A PANOPLY OF SPECIAL LAGRANGIAN SINGULARITIES

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1. 'Gluing constructions of special Lagrangian cones', *Handbook of Geometric Analysis*, (Yau birthday volume) Aug 2008
2. 'Special Lagrangian cones with higher genus link', *Inventiones Mathematicae*, 167, 223–294 (2007)
3. 'Special Lagrangian cones', *American Journal of Mathematics* 126, 845–871 (2004)

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Calibrations – Definitions

A *calibrated geometry* is a distinguished class of minimal submanifolds associated with a differential form.

- A *calibrated form* is a closed differential p-form ϕ on a Riemannian manifold (M, g) satisfying $\phi \leq vol_g$.

  i.e. $$\phi(e_1, \ldots, e_p) \leq 1$$

  for any orthonormal set of p tgt vectors

- For $m \in M$ associate with ϕ the subset $G_m(\phi)$ of oriented p-planes for which equality holds in $(*)$ – the *calibrated* planes.

- A submanifold *calibrated* by ϕ is an oriented p-dim submanifold whose tangent plane at each point m lies in the subset $G_m(\phi)$ of distinguished p-planes.

Lemma: (Harvey–Lawson) Calibrated submanifolds minimize volume in their homology class.

Special Lagrangian Calibration on $\mathbb{C}^n$

On $\mathbb{C}^n$ with standard complex coordinates, let

$$\alpha = \mathrm{Re}(\Omega), \text{ where } \Omega = dz^1 \wedge \ldots \wedge dz^n.$$

- α is a calibrated form.
- Any α-calibrated plane is Lagrangian w.r.t. standard Kähler form
$$\omega = \sum dx^i \wedge dy^i$$
(but not conversely).
- An α-calibrated plane may be obtained from standard "real" $\mathbb{R}^n \subseteq \mathbb{C}^n$ by action of $A \in \mathrm{SU}(n)$.

Thus the name *special* Lagrangian calibration.

Other ambient spaces: Calabi-Yau manifolds

If (M, g) has holonomy in SU(n) can use existence of parallel $(n, 0)$-form Ω to define special Lagrangian calibration.

The Lagrangian angle

Oriented Lagn n-planes in $\mathbb{C}^n \leftrightsquigarrow$ $\mathrm{U}(n)/\mathrm{SO}(n)$.

Map $\det_{\mathbb{C}} : \mathrm{U}(n)/\mathrm{SO}(n) \longrightarrow \mathbb{S}^1$ gives us the *phase* of the Lagrangian n-plane.

Write phase of oriented Lagrangian submfd L of $\mathbb{C}^n$ as $e^{i\theta} : L \to \mathbb{S}^1$.

Locally can lift phase $e^{i\theta}$ to a function $\theta : L \to \mathbb{R}$, the *Lagrangian angle*.

θ not necessarily globally well-defined, but $d\theta$ is and has geometric meaning:

$$d\theta = \iota_H \omega$$

where H is mean curvature vector of L

$H = 0 \longleftrightarrow d\theta_L = 0 \longleftrightarrow \theta_L$ locally constant

$\theta_L = 0 \longleftrightarrow L$ is special Lagrangian.

L is *Hamiltonian stationary* if $d\theta$ is a harmonic 1-form

Examples of Special Lagrangian submanifolds in $\mathbb{C}^n$

<u>SLG level sets</u> (Harvey – Lawson 1982)
(explicit examples with symmetries)

$F : \mathbb{C}^3 \to \mathbb{R}^3$
$F = \Big(|z_1|^2 - |z_2|^2,\ |z_1|^2 - |z_3|^2,\ \mathrm{Im}(z_1 z_2 z_3) \Big)$

$F^{-1}(c)$ is a (possibly singular) SLG 3-fold invariant under a T^2 action.

- For a generic c, $F^{-1}(c)$ is nonsingular and diffeomorphic to $T^2 \times \mathbb{R}$.
- For 1-dimensional set of critical values of c, $F^{-1}(c)$ is a smoothly immersed 3-fold, with two components each diffeomorphic to $S^1 \times \mathbb{R}^2$ and intersecting in a circle.
- $F^{-1}(\mathbf{0})$ is a singular cone C.
  Link of C has 2 (antipodal) components.

