to compute intersection numbers effectively.

$$
\begin{aligned}
&(2g+n-1)(2g+n-2)\langle\prod_{j=1}^{n}\tau_{d_j}\rangle_g \\
&\quad=\frac{2d_1+3}{12}\langle\tau_0^4\tau_{d_1+1}\prod_{j=2}^{n}\tau_{d_j}\rangle_{g-1}-\frac{2g+n-1}{6}\langle\tau_0^3\prod_{j=1}^{n}\tau_{d_j}\rangle_{g-1}\\
&\qquad+\sum_{\{2,\ldots,n\}=I\coprod J}(2d_1+3)\langle\tau_{d_1+1}\tau_0^2\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\tau_0^2\prod_{i\in J}\tau_{d_i}\rangle_{g-g'}\\
&\qquad\quad-\sum_{\{2,\ldots,n\}=I\coprod J}(2g+n-1)\langle\tau_{d_1}\tau_0\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\tau_0^2\prod_{i\in J}\tau_{d_i}\rangle_{g-g'}.
\end{aligned}
$$

It's not difficult to see that when indices $d_j \geq 1$, all non-zero intesection indices on the right hands have genera strictly less than g.

New differential equations

Let

$$F(t_0, t_1, \ldots) = \sum_g \sum_{\mathbf{n}} \langle \prod_{i=0}^{\infty} \tau_i^{n_i} \rangle_g \prod_{i=0}^{\infty} \frac{t_i^{n_i}}{n_i!}.$$

We have

$$\frac{\partial^2 F}{\partial t_1 \partial t_r} = (2r+3)\frac{\partial^2 F}{\partial t_0 \partial t_{r+1}} - \frac{1}{6}\frac{\partial^4 F}{\partial t_0^3 \partial t_r} - \frac{\partial^2 F}{\partial t_0 \partial t_r}\frac{\partial^2 F}{\partial t_0^2},$$

which is equivalent to

$$\begin{aligned}(2g+n-1)\langle \tau_r \prod_{j=1}^{n} \tau_{d_j} \rangle_g &= (2r+3)\langle \tau_0 \tau_{r+1} \prod_{j=1}^{n} \tau_{d_j} \rangle_g \\ &- \frac{1}{6}\langle \tau_0^3 \tau_r \prod_{j=1}^{n} \tau_{d_j} \rangle_{g-1} - \sum_{\underline{n}=I \coprod J} \langle \tau_0 \tau_r \prod_{i\in I} \tau_{d_i} \rangle_{g'} \langle \tau_0^2 \prod_{i \in J} \tau_{d_i} \rangle_{g-g'}.\end{aligned}$$

Tautological rings

Denote by $\mathcal{M}_g$ the moduli space of Riemann surfaces of genus $g \geq 2$. The tautological ring $\mathcal{R}^*(\mathcal{M}_g)$ is defined to be the $\mathbb{Q}$-subalgebra of the Chow ring $\mathcal{A}^*(\mathcal{M}_g)$ generated by the tautological classes κ_i and λ_i.

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$\mathcal{R}^*(\mathcal{M}_g)$ has the following properties:

i) (Mumford) $\mathcal{R}^*(\mathcal{M}_g)$ is in fact generated by the $g-2$ classes $\kappa_1, \ldots, \kappa_{g-2}$;

ii) (Looijenga) $\mathcal{R}^j(\mathcal{M}_g) = 0$ for $j > g-2$ and $\dim \mathcal{R}^{g-2}(\mathcal{M}_g) \leq 1$ (Faber showed that actually $\dim \mathcal{R}^{g-2}(\mathcal{M}_g) = 1$).

Faber's conjecture

Around 1993, Faber proposed a series of remarkable conjectures about the structure of the tautological ring $\mathcal{R}^*(\mathcal{M}_g)$.

Roughly speaking, Faber's conjecture asserts that $R^*(\mathcal{M}_g)$ behaves like the cohomology ring of a $(g-2)$-dimensional complex projective manifold.

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Around 1993, Faber proposed a series of remarkable conjectures about the structure of the tautological ring $\mathcal{R}^*(\mathcal{M}_g)$.

Roughly speaking, Faber's conjecture asserts that $R^*(\mathcal{M}_g)$ behaves like the cohomology ring of a $(g-2)$-dimensional complex projective manifold.

i) (Perfect pairing conjecture) When an isomorphism $R^{g-2}(\mathcal{M}_g) \cong \mathbb{Q}$ is fixed, the following natural pairing is perfect

$$\mathcal{R}^k(\mathcal{M}_g) \times R^{g-2-k}(\mathcal{M}_g) \longrightarrow \mathcal{R}^{g-2}(\mathcal{M}_g) = \mathbb{Q};$$

Faber's perfect paring conjecture is still open to this day.

ii) The [g/3] classes $\kappa_1, \ldots, \kappa_{[g/3]}$ generate the ring, with no relations in degrees $\leq [g/3]$; (proved by Morita and Ionel)

Faber intersection number conjecture

An important part (the only quantitative part) of Faber's conjecture is the famous Faber intersection number conjecture.

