Recent progress in the value distribution of the hyperbolic Gauss map

Yu Kawakami
Kyushu university and OCAMI

December 18, 2008
”Riemann Surfaces, Harmonic Maps and Visualization”

§ 1. Introduction: Value distribution thoery

The little Picard theorem

A noncostant meromorphic function on $\mathbb{C}$ can omit at most 2 values in the Riemann sphere $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$.

※ "2" is the best possible because $f(z) = e^z$ omits 0 and ∞.

Ahlfors showed that

"The geometric meaning of 2 is the Euler number of $\widehat{\mathbb{C}}$"

by using the Nevanlinna theory.

Theorem (Ahlfors)

$f : \mathbb{C} \to N$: a nonconstant holomorphic map

(1) If N is the Riemann sphere, then f can omit at most "2" values.

(2) If N is the 1-dimensional complex torus, then f is surjective, that is, omit "0" values.

(3) If N is the closed Riemann surface with $\gamma \geq 2$, f does not exist.

★ The goal of our talk

The upper bound of the number of exceptional values of the hyperbolic Gauss map of (CMC-1 and flat) surfaces in the hyperbolic 3-space represents the topological data of Riemann surfaces.

§ 2. Preliminaries: The hyperbolic 3-space $\mathbb{H}^3$

$\mathbb{R}^4_1$: the Lorentz-Minkowski 4-space with the Lorentz metric

$$\langle (x_0, x_1, x_2, x_3), (y_0, y_1, y_2, y_3) \rangle = -x_0 y_0 + x_1 y_1 + x_2 y_2 + x_3 y_3 \, .$$

Then the hyperbolic 3-space

$$\mathbb{H}^3 = \{(x_0, x_1, x_2, x_3) \in \mathbb{R}^4_1 \,|\, -(x_0)^2 + (x_1)^2 + (x_2)^2 + (x_3)^2 = -1, x_0 > 0\}$$

with the induced metric from $\mathbb{R}^4_1$, which is a simply connected Riemannian 3-manifold with constant sectional curvature -1.

The Hermite model of $\mathbb{H}^3$

We identify $\mathbb{R}^4_1$ with the set of 2×2 Hermitian matrices Herm(2)$=$ $\{X^* = X\}$ ($X^* := {}^t\overline{X}$) by

$$(x_0, x_1, x_2, x_3) \longleftrightarrow \begin{pmatrix} x_0 + x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_0 - x_3 \end{pmatrix} .$$

In this identification, $\mathbb{H}^3$ is represented as

$$\mathbb{H}^3 = \{aa^* \,|\, a \in SL(2, \mathbb{C})\}$$

(i.e. $\mathbb{H}^3 = SL(2, \mathbb{C})/SU(2)$) with the metric

$$\langle X, Y \rangle = -\frac{1}{2} \operatorname{trace}(X\widetilde{Y}), \quad \langle X, X \rangle = -\det(X),$$

where $\widetilde{Y}$ is the cofactor matrix of Y.

Bryant representation formula for CMC-1 surfaces

Theorem (Bryant, Umehara-Yamada)

$\widetilde{M}$: a simply connected Riemann surf. with a reference point $z_0 \in \widetilde{M}$.
g : a meromorphic function
ω : a holomorphic one-form on $\widetilde{M}$ such that

$$ds^2 = (1 + |g|^2)^2 |\omega|^2$$

is a Riemann metric on $\widetilde{M}$.
Take a holom. imm. $F = (F_{ij})\colon \widetilde{M} \to SL(2, \mathbb{C})$ satisfying $F(z_0) = id$ and

$$F^{-1}dF = \begin{pmatrix} g & -g^2 \\ 1 & -g \end{pmatrix} \omega \,.$$

Then $f\colon \widetilde{M} \to \mathbb{H}^3$ defined by

$$f = FF^*, \quad (F^* := {}^t\overline{F}),$$

is a CMC-1 surface in $\mathbb{H}^3$ and the induced metric of f is ds^2.
The converse is OK!

