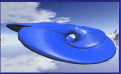
Hamiltonian stationary Lagrangian tori in $\mathbb{C}^2$, revisited

Katrin Leschke

University of Leicester

"Riemann Surfaces, Harmonic Maps and Visualization"
Osaka 2008



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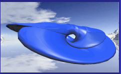

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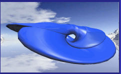

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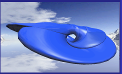
- An immersion $f : M \to \mathbb{R}^4$ of a Riemann surface M into $\mathbb{R}^4 = \mathbb{C}^2$ is called Lagrangian if $f^*\omega = 0$.
- The Gauss map γ of a Lagrangian immersion has values in the space of Lagrangian subspaces $\mathrm{Lag}(\mathbb{R}^4)$:

$$\gamma : M \to \mathrm{Lag}(\mathbb{R}^4)\,.$$



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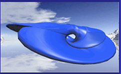

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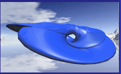

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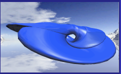
Moreover, when $M = T^2 = \mathbb{C}/\Gamma$ is a 2-torus,

$$\beta(z) = 2\pi < \beta_0, z >$$

where $\beta_0 \in \Gamma^* \subset \mathbb{C}$ is called the Maslov form.



Variational problems

Consider Hamiltonian stationary Lagrangians (HSL), that is immersions $f : M \to \mathbb{C}^2$ which are critical points of the area functional

$$\mathcal{A}(f) = \int_M |df|^2$$

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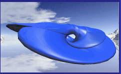

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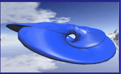
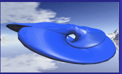
Fact: $f : M \to \mathbb{C}^2$ is Hamiltonian stationary Lagrangian if and only if its Lagrangian angle map β is harmonic.



Results

- Oh: first and second variational formulae of the area functional



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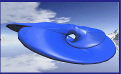

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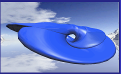

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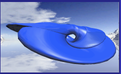
- Castro, Chen, Urbano: non-trivial examples.
- Helein-Romon: complete description of HSL tori by Fourier polynomials; frequencies lie on a circle whose radius is governed by the Maslov class.

Left normal of a HSL surface

Let $f : M \to \mathbb{R}^4$ be a conformal immersion. Then the Gauss map of f is given by

$$(N, R) : M \to S^2 \times S^2 = Gr_2(\mathbb{R}^4)\,.$$



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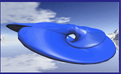

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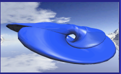

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$$df = e^{\frac{j\beta}{2}}\, dz\, g$$

and the left normal

$$N = e^{j\beta} i$$

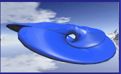
satisfies $*df = Ndf$.

Spectral curves

- Helein-Romon: family of flat connections

$$d^\lambda = \lambda^{-2}\alpha_{-2} + \lambda^{-1}\alpha_{-1} + \alpha_0 + \lambda\alpha_1 + \lambda^2\alpha_2$$

where α_j lie in the eigenspaces of an order 4 autmorphism of the Lie algebra of the group of symplectic isometries of $\mathbb{R}^4$.



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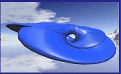

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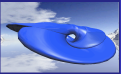

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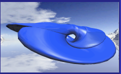
Gives Hamiltonian stationary Lagrangian tori with branch points and "no" control on the branch locus.



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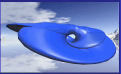
- HSL tori $f : T^2 \to \mathbb{R}^4$ are conformal: have multiplier spectral curve (Schmidt, Taimanov, Bohle-L-Pedit-Pinkall)
- The left normal of a HSL torus $N : T^2 \to S^1$ is harmonic: have spectral curve of the harmonic left normal (Hitchin).