$$\frac{(2g-3+n)!}{2^{2g-1}(2g-1)!\prod_{j=1}^{n}(2d_j-1)!!} = \langle \tau_{2g} \prod_{j=1}^{n} \tau_{d_j} \rangle_g$$

$$- \sum_{j=1}^{n} \langle \tau_{d_j+2g-1} \prod_{i \neq j} \tau_{d_i} \rangle_g + \frac{1}{2} \sum_{j=0}^{2g-2} (-1)^j \langle \tau_{2g-2-j} \tau_j \prod_{i=1}^{n} \tau_{d_i} \rangle_{g-1}$$

$$+ \frac{1}{2} \sum_{\underline{n}=I \coprod J} \sum_{j=0}^{2g-2} (-1)^j \langle \tau_j \prod_{i \in I} \tau_{d_i} \rangle_{g'} \langle \tau_{2g-2-j} \prod_{i \in J} \tau_{d_i} \rangle_{g-g'},$$

$\lambda_g\lambda_{g-1}$ theorem

Using Mumford's Chern character formula for Hodge bundles

$$\mathrm{ch}_{2g-1}(\mathbb{E}) = \frac{B_{2g}}{(2g)!}\left[\kappa_{2g-1} - \sum_{i=1}^{n}\psi_i^{2g-1} + \frac{1}{2}\sum_{\xi\in\Delta} l_{\xi_*}\left(\sum_{i=0}^{2g-2}\psi_1^i(-\psi_2)^{2g-2-i}\right)\right]$$

Faber intersection number conjecture is equivalent to

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1}\dots\psi_n^{d_n}\lambda_g\lambda_{g-1} = \frac{(2g-3+n)!|B_{2g}|}{2^{2g-1}(2g)!\prod_{j=1}^{n}(2d_j-1)!!},$$

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This was proved by Getzler and Pandharipande (1998) under the assumption that degree 0 Virasoro conjecture for $\mathbb{P}^2$ holds. Givental announced a proof of Virasoro conjecture for $\mathbb{P}^n$. Recently Teleman proved the Virasoro conjecture for all manifolds with semi-simple quantum cohomology.

Tautological relations in $\mathcal{R}^*(\mathcal{M}_g)$

Faber intersection number conjecture is equivalent to the following tautological relation in $\mathcal{R}^*(\mathcal{M}_g)$

$$\pi_*(\psi_1^{d_1+1}\dots\psi_n^{d_n+1}) = \sum_{\sigma\in S_n}\kappa_\sigma = \frac{(2g-3+n)!(2g-1)!!}{(2g-1)!\prod_{j=1}^n(2d_j+1)!!}\kappa_{g-2},$$

where $\pi : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_g$ is the forgetful morphism.

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In 2006, Goulden, Jackson and Vakil give an enlightening proof of this relation for up to three points. Their remarkable proof relied on relative virtual localization in Gromov-Witten theory and some tour de force combinatorial computations.

Importance

The Faber intersection number conjecture computes all top intersections in $\mathcal{R}^{g-2}(\mathcal{M}_g)$ explicitly.

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The Faber intersection number conjecture computes all top intersections in $\mathcal{R}^{g-2}(\mathcal{M}_g)$ explicitly.

$\mathbf{m} = (m_1, m_2, \dots) \in N^\infty$, define $|\mathbf{m}| := \sum_{i \geq 1} i \cdot m_i$, $\quad ||\mathbf{m}|| := \sum_{i \geq 1} m_i$.
If $|\mathbf{m}| = g - 2$, $\kappa(\mathbf{m}) = \prod_{j \geq 1} \kappa_j^{m_j}$, then

$$\kappa(\mathbf{m}) = C(\mathbf{m}) \cdot \kappa_{g-2}$$

where

$$C(\mathbf{m}) = \sum_{r=1}^{||\mathbf{m}||} \frac{(-1)^{||\mathbf{m}||-r}}{r!} \sum_{\substack{\mathbf{m}=\mathbf{m_1}+\cdots+\mathbf{m_r} \\ \mathbf{m_i} \neq \mathbf{0}}} \binom{\mathbf{m}}{\mathbf{m_1}, \dots, \mathbf{m_r}} \times \frac{(2g-3+r)!}{(2g-2)!! \prod_{j=1}^{r} (2|\mathbf{m}_j| + 1)!!}.$$

Proof of Faber intersection number conjecture

Now we describe our proof.

Since one and two-point functions in genus 0 are

$$F_0(x) = \frac{1}{x^2}, \qquad F_0(x,y) = \frac{1}{x+y} = \sum_{k=0}^{\infty}(-1)^k \frac{x^k}{y^{k+1}},$$

it is consistent to define the virtual intersection numbers

$$\langle \tau_{-2} \rangle_0 = 1, \qquad \langle \tau_k \tau_{-1-k} \rangle_0 = (-1)^k, \ k \geq 0.$$

Relations with n-point functions

i) $$\frac{1}{2}\sum_{j=0}^{2g-2}(-1)^j\langle\tau_{2g-2-j}\tau_j\prod_{i=1}^{n}\tau_{d_i}\rangle_{g-1}=[F_{g-1}(y,-y,x_1,\ldots,x_n)]_{y^{2g-2}}$$