  Each component is flat 2-torus (with equilateral conformal structure)

Singularities of SLG varieties

Q: What can we say about singularities of SLG varieties?

A: In complete generality, very little!

- Generalize from SLG manifolds to SLG rectifiable currents – objects with measure-theoretic oriented tgt plane a.e.
- SLG rectifiable currents in $\mathbb{C}^n$ are *mass-minimizing* currents
- Almgren's *Codimension Two* or *Big* regularity result for general mass minimizing currents applies
- Very hard and long (1000 page book)!!
- $\Rightarrow$ Singular set S has Hausdorff codim ≥ 2.
- Not enough information about S for geometric applications.

Want to find interesting but more manageable classes of singular SLG varieties.

Isolated conical singularities of SLG n-folds

SLG n-fold with *isolated conical singularities* means:

- Compact singular SLG n-fold X in a Calabi-Yau M.
- Finite number of distinct singular points $x_1, \ldots, x_k$.
- Near the singular point x_i, X looks like a SLG cone C_i in $\mathbb{C}^n$ with isolated singularity.

Joyce studied the deformation theory of these singular SLG varieties

- Keep number and local model of singular points fixed but not their locations.
- Singular deformation theory is obstructed.
- Obstruction space depends on geometry of each SLG cone C_i.

$\Rightarrow$ To understand this class of singular SL n-folds need to study the geometry of SL cones.

SLG Cones and Their Links

Def$^{\mathbf{n}}$: A *cone* $C \subset \mathbb{C}^n$ is a subset invariant under all dilations.

For $\Sigma \subset \mathbb{S}^{2n-1} \subset \mathbb{C}^n$, let $C(\Sigma)$ denote the cone on Σ:

$$C(\Sigma) = \{tx : t \geq 0, x \in \Sigma\}$$

Def$^{\mathbf{n}}$: A cone is *regular* if $C = C(\Sigma)$ where Σ is compact, connected, embedded and oriented submanifold of $\mathbb{S}^{2n-1}$.

Call Σ the *link* of the regular cone C.

Prop$^{\mathbf{n}}$ A: C is SLG in $\mathbb{C}^n$ (up to a phase) $\iff$ Σ is minimal and Legendrian in $\mathbb{S}^{2n-1}$.

SLG cones in dimension 3

Link of a regular SLG cone is a cpt oriented Riemannian surface of genus g.

The trichotomy:

1. $g = 0$. Only example is standard round $\mathbb{S}^2$

2. $g = 1$
   - $\exists$ infinitely many explicit examples with cts symmetries
   - *Integrable systems* (loop group/spectral curve) methods produce all examples

3. $g > 1$
   - Cts symmetry not possible
   - Integrable systems methods not currently effective
   - Can use geometric PDE *gluing* methods to construct many examples (Haskins & Kapouleas)

SL cones in higher dimensions

Th$^{\text{m}}$: (Haskins-Kapouleas 2008-9)
(i) For every $n \geq 3$ there are infinitely many topological types of special Lagrangian cone in $\mathbb{C}^n$ each of which admits infinitely many geometrically distinct representatives.

(ii) For every $n \geq 5$ statement (i) also holds for Hamiltonian stationary (non SL) cones.

(iii) For every $n \geq 6$ there are infinitely many topological types of SL cone in $\mathbb{C}^n$ which can occur in continuous families of arbitrarily large dimension.

(iv) For $n \geq 7$ statement (iii) also holds for Hamiltonian stationary (non SL) cones in $\mathbb{C}^n$.

Ingredients of proof:
A. Integrable systems constructions of SL T^2 cones
B. Gluing constructions of infinitely many topological types of SL cones
C. A *twisted product* construction for pairs of SL cones

A: SLG T^2 cones using integrable systems

Sharipov (1991) studied minimal Legendrian tori in $\mathbb{S}^5$ using *finite gap* methods.
• Gave formulae for immersions in terms of theta functions associated to *spectral data*.
• Did not solve period conditions needed to obtain tori; so could not produce tori.
• Did not prove uniqueness of spectral data.