The family of flat connections

A map $N : M \to S^2 \subset \mathrm{Im}\,\mathbb{H}$ is harmonic if and only if the family of complex connections

$$d^\mu = d + (\mu - 1)A^{1,0} + (\mu^{-1} - 1)A^{0,1}$$

on the trivial bundle $\underline{\mathbb{H}}$ is flat, where $A = \frac{1}{4}(*dN + NdN)$ and $\mu \in \mathbb{C}_*$.



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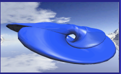

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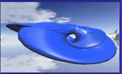
Here $\mathbb{C} = \mathrm{span}\{1, I\}$ where the complex structure I on $\underline{\mathbb{H}}$ is defined by right multiplication by i.
Moreover, for $\omega \in \Omega^1$

$$\omega^{1,0} = \frac{1}{2}(\omega - I * \omega), \quad \omega^{0,1} = \frac{1}{2}(\omega + I * \omega)$$

denote the $(1,0)$ and $(0,1)$ parts with respect to the complex structure I.

The spectral curve of a harmonic map [Hitchin]

Let $N : M \to S^2$ and d^μ the associated family of flat connections.



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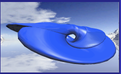
  are the eigenvectors of the monodromy of d^μ.



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  are the eigenvectors of the monodromy of d^μ.
- The spectral curve Σ_e of $N : T^2 \to S^2$ is the normalization of

$$\text{Eig} := \{(\mu, h) \mid \exists \alpha : d^\mu\alpha = 0, \gamma^*\alpha = \alpha h_\gamma, \gamma \in \Gamma\}$$

The eigenline bundle [Hitchin]

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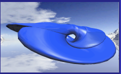
- Generically, the space of parallel sections of d^μ with a given multiplier is 1-dimensional, and one obtains the eigenline bundle $\mathcal{E} \to \Sigma_e$.



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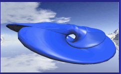
- The harmonic map can be reconstructed by linear flow in the Jacobian of Σ_e.

The spectral curve of the left normal

Let $f : T^2 \to \mathbb{C}^2$ be a Hamiltonian stationary Lagrangian torus with harmonic left normal N and family of flat connections d^μ.



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Theorem (L-Romon, Moriya)

All parallel sections with multiplier can be computed explicitely:

$$\alpha^\mu_\pm = e^{j\frac{\beta}{2}}(1 \mp k\sqrt{\mu}^{-1})e^{\pm 2\pi(<A^\mu,.>+i<C^\mu,.>)}$$

with $A^\mu = \frac{i\beta_0}{4}(\sqrt{\mu}^{-1} - \overline{\sqrt{\mu}})$, $C^\mu = \frac{\beta_0}{4}(\sqrt{\mu}^{-1} + \overline{\sqrt{\mu}})$.

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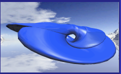

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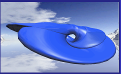

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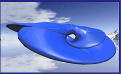
- $\mu : \bar{\Sigma}_e \to \mathbb{CP}^1, (\mu, h) \mapsto \mu$ *is a 2-fold covering, branched over* $0, \infty$.



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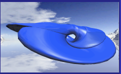

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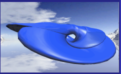
- *The eigenline bundle $\mathcal{E}$ extends holomorphically to $\bar{\Sigma}_e$.*
- *Let $J \in \Gamma(\mathrm{End}(\underline{\mathbb{H}})), J^2 = -1$, be the complex structure given by the quaternionic extension of*

$$J|_{\mathcal{E}_{x_\infty}} = I|_{\mathcal{E}_{x_\infty}}, \quad \mu(x_\infty) = \infty .$$



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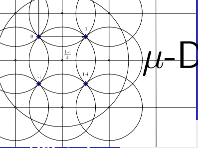
$$J|_{\mathcal{E}_{x_\infty}} = I|_{\mathcal{E}_{x_\infty}}, \quad \mu(x_\infty) = \infty\,.$$

Then J is in fact the complex structure given by left multiplication by N.