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ii) $$\frac{1}{2}\sum_{\underline{n}=I\coprod J}\sum_{j=0}^{2g-2}(-1)^j\langle\tau_j\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\tau_{2g-2-j}\prod_{i\in J}\tau_{d_i}\rangle_{g-g'}+\langle\prod_{j=1}^{n}\tau_{d_j}\tau_{2g}\rangle_g$$
$$-\sum_{j=1}^{n}\langle\tau_{d_j+2g-1}\prod_{i\neq j}\tau_{d_i}\rangle_g$$
$$=\frac{1}{2}\sum_{\underline{n}=I\coprod J}\sum_{j\in\mathbb{Z}}(-1)^j\langle\tau_j\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\tau_{2g-2-j}\prod_{i\in J}\tau_{d_i}\rangle_{g-g'}$$
$$=\left[\sum_{g'=0}^{g}\sum_{\underline{n}=I\coprod J}F_{g'}(y,x_I)F_{g-g'}(-y,x_J)\right]_{y^{2g-2}\prod_{i=1}^{n}x_i^{d_i}}.$$

Byproduct

We proved the following Hodge integral identity.

$$
\begin{aligned}
&-\frac{(2g-2)!}{|B_{2g-2}|}\langle\tau_{d_1}\cdots\tau_{d_n} \mid \mathrm{ch}_{2g-3}(\mathbb{E})\rangle_g \\
=&\frac{2g-2}{|B_{2g-2}|}\left(\langle\tau_{d_1}\cdots\tau_{d_n} \mid \lambda_{g-1}\lambda_{g-2}\rangle_g - 3\langle\tau_{d_1}\cdots\tau_{d_n} \mid \lambda_{g-3}\lambda_g\rangle_g\right) \\
=&\frac{1}{2}\sum_{j=0}^{2g-4}(-1)^j\langle\tau_{2g-4-j}\tau_j\tau_{d_1}\cdots\tau_{d_n}\rangle_{g-1} + \frac{(2g-3+n)!}{2^{2g+1}(2g-3)!}\cdot\frac{1}{\prod_{j=1}^n(2d_j-1)!!}.
\end{aligned}
$$

Mirzakhani's recursion formula of Weil-Petersson volumes

Mirzakhani proved a beautiful recursion formula for the Weil-Petersson volume of the moduli space $\mathcal{M}_{g,n}(\mathbf{L})$ of genus g hyperbolic surfaces with n geodesic boundary components of specified length $\mathbf{L} = (L_1, \dots, L_n)$.

$$\begin{aligned}
\mathrm{Vol}_{g,n}(\mathbf{L}) = \frac{1}{2L_1} \sum_{\substack{g_1+g_2=g \\ \underline{n}=I \coprod J}} \int_0^{L_1} \int_0^{\infty} \int_0^{\infty} xyH(t, x+y) \\
\times \mathrm{Vol}_{g_1,n_1}(x, \mathbf{L}_I) \mathrm{Vol}_{g_2,n_2}(y, \mathbf{L}_J) dxdydt \\
+ \frac{1}{2L_1} \int_0^{L_1} \int_0^{\infty} \int_0^{\infty} xyH(t, x+y) \mathrm{Vol}_{g-1,n+1}(x, y, L_2, \dots, L_n) dxdydt \\
+ \frac{1}{2L_1} \sum_{j=2}^{n} \int_0^{L_1} \int_0^{\infty} x\big(H(x, L_1+L_j) + H(x, L_1-L_j)\big) \\
\times \mathrm{Vol}_{g,n-1}(x, L_2, \dots, \hat{L}_j, \dots, L_n) dxdt,
\end{aligned}$$

where the kernel function

$$H(x,y) = \frac{1}{1+e^{(x+y)/2}} + \frac{1}{1+e^{(x-y)/2}}.$$

Mulase-Safnuk differential form of Mirzakhani's recursion

Mulase and Safnuk obtained an enlightening differential version of Mirzakhani's recursion.

$$
\begin{aligned}
&(2d_1+1)!!\langle\tau_{d_1}\cdots\tau_{d_n}\kappa_1^a\rangle_g \\
&=\sum_{j=2}^{n}\sum_{b=0}^{a}\frac{a!}{(a-b)!}\frac{(2(b+d_1+d_j)-1)!!}{(2d_j-1)!!}\beta_b\langle\kappa_1^{a-b}\tau_{b+d_1+d_j-1}\prod_{i\neq 1,j}\tau_{d_i}\rangle_g \\
&+\frac{1}{2}\sum_{b=0}^{a}\sum_{r+s=b+d_1-2}\frac{a!}{(a-b)!}(2r+1)!!(2s+1)!!\beta_b\langle\kappa_1^{a-b}\tau_r\tau_s\prod_{i\neq 1}\tau_{d_i}\rangle_{g-1} \\
&\quad+\frac{1}{2}\sum_{b=0}^{a}\sum_{\substack{c+c'=a-b\\ I\coprod J=\{2,\ldots,n\}}}\sum_{r+s=b+d_1-2}\frac{a!}{c!c'!}(2r+1)!!(2s+1)!!\beta_b \\
&\qquad\times\langle\kappa_1^{c}\tau_r\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\kappa_1^{c'}\tau_s\prod_{i\in J}\tau_{d_i}\rangle_{g-g'},
\end{aligned}
$$

$$\beta_b=(-1)^{b-1}2^b(2^{2b}-2)\frac{B_{2b}}{(2b)!}$$

Generalization to higher Weil-Petersson volumes

Motivated by Mulase and Safnuk's work, we generalize Mulase-Safnuk differential form of Mirzakhani's recursion to the following recursion formula of higher WP volumes.