Minimal Legendrian surfaces in $\mathbb{S}^5$ project via Hopf map $\mathbb{S}^5 \to \mathbb{CP}^2$ to minimal Lagrangians. Conversely, minimal Lagrangian *tori* in $\mathbb{CP}^2$ lift (3-fold cover perhaps) to minimal Legendrian tori in $\mathbb{S}^5$.

• Minimal tori in $\mathbb{CP}^2$ all come from primitive harmonic maps to a flag manifold.
• McIntosh (1995-96) used this to give a spectral curve construction of these tori.
• Lagrangian condition $\Rightarrow$ spectral curve has an extra involution.

A: SLG T^2 cones using integrable systems

McIntosh (2003): Minimal Lagrangian tori in $\mathbb{C}P^2$ (up to congruence) are in bijective correspondence with certain *spectral data*, $(X, \lambda, \rho, \mu, \mathcal{L})$.

- X is a compact Riemann surface of even genus $g = 2p$, $\mathcal{L}$ is a holo line bundle on X.
- $p = 0 \Rightarrow$ must get Clifford torus.
  $p = 1 \Rightarrow$ get S^1-invariant tori (H_ 2000).
  $p = 2$, $\exists$ examples due to Joyce (2001).
- Spectral data gives rise to torus $\Rightarrow$ strong restrictions on spectral data – a kind of "rationality condition".
- Not obvious that for $p > 2$ there are any spectral curves giving rise to tori.

McIntosh-Carberry (2004): For each $p > 2$, there exist countably many spectral curves of genus $2p$, each yielding a $p-2$-dimensional real family of minimal Lagrangian tori.

• periodicity conditions don't change if we vary $\mathcal{L}$ in Jacobian variety
• Lagrangian deformations $\longleftrightarrow$ move $\mathcal{L}$ in Prym

B: SL cones via gluing constructions

Th$^{\mathbf{m}}$ 1: (Haskins-Kapouleas 2007)
For any $d \in \mathbb{N}$ there exist infinitely many SLG cones C with a link a surface of genus $g = 2d+1$. Also true for genus 4.

Th$^{\mathbf{m}}$ 2: (Haskins-Kapouleas 2008-9)
For every $n \geq 3$ there are infinitely many topological types of special Lagrangian cone in $\mathbb{C}^n$ each of which admits infinitely many geometrically distinct representatives.

- Proofs use delicate *gluing* techniques – singular perturbation theory for geometric PDE à la Schoen (singular Yamabe problem), Kapouleas (CMC surfaces) . . .
- Links composed of *large* number of *almost spherical regions* connected to each other via small highly curved regions.
- Reminiscent of use of Delaunay surfaces to construct new CMC surfaces in $\mathbb{R}^3$ by Kapouleas.

Outline of proofs of Theorems 1 & 2

1. Use ODE methods to construct SL analogues of Delaunay cylinders:
   • highly symmetric SL *necklaces* w/ many almost spherical regions connected by highly curved neck regions.
   • $\exists$ finitely many topological types of necklace in each dimension.

2. Combine many SL necklaces to form topologically more complicated Legendrian submanifolds that are approximately minimal.

3. Linear theory.
   (a) Understand linearization $\mathcal{L}$ of condition that graph(f) over Σ is minimal.
   (b) Understand (approx) kernel of $\mathcal{L}$.
   (c) Understand how to compensate for the (approx) kernel of $\mathcal{L}$.

4. Deal with nonlinear terms.
   (a) Set up iteration/ fixed point scheme.
   (b) Prove nonlinear estimates to guarantee convergence.

The Building Blocks for $n = 3$: SL analogues of Delaunay cylinders in $\mathbb{S}^5$

$SO(2)$-invariant SL cylinders in $\mathbb{S}^5$.

SO(2) $\subset$ SO(3): std SO(3) action on $\mathbb{C}^3$
SO(2): stabilizer of $(1,0,0) \in \mathbb{S}^5$.

Thm: (H 2000) There exists a 1-parameter family of SO(2)-invariant SL cylinders

$$X_\tau : \mathbb{R} \times \mathbb{S}^1 \to \mathbb{S}^5$$

interpolating between:

- a flat cylinder when $\tau = \tau_{\text{max}}$, and
- the round sphere $\mathbb{S}^2 \setminus (\pm 1, 0, 0)$ when $\tau = 0$.