Definition

(a) The mero. fct. g is called a secondary Gauss map of f.
(b) The pair (g, ω) is called Weierstrass data of f
(c) The holom. 2-diff. $Q = \omega dg$ is called the Hopf differential of f.
(d) F is called a holom. null lift of f. ("null" means $\det(F^{-1}dF) = 0$))

※ Remark

$f : M \to \mathbb{H}^3$: a CMC-1 surface of a (not necessarily simply connected) Riemann surface M.
The holom. null lift F is defined only the universal cover $\widetilde{M}$ of M.
So the W-data (g, ω) is not single-valued on M.

However, the hyperbolic Gauss map and the Hopf differential of f are well-defined on M and have geometric meanings.

The hyperbolic Gauss map of CMC-1 surfaces.

$f\colon M \to \mathbb{H}^3$: a CMC-1 surface, $F\colon \widetilde{M} \to \mathbb{H}^3$: its holom. null lift
The hyperbolic Gauss map G of f is defined by,

$$G = \frac{dF_{11}}{dF_{21}} = \frac{dF_{12}}{dF_{22}}, \quad \text{where } F(z) = \begin{pmatrix} F_{11}(z) & F_{12}(z) \\ F_{21}(z) & F_{22}(z) \end{pmatrix}.$$

Geometric meaning of G

Identifying the ideal boundary $\mathbb{S}^2_\infty$ of $\mathbb{H}^3$ with $\hat{\mathbb{C}}$. Then, G sends each $p \in M$ to the point $G(p)$ at $\mathbb{S}^2_\infty$ reached by the oriented normal geodesics of $\mathbb{H}^3$ that starts at $f(p)$.
In particular, G is a meromorphic function on M!

● The dual surface $f^\sharp$ of f

The inverse matrix F^{-1} is also a holomorphi null immersion.
Thus it produces a new CMC-1 surface $f^\sharp = F^{-1}(F^{-1})^*\colon \widetilde{M} \to \mathbb{H}^3$, the surface is called the dual of f.
By definition, we can define the W-data $(g^\sharp, \omega^\sharp)$ of $f^\sharp$.

Theorem (Umehara-Yamada)

$$g^\sharp = G, \quad \omega^\sharp = -\frac{Q}{dG}, \quad Q^\sharp = -Q, \quad G^\sharp = g\,.$$

So this duality between f and $f^\sharp$ interchanges the roles of the hyperbolic Gauss map and secondary Gauss map.
We call the pair $(G, \omega^\sharp)$ the dual Weierstrass data of f.

The induced metric $ds^{2\sharp}$ of $f^\sharp$ is given by

$$ds^{2\sharp} = (1+|g^\sharp|^2)^2|\omega^\sharp|^2 = (1+|G|^2)^2\left|\frac{Q}{dG}\right|^2.$$

We call the metric $ds^{2\sharp}$ the dual metric of f.

Since the dual metric $ds^{2\sharp}$ is single-valued on M,

we can define the dual total absolute curvature

$$\mathrm{TA}(f^\sharp) := \int_M (-K^\sharp)dA^\sharp = \int_M \frac{4|dG|^2}{(1+|G|^2)^2},$$

$K^\sharp(\leq 0)$: the Gaussian curvature of $ds^{2\sharp}$

$dA^\sharp$: the area element of $ds^{2\sharp}$

$\Rightarrow$ **TA($f^\sharp$) is the area of G on $\widehat{\mathbb{C}}$.**

● The definition of algebraic CMC-1 surfaces

When the dual total absolute curvature of a complete CMC-1 surface is finite, the surface is called an algebraic CMC-1 surface.