$$
\begin{aligned}
&(2d_1+1)!!\langle\kappa(\mathbf{b})\tau_{d_1}\cdots\tau_{d_n}\rangle_g \\
&=\sum_{j=2}^{n}\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}}\alpha_{\mathbf{L}}\binom{\mathbf{b}}{\mathbf{L}}\frac{(2(|\mathbf{L}|+d_1+d_j)-1)!!}{(2d_j-1)!!}\langle\kappa(\mathbf{L}')\tau_{|\mathbf{L}|+d_1+d_j-1}\prod_{i\neq 1,j}\tau_{d_i}\rangle_g \\
&+\frac{1}{2}\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}}\sum_{r+s=|\mathbf{L}|+d_1-2}\alpha_{\mathbf{L}}\binom{\mathbf{b}}{\mathbf{L}}(2r+1)!!(2s+1)!!\langle\kappa(\mathbf{L}')\tau_r\tau_s\prod_{i=2}^{n}\tau_{d_i}\rangle_{g-1} \\
&\quad+\frac{1}{2}\sum_{\substack{\mathbf{L}+\mathbf{e}+\mathbf{f}=\mathbf{b}\\ I\coprod J=\{2,\ldots,n\}}}\sum_{r+s=|\mathbf{L}|+d_1-2}\alpha_{\mathbf{L}}\binom{\mathbf{b}}{\mathbf{L},\mathbf{e},\mathbf{f}}(2r+1)!!(2s+1)!! \\
&\qquad\qquad\times\langle\kappa(\mathbf{e})\tau_r\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\kappa(\mathbf{f})\tau_s\prod_{i\in J}\tau_{d_i}\rangle_{g-g'}.
\end{aligned}
$$

Tautological Constants

These tautological constants $\alpha_{\mathbf{L}}$ can be determined recursively from the following formula

$$\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} \frac{(-1)^{||\mathbf{L}||}\alpha_{\mathbf{L}}}{\mathbf{L}!\mathbf{L}'!(2|\mathbf{L}'|+1)!!} = 0, \qquad \mathbf{b} \neq 0,$$

namely

$$\alpha_{\mathbf{b}} = \mathbf{b}! \sum_{\substack{\mathbf{L}+\mathbf{L}'=\mathbf{b} \\ \mathbf{L}'\neq\mathbf{0}}} \frac{(-1)^{||\mathbf{L}'||-1}\alpha_{\mathbf{L}}}{\mathbf{L}!\mathbf{L}'!(2|\mathbf{L}'|+1)!!}, \qquad \mathbf{b} \neq 0,$$

with the initial value $\alpha_{\mathbf{0}} = 1$.

The crucial idea of inversion comes from Mulase and Safnuk's work.

Eynard-Orantin's recursion

Witten-Kontsevich theorem, Mirzakhani's recursion relation and its generalizations by Mulase-Safuk and Liu-X are rederived from Eynard and Orantin's theory of symplectic invariants of curves.

Bouchard-Klemm-Marino-Pasquetti (BKMP) further conjectures that Eynard-Orantin's recursion can be used to compute the Gromov-Witten invariants of toric Calabi-Yau 3-folds through mirror symmetry.

See their paper "Remodeling the B-model", arXiv:0709.1453.

Recursion for higher WP volumes

For $\mathbf{b} \in N^\infty$, we denote by $V_{g,n}(\mathbf{b})$ the higher Weil-Petersson volume

$$V_{g,n}(\mathbf{b}) \triangleq \langle \tau_0^n \kappa(\mathbf{b}) \rangle_g = \int_{\overline{\mathcal{M}}_{g,n}} \kappa(\mathbf{b}).$$

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Let $n \geq 1$. Then

$$\left(2g - 1 + ||\mathbf{b}||\right) V_{g,n}(\mathbf{b}) = \frac{1}{12} V_{g-1,n+3}(\mathbf{b}) - \sum_{\substack{\mathbf{L}+\mathbf{L}'=\mathbf{b} \\ ||\mathbf{L}'|| \geq 2}} \binom{\mathbf{b}}{\mathbf{L}} V_{g,n}(\mathbf{L} + \delta_{|\mathbf{L}'|})$$

$$+ \frac{1}{2} \sum_{\substack{\mathbf{L}+\mathbf{L}'=\mathbf{b} \\ \mathbf{L} \neq \mathbf{0}, \mathbf{L}' \neq \mathbf{0}}} \sum_{r+s=n-1} \binom{\mathbf{b}}{\mathbf{L}} \binom{n-1}{r} V_{g',r+2}(\mathbf{L}) V_{g-g',s+2}(\mathbf{L}').$$

Compatibility of κ and ψ classes

From the translational difference between generating functions, recursion formulae of ψ classes can always be generalized to include κ classes.

1. Generalization of the string equation

$$\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} (-1)^{||\mathbf{L}||} \binom{\mathbf{b}}{\mathbf{L}} \langle \tau_{|\mathbf{L}|} \prod_{j=1}^{n} \tau_{d_j} \kappa(\mathbf{L}') \rangle_g = \sum_{j=1}^{n} \langle \tau_{d_j - 1} \prod_{i \neq j} \tau_{d_i} \kappa(\mathbf{b}) \rangle_g$$

2. Generalization of the dilaton equation

$$\sum_{\mathbf{L}+\mathbf{L}'=\mathbf{b}} (-1)^{||\mathbf{L}||} \binom{\mathbf{b}}{\mathbf{L}} \langle \tau_{|\mathbf{L}|+1} \prod_{j=1}^{n} \tau_{d_j} \kappa(\mathbf{L}') \rangle_g = (2g - 2 + n) \langle \prod_{j=1}^{n} \tau_{d_j} \kappa(\mathbf{b}) \rangle_g$$

Push-forward and pull-back of κ

The following results can be found in Arbarello and Cornalba's paper "*Combinatorial and Algebro-Geometric cohomology classes on the Moduli Spaces of Curves*", arXiv:9406008.