μ-Darboux transforms

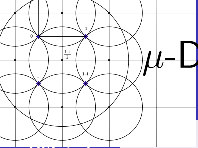
Let $f : M \to \mathbb{C}^2$, be a Hamiltonian stationary Lagrangian torus with harmonic left normal N.

Theorem (L-Romon)

Let $\alpha \in \Gamma(\underline{\mathbb{H}})$ be a parallel section of d^μ and put

$$T^{-1} = \frac{1}{2}(N\alpha(a-1)\alpha^{-1} + \alpha b \alpha^{-1})$$

$a = \frac{\mu+\mu^{-1}}{2}$, $b = i\frac{\mu^{-1}-\mu}{2}$.



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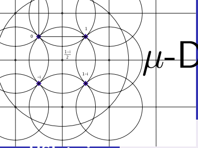
Then

$$\hat{N} = -TNT^{-1}$$

is a harmonic map $\hat{N} : M \to S^2$ of M into the 2-sphere.



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Let $f : M \to \mathbb{C}^2$, be a Hamiltonian stationary Lagrangian torus with harmonic left normal N.

Theorem (L-Romon)

Let α be a parallel section with multiplier and put

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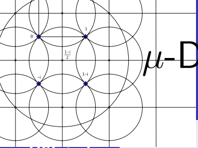

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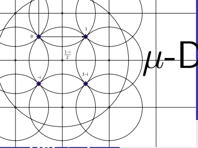

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Let $f : T^2 \to \mathbb{C}^2$, be a Hamiltonian stationary Lagrangian torus with harmonic left normal N and $df = e^{\frac{j\beta}{2}} dzg$.

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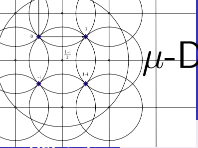
If $\alpha \in \Gamma(\underline{\mathbb{H}})$ is a parallel section with multiplier, than $\hat{N}$ is the left normal of a HSL torus

$$\hat{f} = f + TH^{-1},$$

where $H = \pi g^{-1} \bar{\beta}_0 e^{\frac{j\beta}{2}} k$.



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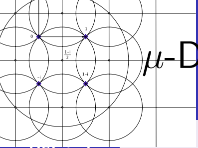
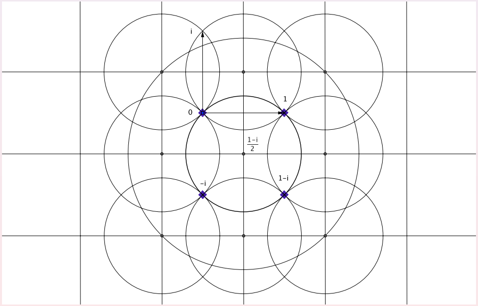
We call $\hat{f}$ a μ-Darboux transform of f.



μ-Darboux transforms

Remark

- *Locally, a μ-Darboux transform is always at least constrained Willmore .*



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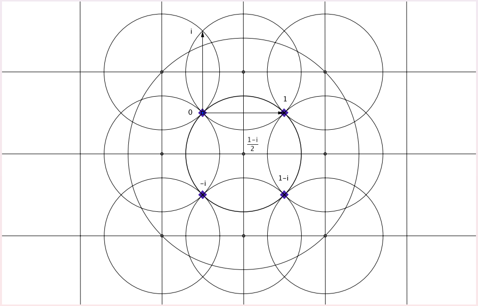
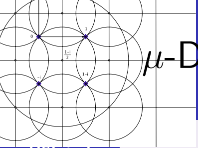


- *A similar theorem holds both for μ-Darboux transforms of CMC tori (Carberry-L-Pedit), and (constrained) Willmore tori (Bohle).*



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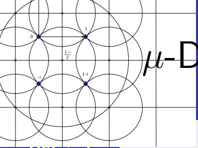

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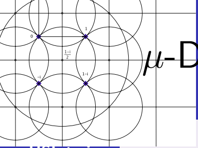