For $\tau \sim 0$, X_τ is periodic and consists of infinitely many almost spherical regions (ASRs) connected by small highly curved necks.

For any $m \gg 0$, $\exists !\ \tau_m \sim 0$ s.t. X_{τ_m} factors through an embedded SL torus with exactly m ASRs.

$\Rightarrow$ $\exists$ **embedded SL toroidal necklaces in $\mathbb{S}^5$ with any (sufficiently large) number of almost spherical beads.**

Higher dimensional building blocks I: SO$(n-1)$-invariant SL cylinders in $\mathbb{S}^{2n-1}$

SO$(n-1) \subset$ SO(n): std SO(n) action on $\mathbb{C}^n$

SO$(n-1)$ is stabilizer of $(1, 0, \ldots, 0) \in \mathbb{S}^{2n-1}$.

Thm: (HK 06) There exists a 1-parameter family of SO$(n-1)$-invariant SL cylinders

$$X_\tau : \mathbb{R} \times \mathbb{S}^{n-2} \to \mathbb{S}^{2n-1}$$

interpolating between:

- a "product" cylinder $\mathbb{R} \times \mathbb{S}^{n-2}$ ($\tau = \tau_{\max}$),
- the round sphere $\mathbb{S}^{n-1} \backslash (\pm 1, 0 \ldots, 0)$ ($\tau = 0$).

For $\tau > 0$, X_τ is periodic and for $\tau \sim 0$ consists of infinitely many almost spherical regions (ASRs) connected by small highly curved necks.

For any $m \gg 0$, $\exists!$ $\tau_m \sim 0$ s.t. X_{τ_m} factors through an SL submfd $\mathbb{S}^1 \times \mathbb{S}^{n-2}$ with exactly m ASRs.

Higher dimensional building blocks II: $\mathbf{SO}(p) \times \mathbf{SO}(q)$-invariant SL cylinders

Choose positive integers p and q, s.t. $p+q=n$.
$\mathrm{SO}(p) \times \mathrm{SO}(q) \subset \mathrm{SO}(n)$: std $\mathrm{SO}(n)$ action

Thm: (HK 06) There exists a 1-parameter family of $\mathrm{SO}(p) \times \mathrm{SO}(q)$-invariant SL cylinders

$$X_\tau : \mathbb{R} \times \mathbb{S}^{p-1} \times \mathbb{S}^{q-1} \to \mathbb{S}^{2n-1}$$

interpolating between:

- a "product" cylinder $\mathbb{R} \times \mathbb{S}^{p-1} \times \mathbb{S}^{q-1}$ when $\tau = \tau_{\max}$,
- the round sphere $\mathbb{S}^{n-1} \setminus (\mathbb{S}^{p-1}, 0) \cup (0, \mathbb{S}^{q-1})$ when $\tau = 0$.

For $\tau > 0$, X_τ is periodic and for $\tau \sim 0$ consists of infinitely many almost spherical regions (ASRs) connected by small highly curved necks.
For any $m \gg 0$, $\exists!\ \tau_m \sim 0$ s.t. X_{τ_m} factors through an SL submfd $\mathbb{S}^1 \times \mathbb{S}^{p-1} \times \mathbb{S}^{q-1}$ with exactly m ASRs.

$\Rightarrow$ **for fixed n get SL necklaces of type (p,q) with any (suff large) number of almost spherical beads.**

Geometry of SL cylinders.

I. SO$(n-1)$-invariant SL cylinders:

• for $\tau \sim 0$ curvature of X_τ blows up near 2 antipodal points $\pm e_1$.

• Near $\pm e_1$, $X_\tau \sim$ Lagrangian catenoid in $\mathbb{C}^{n-1}$.