Lemma

Let $\pi_{n+1} : \overline{\mathcal{M}}_{g,n+1} \longrightarrow \overline{\mathcal{M}}_{g,n}$ be the morphism that forgets the last marked point.

i) $\pi_{n*}(\psi_1^{a_1} \cdots \psi_{n-1}^{a_{n-1}} \psi_n^{a_n+1}) = \psi_1^{a_1} \cdots \psi_{n-1}^{a_{n-1}} \kappa_{a_n}$ *for* $a_j \geq 0$;

ii) $\kappa_a = \pi_{n+1}^*(\kappa_a) + \psi_{n+1}^a$ *on* $\overline{\mathcal{M}}_{g,n+1}$;

iii) $\kappa_0 = 2g - 2 + n$ *on* $\overline{\mathcal{M}}_{g,n}$.

Formal pseudo-differential operators

A pseudo-differential operator is

$$L = \sum_{i=-\infty}^{N} u_i(x)\partial^i, \quad \text{where } \partial = \frac{d}{dx}.$$

For $k \in Z$,

$$\partial^k f = \sum_{j\geq 0} \binom{k}{j} f^{(j)} \partial^{k-j}, \quad \text{where } f^{(j)} = \frac{d^j f}{dx^j}.$$

In particular, $\partial f = f' + f\partial$. All pseudo-differential operators form an algebra.

Fix an integer $r \geq 2$. Consider the differential operator

$$Q \triangleq \partial^r + \sum_{i=0}^{r-2} \gamma_i(x)\partial^i. \tag{1}$$

Note that the coefficient of ∂^{r-1} in Q is zero.

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It is easy to see that there is a pseudo-differential operator

$$Q^{1/r} = \partial + \sum_{i>0} w_i \partial^{-i}, \tag{2}$$

where $\{w_i\}$ are polynomials in derivatives of $\{\gamma_i\}$.

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Let $k \geq 1$ and

$$Q^{k/r} = \partial^k + \sum_{i=0}^{k-2} \gamma_i^k \partial^i + \sum_{i=1}^{\infty} \gamma_{-i}^k \partial^{-i}. \tag{3}$$

We have $\gamma_{-i}^r = 0$ for $i \geq 1$. These γ_i^k can be written as polynomials in derivatives of $\{w_i\}$.

A structure lemma of PDO

For $0 \leq m \leq r-2$, we use the notation

$$z_m^{(j)} \triangleq \frac{r}{m+1} \cdot \frac{d^j \gamma_{-1}^{m+1}}{dx^j}.$$

We have

Proposition

$$\frac{r^2}{r+1}\gamma_{-1}^{r+1} = \frac{1}{2}\sum_{j=0}^{r-2} z_j z_{r-2-j} + DIF(z),$$

where $DIF(z)$ represents the terms containing derivatives of z_m. Both sides are regarded as polynomials in derivatives of $\{w_i\}$.

Proof

Let

$$f = 1 + \sum_{j=2}^{\infty} a_j x^j \in \mathbb{C}[[x]].$$

Then for any $k \geq 1$, we have

$$\frac{[x^{k+2}]f^{k+1}}{k+1} = \frac{1}{2}\sum_{j=1}^{k-1} \frac{[x^{j+1}]f^j}{j} \cdot \frac{[x^{k-j+1}]f^{k-j}}{k-j} + \frac{[x^{k+2}]f^k}{k}.$$

There is an equivalent formulation using *partitions*.

$$\sum_{\ell(\mu)\geq 2} \frac{(|\mu| + \ell(\mu) - 3)!(\ell(\mu) - 1)p_\mu}{(|\mu| - 1)!z_\mu} = \frac{1}{2}\left(\sum_{\mu\neq 0} \frac{(|\mu| + \ell(\mu) - 2)!p_\mu}{(|\mu| - 1)!z_\mu}\right)^2,$$

where $z_\mu = \prod_j m_j(\mu)! j^{m_j(\mu)}$, $\qquad p_\mu = \prod_j p_j^{m_j(\mu)}$.

Moduli spaces of r-spin curves

Let Σ be a Riemann surface of genus g with marked points $x_1, x_2, \dots, x_s$. Fix an integer $r \geq 2$. Label each marked point x_i by an integer m_i, $0 \leq m_i \leq r-1$. Consider the line bundle over Σ,

$$\mathcal{S} = K \otimes (\otimes_{i=1}^{s} \mathcal{O}(x_i)^{-m_i}).$$

If $2g - 2 - \sum_{i=1}^{s} m_i$ is divisible by r, then there are r^{2g} isomorphism classes of line bundles $\mathcal{T}$ such that $\mathcal{T}^{\otimes r} \cong \mathcal{S}$.