Theorem (Braynt, Huber, Z.Yu)
An algebraic CMC-1 surface $f\colon M \to \mathbb{H}^3$ satisfies:
(i) M is biholomorphic to $\overline{M}_\gamma \setminus \{p_1, \ldots, p_k\}$, where $\overline{M}_\gamma$ is a closed Riemann surface of genus γ and $p_j \in \overline{M}_\gamma$ $(j = 1, \ldots, k)$.
(ii) The dual W-data $(G, \omega^\sharp)$ can be extended meromorphically to $\overline{M}_\gamma$.

※ Remark
Theorem (Umehara-Yamada, Z.Yu)
The $ds^{2\sharp}$ is complete (nondeg.) $\Leftrightarrow$ ds^2 is complete (nondeg.)

§ 3. Main result: Ramification estimate for G

Main Result (1) (Ka,2008)

$f\colon M = \overline{M}_\gamma \setminus \{p_1, \ldots, p_k\} \to \mathbb{H}^3$ a non-flat algebraic CMC-1 surface.
d : the degree of G considered as a map on $\overline{M}_\gamma$.
$a_1, \ldots, a_q \in \widehat{\mathbb{C}}$
For each $a_j \in \widehat{\mathbb{C}}$, ν_j : the minimum of the multiplicity of G at $G^{-1}(a_j)$.
(If a_j is the exceptional value of G, then $\nu_j = \infty$.)
Then we have

$$\sum_{j=1}^{q}\left(1 - \frac{1}{\nu_j}\right) \leq 2 + \frac{2}{R}, \quad \frac{1}{R} = \frac{\gamma - 1 + k/2}{d} < 1\,.$$

Corollary (Ka, 2008)
The hyperbolic Gauss map of a non-flat algebraic CMC-1 surface can omit at most 3 values.

※ Remark
However, we do not know the best possible upper bound of the excep. values of G. (The Osserman problem for the hyperbolic Gauss map)
(The case of "2")

● The hyperbolic Gauss map of a catenoid cousin omits 2 values.
In fact, the surface $f\colon M = \mathbb{C} \setminus \{0\} \to \mathbb{H}^3$ is given by

$$g = \frac{1-l^2}{4l} z^l, \qquad Q = \frac{1-l^2}{4z^2} dz^2, \qquad G = z\,,$$

where $l\,(\neq 1)$ is a positive number.

● For some special topological cases, we show that 2 is the best possible upper bound. (Ka, 2008)

Geometric meaning of the upper bound

"$2+2/R$" has the following geometric meaning:

(1) The geometric meaning of "2" is the Euler number of $\widehat{\mathbb{C}}$.

(2) The geometric meaning of "$1/R$" is as follows:

If the universal cover of $M = \overline{M}_\gamma \setminus \{p_1, \dots, p_k\}$ is the unit disk $\mathbb{D}$,

$A_{hyp}(M)$: the area of M w.r.t. the hyperbolic metric on $\mathbb{D}$

$A_{FS}(M)$: the area of M w.r.t. the induced metric $G^*\omega_{FS}$

then

$$\frac{1}{R} = \frac{\gamma - 1 + k/2}{d} = \frac{A_{hyp}(M)}{A_{FS}(M)}.$$

● Proof of Ramification estimate

★ **Point 1.** $R^{-1} < 1$ (Osserman-type inequality)

$\omega^\sharp(= -Q/dG)$ is well-defined on $M = \overline{M}_\gamma \setminus \{p_1, \ldots, p_k\}$.

(1) $ds^{2\sharp}$ is nondegenerate $\Leftrightarrow$ $(\omega^\sharp)_0 = 2(g)_\infty$

(2) $ds^{2\sharp}$ is cplt $\Leftrightarrow$ $\omega^\sharp$ has a pole at each p_j $\Leftrightarrow$ ${\mu_j}^\sharp - d_j \geq 1$

(${\mu_j}^\sharp \in \mathbb{Z}$ is the branching order of G at p_j and $d_j := \mathrm{ord}_{p_j} Q$)