II. SO$(p) \times$ SO(q)-invariant SL cylinders:

• for $\tau \sim 0$ curvature blows along an *orthogonal generalized Hopf link*

$$(\mathbb{S}^{p-1}, 0) \cup (0, \mathbb{S}^{q-1}).$$

• Now have 2 types of highly curved region:

(a) near $(\mathbb{S}^{p-1}, 0)$ X_τ resembles

$$\mathbb{S}^{p-1} \times \text{Lagn catenoid in } \mathbb{C}^q$$

(b) near $(0, \mathbb{S}^{q-1})$ X_τ resembles

$$\text{Lagn catenoid in } \mathbb{C}^p \times \mathbb{S}^{q-1}$$

Adjacent spheres in necklace as $\tau \to 0$.

Case I: consecutive spheres meet at a point.

Case II: consecutive spheres meet along an equatorial sphere of dimension $p-1$ or $q-1$

Construction of initial submanifolds

Basic idea: fuse a finite number of SL necklaces at one common central sphere.

Initial configuration determined by

1. type (p, q) of SL necklaces used
2. position of attachment sets on central sphere
3. number of beads in SL necklace

To simplify treatment of approximate kernel look only at "maximally symmetric" configurations.

$\Rightarrow$

A. only use one necklace type per construction and same number m of beads for all fused necklaces

B. attachment sets must be highly symmetrically arranged

Describe two families of gluing constructions
A. using type $(1, n-1)$ necklaces,
• each attachment set is pair of *antipodal pts*

B. using type (p, p) necklaces
• each attachment set is an *orthogonal generalized Hopf link*

Attachment sets Δ in case A:

(i) $\Delta \subset \mathbb{S}^1 \subset \mathbb{S}^{n-1} \subset \mathbb{S}^{2n-1}$
$\Delta =$ vertices of regular g-gon, g odd plus all its antipodal points.
(ii) $\Delta \subset \mathbb{S}^{k-1} \subset \mathbb{S}^{n-1} \subset \mathbb{S}^{2n-1}$
$\Delta =$ vertices of a regular k-simplex plus all its antipodal points.

Attachment sets Δ in case B:

$$\Delta = \Delta_0 \cup \ldots \cup \Delta_{g-1}, \quad g \text{ odd}$$

$$\Delta_0 = (\mathbb{S}^{p-1}, 0) \cup (0, \mathbb{S}^{p-1})$$

$$\Delta_k = (R_g)^k \Delta_0$$

where

$$R_g = \begin{pmatrix} \cos(2\pi/g) Id_p & \sin(2\pi/g) Id_p \\ -\sin(2\pi/g) Id_p & \cos(2\pi/g) Id_p \end{pmatrix}$$

B: Main Gluing Theorem

Thm: (HK 08-09) For any m sufficiently large the initial Legn submfds constructed using the attachment sets Δ described above and SL necklaces with m beads can be corrected to be special Legendrian.

Cor: For any $n \geq 3$ $\exists$ infinitely many topological types of SL cone in $\mathbb{C}^n$ each of which has infinitely many geometrically distinct representatives.

Source of main technical difficulties in proof:

1. Initial submfds contain $\sim gm$ almost spherical regions where g is number of necklaces fused.

2. Each ASR contributes

$$\dim(SU(n)) - \dim(SO(n))$$

small eigenvalues to $\mathcal{L}$

3. Construction works for m large.

$1-3 \Rightarrow$ linearization $\mathcal{L}$ has huge approx kernel.

C: Twisted products of SL cones

Product cones: C_1, C_2 regular cones in $\mathbb{C}^p$, $\mathbb{C}^q$ $\Rightarrow$

- $C_1 \times C_2 \subset \mathbb{C}^p \times \mathbb{C}^q$ also a cone,
- C_i both Lagn cones $\Rightarrow C_1 \times C_2$ also Lagn
- C_i both SL cones $\Rightarrow C_1 \times C_2$ also SL

But $\Sigma_{1,2}$ the link of $C_1 \times C_2$ is singular.

$$\Sigma_{1,2} = \{\cos t\, \Sigma_1, \sin t\, \Sigma_2\} \subset \mathbb{S}^{2(p+q)-1}.$$

Map from $[0, \frac{\pi}{2}] \times \Sigma_1 \times \Sigma_2 \to \Sigma_{1,2}$ given by

$$(t, \sigma_1, \sigma_2) \mapsto (\cos t\, \sigma_1, \sin t\, \sigma_2)$$

fails to be immersion at $t = 0$ and $t = \frac{\pi}{2}$.