The choice of an isomorphism class of $\mathcal{T}$ determines a cover

$$\pi : \mathcal{M}_{g,s}^{1/r} \longrightarrow \mathcal{M}_{g,s}$$

.

Witten's r-spin numbers

We associate with each marked point x_i an integer $n_i \geq 0$. Witten's r-spin intersection numbers are defined by

$$\langle \tau_{n_1,m_1} \dots \tau_{n_s,m_s} \rangle_g = \frac{1}{r^g} \int_{\overline{\mathcal{M}}_{g,s}^{1/r}} \prod_{i=1}^{s} \psi(x_i)^{n_i} \cdot c_{top}(\mathcal{V}),$$

where $\mathcal{V}$ is a vector bundle over $\overline{\mathcal{M}}_{g,s}^{1/r}$ whose fiber is the dual space to $H^1(\Sigma, \mathcal{T})$.

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Consider the formal series F in variables $t_{n,m}$, $n \geq 0$ and $0 \leq m \leq r-1$,

$$F(t_{0,0}, t_{0,1}, \dots) = \sum_{d_{n,m}} \langle \prod_{n,m} \tau_{n,m}^{d_{n,m}} \rangle \prod_{n,m} \frac{t_{n,m}^{d_{n,m}}}{d_{n,m}!}.$$

Gelfand-Dickii hierarchy

$$Q = D^r + \sum_{i=0}^{r-2} \gamma_i(x) D^i, \qquad \text{where } D = \frac{\sqrt{-1}}{\sqrt{r}} \frac{\partial}{\partial x}.$$

The Gelfand–Dikii equations read

$$i \frac{\partial Q}{\partial t_{n,m}} = [Q_+^{n+(m+1)/r}, Q] \cdot \frac{c_{n,m}}{\sqrt{r}},$$

where

$$c_{n,m} = \frac{(-1)^n r^{n+1}}{(m+1)(r+m+1)\cdots(nr+m+1)}.$$

Generalized Witten conjecture

The conjecture of Witten is that the generating function F is the string solution of the r-Gelfand–Dikii hierarchy, namely

$$\frac{\partial F}{\partial t_{0,0}} = \frac{1}{2}\sum_{i,j=0}^{r-2}\delta_{i+j,r-2}t_{0,i}t_{0,j} + \sum_{n=0}^{\infty}\sum_{m=0}^{r-2}t_{n+1,m}\frac{\partial F}{\partial t_{n,m}},$$

$$\frac{\partial^2 F}{\partial t_{0,0}\partial t_{n,m}} = -c_{n,m}\mathrm{res}(Q^{n+\frac{m+1}{r}}),$$

where Q satisfies the Gelfand-Dikii equations and $t_{0,0}$ is identified with x.

Recursion of r-spin numbers

The following formula gives an effective recursion formula to compute all r-spin intersection numbers.

For fixed $r \geq 2$, we have

$$\langle\langle \tau_{1,0}\tau_{0,0}\rangle\rangle_g = \frac{1}{2}\langle\langle \tau_{0,0}\tau_{0,m'}\rangle\rangle_{g'}\eta^{m'm''}\langle\langle \tau_{0,m''}\tau_{0,0}\rangle\rangle_{g-g'} + Low(r),$$

where $Low(r)$ consists of $\langle\langle \ldots \rangle\rangle$ with strictly lower genera.

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When $r = 4$, we have

$$\langle\langle\tau_{1,0}\tau_{0,0}\rangle\rangle_g = \langle\langle\tau_{0,0}\tau_{0,2}\rangle\rangle_{g'}\langle\langle\tau_{0,0}^2\rangle\rangle_{g-g'} + \frac{1}{2}\langle\langle\tau_{0,0}\tau_{0,1}\rangle\rangle_{g'}\langle\langle\tau_{0,1}\tau_{0,0}\rangle\rangle_{g-g'}$$
$$+\frac{1}{4}\langle\langle\tau_{0,0}^3\tau_{0,2}\rangle\rangle_{g-1}+\frac{1}{48}\langle\langle\tau_{0,0}^2\rangle\rangle_{g'}\langle\langle\tau_{0,0}^4\rangle\rangle_{g-1-g'}+\frac{1}{32}\langle\langle\tau_{0,0}^3\rangle\rangle_{g'}\langle\langle\tau_{0,0}^3\rangle\rangle_{g-1-g'}$$
$$+\frac{1}{480}\langle\langle\tau_{0,0}^6\rangle\rangle_{g-2}$$

Gromov-Witten invariants

Fix a basis $\{\gamma_0 = 1, \dots, \gamma_N\}$ for $H^*(X, \mathbb{Q})$, we may use $\eta_{ab} = \int_X \gamma_a \cup \gamma_b$ and its inverse η^{ab} to lower and raise indices

$$\gamma^a = \eta^{ab}\gamma_b.$$

If $\gamma_{a_1}, \dots, \gamma_{a_n} \in H^*(X, \mathbb{Q})$, the Gromov-Witten invariants are defined by

$$\langle \tau_{d_1}(\gamma_{a_1}) \dots \tau_{d_n}(\gamma_{a_n}) \rangle_{g,\beta}^X = \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{virt}}} \Psi_1^{d_1} \cdots \Psi_n^{d_n} \cup \mathrm{ev}^*(\gamma_{a_1} \boxtimes \cdots \boxtimes \gamma_{a_n}).$$

Gromov-Witten invariants are rational numbers.