Applying the Riemann-Roch formula to $\omega^\sharp$ on $\overline{M}_\gamma$, we obtain

$$2d - \sum_{j=1}^{k} ({\mu_j}^\sharp - d_j) = 2\gamma - 2\,.$$

Umehara and Yamada showed that ${\mu_j}^\sharp - d_j \geq 2$. Thus we get

$$d = \gamma - 1 + \frac{1}{2}\sum_{j=1}^{k} ({\mu_j}^\sharp - d_j) > \gamma - 1 + \frac{k}{2}, \text{ s.t. } R^{-1} < 1\,.$$

★ **Point 2.** $\sum_{j=1}^{q}\left(1-\frac{1}{\nu_j}\right) \leq 2+\frac{2}{R}$ (Ka-Kobayashi-Miyaoka inequality)

G omits r_0 values

n_0 : the sum of the branching orders at the inverse image of these exceptional values of G

Then

$$k \geq dr_0 - n_0 \Leftrightarrow r_0 \leq \frac{n_0 + k}{d}\,.$$

G has l_0 totally ramified values which are not exceptional values.

$$dl_0 - n_r \leq \sum_{i=1}^{l_0} \frac{d}{\nu_i} \Leftrightarrow l_0 - \sum_{i=1}^{l_0} \frac{1}{\nu_i} \leq \frac{n_r}{d}\,.$$

n_G : the total branching order of G on $\overline{M}_\gamma$

Then applying the Riemann-Hurwitz theorem to the meromorphic function G on $\overline{M}_\gamma$, we obtain

$$n_G = 2(d + \gamma - 1)\,.$$

Therefore

$$\sum_{j=1}^{q}\left(1 - \frac{1}{\nu_j}\right) \leq \frac{n_0 + k + n_r}{d} \leq \frac{n_G + k}{d} \leq 2 + \frac{2}{R}\,.$$

§4. Further Topic: The hyperbolic Gauss map of flat fronts in $\mathbb{H}^3$

The hyperbolic Gauss map in $\mathbb{H}^3$

For each $p \in M$, $\exists (G(p), G_*(p)) \in \widehat{\mathbb{C}} \times \widehat{\mathbb{C}}$ a pair of distinct points on $\mathbb{S}^2_\infty$ s.t. the geod. in $\mathbb{H}^3$ starting from $G^*(p)$ toward $G(p)$ coincides with the oriented normal geodesic at $f(p)$.

Fact 1. (Kokubu-Takahashi-Umehara-Yamada)
If the surface(or front) in $\mathbb{H}^3$ is flat, then G and G_* are holomorphic.
Fact 2. (Ossreman type inequality, Kokubu-Umahara-Yamada)
A cplt flat front $f \colon \overline{M}_\gamma \setminus \{p_1, \ldots, p_k\} \to \mathbb{H}^3$ with regular ends satisfies the inequality

$$\mathrm{deg} G + \mathrm{deg} G_* \geq k \,,$$

where $\mathrm{deg}G$ ($\mathrm{deg}G_*$) is the degree of the holom. map $G(G_*) \colon \overline{M}_\gamma \to \widehat{\mathbb{C}}$
.

Main Result (2) (Ka, 2008 $\boxed{\text{NEW}}$)

$f\colon \overline{M}_\gamma \setminus \{p_1, \ldots, p_k\} \to \mathbb{H}^3$: a cplt flat front with regular ends
If G omits p values and G^* omits q values, then $p \leq 2$ or $q \leq 2$ or

$$\frac{1}{p-2} + \frac{1}{q-2} \geq \frac{k}{2\gamma - 2 + k}.$$

※ The inequality is similar to the case of the Gauss map of minimal surfaces in $\mathbb{R}^4$. (Fujimoto, Ka 2008)

Corollary (Ka, 2008 $\boxed{\text{NEW}}$)

($\gamma = 0$) If $p \geq 4$ and $q \geq 4$, then f is horosphere.
($\gamma = 1$) If $p \geq 5$ and $q \geq 5$, then f is horosphere.
($\gamma \geq 2$) ?

Thank you very much! Let's enjoy the dinner time!!