- near $t = 0$ link singularity looks like $\Sigma_1 \times C_2$
- near $t = \frac{\pi}{2}$ singularity looks like $C_1 \times \Sigma_2$

Twisted product cones: look for deformations of $C_1 \times C_2$ to a regular SL cone.

C: Twisted products of SL cones

If $w = (w_1, w_2) : I \to \mathbb{S}^3 \subset \mathbb{C}^2$ define the w-twisted product of Σ_1 with Σ_2 by

$$(t, \sigma_1, \sigma_2) \to (w_1(t)\sigma_1, w_2(t)\sigma_2).$$

Lemma: If w is a Legendrian curve in $\mathbb{S}^3$ and Σ_1, Σ_2 are both Legendrian then away from zeros of w_i the w-twisted product $\Sigma_1 *_w \Sigma_2$ is also Legendrian with phase $e^{i\theta}$ given by

$$(-1)^{p-1} e^{i\theta_1} e^{i\theta_2} e^{i\theta_w + i(p-1)\arg w_1 + i(q-1)\arg w_2}$$

where θ_i, θ_w are Lagn phases of Σ_i and w

Defn: A Legn curve w in $\mathbb{S}^3$ is a (p,q)-*twisted SL curve* if its phase satisfies

$$e^{i\theta_w} = (-1)^{p-1} e^{-i(p-1)\arg w_1 - i(q-1)\arg w_2}.$$

Cor: If Σ_i are both SL then the w-twisted product is SL $\Longleftrightarrow$ w is a (p,q)-twisted SL curve.

Main point: want to find closed (p,q)-twisted SL curves.

C: Twisted products of SL cones

Prop: Any curve $w : I \to \mathbb{C}^2$ satisfying

$$\overline{w}_1 \dot{w}_1 = -\overline{w}_2 \dot{w}_2 = (-1)^p \overline{w}_1^p \, \overline{w}_2^q,$$

with initial condition $|w(0)| = 1$, is a (p,q)-twisted SL curve in $\mathbb{S}^3$.

Conversely, any (p,q)-twisted SL curve in $\mathbb{S}^3$ containing no points with $w_i(t) = 0$ admits a parametrization satisfying these ODEs.

Thm: (HK 08) Up to symmetry $\exists$ a 1-parameter family of (p,q)-twisted SL curves w_τ. For a countable dense set of values of τ these curves are closed.

Cor: HK 08 (i) There exist infinitely many topological types of SL cone in $\mathbb{C}^n$ for $n \geq 3$, each of which admits infinitely many distinct geometric representatives.

(ii) For every $n \geq 6$ there are infinitely many topological types of SL cone in $\mathbb{C}^n$ which can occur in continuous families of arbitrarily large dimension.

Proof: (needs gluing results in $n = 3$ and integrable systems construction of families of SL 2-tori)

(i) For $n = 4$ any closed $(1,3)$-twisted SL curve and any SL surface in $\mathbb{S}^5$ give rise to a SL submanifold of $\mathbb{S}^7$ with link $\mathbb{S}^1 \times \Sigma$.

For $n > 4$ any closed $(n-3,3)$-twisted SL curve, any SL surface Σ in $\mathbb{S}^5$ and any SL submfd $\Sigma_{2(n-3)}$ in $\mathbb{S}^{2(n-3)-1}$ (e.g. the equatorial sphere) gives rise to a SL submfd of $\mathbb{S}^{2n-1}$ with link $\mathbb{S}^1 \times \Sigma \times \Sigma_{2(n-3)}$.

(ii) To get infinitely many top types of SL cones that come in at least a p dimensional cts family of SL cones for $n \geq 6$:

Take any closed $(n-3,3)$-twisted SL curve, Σ to be any SL T^2 of sufficiently high spectral curve genus and $\Sigma_{2(n-3)}$ to be any of the infinitely many top types of SL cones already constructed in $\mathbb{C}^{n-3}$
$\Rightarrow$ get at least at p-diml cts family of SL submfds of $\mathbb{S}^{2n-1}$ with topology $\mathbb{S}^1 \times T^2 \times \Sigma_{2(n-3)}$.