Universal relations of Gromov-Witten invariants

String equation

$$\langle \tau_{0,0}\tau_{k_1,a_1}\cdots\tau_{k_n,a_n}\rangle_{g,\beta}^X = \sum_{i=1}^n \langle \tau_{k_1,a_1}\cdots\tau_{k_i-1,a_i}\cdots\tau_{k_n,a_n}\rangle_{g,\beta}^X.$$

Dilaton equation

$$\langle \tau_{1,0}\tau_{k_1,a_1}\cdots\tau_{k_n,a_n}\rangle_{g,\beta}^X = (2g-2+n)\langle \tau_{k_1,\alpha_1}\cdots\tau_{k_n,\alpha_n}\rangle_{g,\beta}^X.$$

Divisor equation

$$\begin{aligned}\langle \tau_0(\omega)\tau_{k_1,a_1}\cdots\tau_{k_n,a_n}\rangle_{g,\beta}^X &= \left(\omega\cap\beta\right)\langle \tau_{k_1,a_1}\cdots\tau_{k_n,a_n}\rangle_{g,\beta}^X \\ &+ \sum_{i=1}^n \langle \tau_{k_1,a_1}\cdots\tau_{k_i-1}(\omega\cup\gamma_a)\cdots\tau_{k_n,a_n}\rangle_{g,\beta}^X,\end{aligned}$$

where $\omega\in H^2(X,\mathbb{Q})$.

Topological recursion relations

We may pull back tautological relations on $\overline{\mathcal{M}}_{g,n}$ via the forgetful map

$$\pi : \overline{\mathcal{M}}_{g,n+1}(X,\beta) \to \overline{\mathcal{M}}_{g,n}$$

to get universal equations for Gromov-Witten invariants by the splitting axiom and cotangent line comparison equations.

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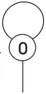
On $\overline{\mathcal{M}}_{1,1}$, the relation

$$\psi_1 = \frac{1}{12}$$ 

implies the genus 1 TRR

$$\langle\langle \tau_k(x) \rangle\rangle_1 = \langle\langle \tau_{k-1}(x)\gamma_\alpha \rangle\rangle_0 \langle\langle \gamma^\alpha \rangle\rangle_1 + \frac{1}{24}\langle\langle \tau_{k-1}(x)\gamma_\alpha\gamma^\alpha \rangle\rangle_0.$$

Xiaobo Liu's T operator

Define the operator T on the space of vector fields by

$$T(\mathcal{W}) = \tau_+(\mathcal{W}) - \langle\langle \mathcal{W}\,\gamma^\alpha \rangle\rangle_0 \gamma_\alpha$$

for any vector field $\mathcal{W}$.

In the process of translating relations in $R^*(\overline{\mathcal{M}}_{g,n})$ into universal equations for Gromov-Witten invariants, each marked point corresponds to a vector field, and the cotangent line class corresponds to the operator T. Each node is translated into a pair of primary vector fields γ_α and γ^α.

WDVV equation

From the simple fact that the three boundary divisors of $\overline{\mathcal{M}}_{0,4} \cong \mathbb{P}^1$ are equal, there is the well-known

WDVV equation

$$\langle\langle\tau_{k_1,a_1}\tau_{k_2,a_2}\gamma_\alpha\rangle\rangle_0\langle\langle\gamma^\alpha\tau_{k_3,a_3}\tau_{k_4,a_4}\rangle\rangle_0 = \langle\langle\tau_{k_1,a_1}\tau_{k_3,a_3}\gamma_\alpha\rangle\rangle_0\langle\langle\gamma^\alpha\tau_{k_2,a_2}\tau_{k_4,a_4}\rangle\rangle_0$$

which is the associativity condition of the quantum cohomology ring.

Vanishing Identities of Gromov-Witten Invariants

Our proof of Faber intersection number conjecture ($X = pt$) motivates us to formulate the following conjecture

Conjecture

Let $x_i, y_i \in H^*(X)$ and $k \geq 2g - 3 + r + s$. Then

$$\sum_{g'=0}^{g} \sum_{j \in \mathbb{Z}} (-1)^j \langle\langle \tau_j(\gamma_a) \prod_{i=1}^{r} \tau_{p_i}(x_i) \rangle\rangle_{g'} \langle\langle \tau_{k-j}(\gamma^a) \prod_{i=1}^{s} \tau_{q_i}(y_i) \rangle\rangle_{g-g'} = 0.$$

Note that j runs over all integers.

where we adopt Gathmann's convention $\langle \tau_{-2}(pt) \rangle_{0,0}^{X} = 1$ and

$$\langle \tau_m(\gamma_1) \tau_{-1-m}(\gamma_2) \rangle_{0,0}^{X} = (-1)^{\max(m,-1-m)} \int_X \gamma_1 \cdot \gamma_2, \quad m \in \mathbb{Z}.$$

The conjecture has recently been proved recently by Xiaobo Liu and R. Pandharipande.

J. Zhou's Work on Hurwitz-Hodge Integrals

Our results on intersection numbers and Faber intersection number conjecture have found applications in Professor Jian Zhou's recent works on Hurwitz-Hodge integrals and orbifold Gromov-Witten theory.

J. Zhou proves the crepant resolution conjecture of type A surface singularities for all genera.

Quantum cohomology

Let $\gamma = \sum_{i=0}^{N} t_i \gamma_i$.

$$\Phi(\gamma) = \sum_{n=0}^{\infty} \sum_{\beta \in H_2^+(X,\mathbb{Z})} \langle \gamma^n \rangle_{0,\beta}^X q^\beta.$$

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Define Φ_{ijk} to be the partial derivative:

$$\Phi_{ijk} = \frac{\partial^3 \Phi}{\partial t_i \, \partial t_j \partial t_k}, \quad 0 \leq i, j, k \leq m.$$

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The quantum product $*$ is defined by the rule:

$$\gamma_i \, * \, \gamma_j = \sum_{e,\, f} \Phi_{ije} \, \eta^{ef} \, \gamma_f$$

which makes $A^* X \otimes \mathbb{Q}[[t_0, \ldots, t_N]]$ into a commutative, associative $\mathbb{Q}[[t_0, \ldots, t_N]]$-algebra, with unit $\gamma_0 = 1$.

Kontsevich's formula of plane curves

let N_d be the number of degree d, rational plane curves passing through $3d-1$ general points in $\mathbb{P}^2$. Since there is a unique line passing through 2 points, $N_1 = 1$.

The associativity of quantum cohomology ring of $\mathbb{P}^2$ yields the following relation determining all N_d for $d \geq 2$:

$$N_d = \sum_{d_1+d_2=d,\ d_1,d_2>0} N_{d_1} N_{d_2} \left(d_1^2 d_2^2 \binom{3d-4}{3d_1-2} - d_1^3 d_2 \binom{3d-4}{3d_1-1} \right).$$

Degree zero Gromov-Witten invariants

The moduli space of degree 0 maps to X has a very simple form:

$$\overline{\mathcal{M}}_{g,n}(X,0) = X \times \overline{\mathcal{M}}_{g,n}.$$

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The virtual fundamental class

$$[\overline{\mathcal{M}}_{g,n}(X,0)]^{\mathrm{vir}} = e(T_X \boxtimes \mathbb{E}^\vee) \cap [X \times \overline{\mathcal{M}}_{g,n}].$$

So the degree 0 Gromov-Witten invariants of X are the integrals:

$$\langle \tau_{k_1}(\gamma_{a_1}) \dots \tau_{k_n}(\gamma_{a_n}) \rangle_{g,0}^X = \int_{X \times \overline{\mathcal{M}}_{g,n}} \gamma_{a_1} \dots \gamma_{a_n} \psi_1^{k_1} \dots \psi_n^{k_n} \cup e(T_X \boxtimes \mathbb{E}^\vee).$$

Let $r = \dim X$, $R_a^b \gamma_b = c_1(X) \cup \gamma_a$, $\tilde{t}_k^a = t_k^a - \delta_{a0}\delta_{k1}$ and $[x]_i^k = e_{k+1-i}(x, x+1, \ldots, x+k)$. If $\gamma_a \in H^{p_a,q_a}(X)$, $b_a = p_a + (1-r)/2$.

$$
\begin{aligned}
L_k = \sum_{m=0}^{\infty} \sum_{i=0}^{k+1} \Big(& [b_a + m]_i^k (R^i)_a^b \tilde{t}_m^a \partial_{b,m+k-i} \\
& + \frac{\hbar}{2} (-1)^{m+1} [b_a - m - 1]_i^k (R^i)^{ab} \partial_{a,m} \partial_{b,k-m-i-1} \Big) \\
& + \frac{1}{2\hbar} (R^{k+1})_{ab} t_0^a t_0^b + \frac{\delta_{k0}}{48} \int_X \big((3-r) c_r(X) - 2 c_1(X) c_{r-1}(X) \big).
\end{aligned}
$$

Virasoro conjecture (Eguchi, Hori and Xiong and also by Katz)

$$
L_k(\exp F) = 0, \quad k \geq -1.
$$

Another Witten's conjecture

For Gromov-Witten potential, Witten suggested

$$U_\alpha = \frac{\partial^2 F^X}{\partial t_{0,0} \partial t_{0,\sigma}}, \; U'_\alpha = \frac{\partial^3 F^X}{\partial t_{0,0}^2 \partial t_{0,\sigma}}, \; \cdots, \; U_\sigma^{(k)} = \frac{\partial^{k+2} F^X}{\partial t_{0,0}^{k+1} \partial t_{0,\sigma}}, \quad \text{for} \;\; k \geq 0$$

Generalized Witten Conjecture: *For every $g \geq 0$, there are differential functions $G_{m,\alpha;n,\beta}(U_\alpha, {U_\alpha}', {U_\alpha}'', \cdots)$ such that*

$$\frac{\partial^2 F_g}{\partial \tau_{m,\alpha} \partial \tau_{n,\beta}} = G_{m,\alpha;n,\beta}(U_\alpha, {U_\alpha}', {U_\alpha}'', \cdots)$$

up to terms of genus g.

This conjecture was affirmed in case $X = pt$ or $g = 0$.

W-constraints governing the Gromov-Witten theory

Virasoro conjecture is not effective in computing Gromov-Witten invariants. It has to be combined with topological recursions, as presented by Gathmann.

For the r-spin intersection numbers, there is the W-algebra contraints, which uniquely determines these numbers.

A generally expected conjecture is to find the W-constraints for Gromov-Witten theory.

Thanks a lot!