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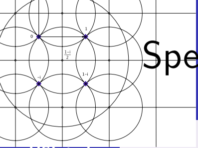
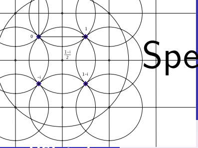
- *The μ-Darboux transformation satisfies a weaker enveloping condition.*



Special cases

For special parameter μ the transform on the left normal is trivial



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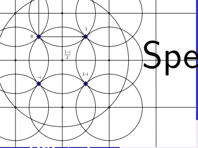

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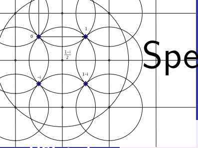

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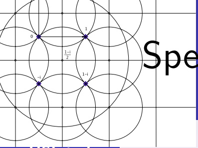

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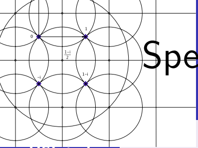

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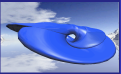
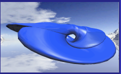
What is the condition for the existence of a HSL torus with Lagrangian angle β?



Holomorphic sections with multiplier

Recall: A Hamiltonian stationary Lagrangian immersion f has Lagrangian angle $\beta \iff *df = Ndf$ with $N = e^{j\beta}i$.



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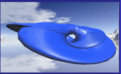
The operator $D : \Gamma(\underline{\mathbb{H}}) \to \Gamma(\bar{K}\underline{\mathbb{H}})$

$$D := \frac{1}{2}(d + J * d)$$

is a (quaternionic) holomorphic structure where the complex structure J on $\underline{\mathbb{H}}$ is given by left multiplication by N.



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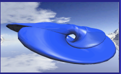

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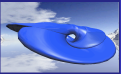

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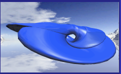

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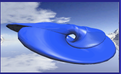
Let $f : \mathbb{C}/\Gamma \to \mathbb{R}^4$ be a Hamiltonian stationary torus. For $(A, B) \in \mathbb{C}^2$ consider

$$|\delta - B|^2 - |A|^2 = \frac{|\beta_0|^2}{4}, \quad < A, \delta - B >= 0 \tag{1}$$

with $\delta \in \Gamma^* + \frac{\beta_0}{2}$.



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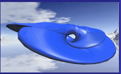
the set of admissible frequencies.



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Let $f : \mathbb{C}/\Gamma \to \mathbb{R}^4$ be a Hamiltonian stationary torus, and D the quaternionic holomorphic structure given by the complex structure J.

Theorem (L-Romon)

- *Multipliers of holomorphic sections are exactly given by*

$$h^{A,B} = e^{2\pi(<A,\cdot>-i<B,\cdot>)}$$

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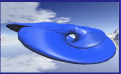

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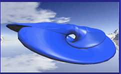
$$\alpha_\delta = e^{\frac{j\beta}{2}}(1 - k\lambda_\delta)e^{2\pi(<A,\cdot>+<\delta-B,\cdot>)}$$

  with $\lambda_\delta \in \mathbb{C}_*$ *is a (monochromatic) holomorphic section.*



HSL tori with prescribed Lagrangian angle [Helein-Romon, L-Romon]

Let Γ be a lattice in $\mathbb{C}$, and let $\beta_0 \in \Gamma^*$. Then $\beta = 2\pi < \beta_0, >$ is a Lagrangian angle of a Hamiltonian stationary torus f if and only if

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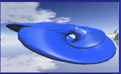

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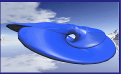

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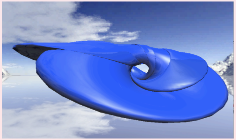

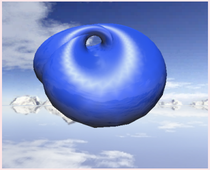

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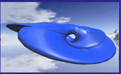
- $|\Gamma^*_{A,B}| = 1$ *away from a discrete set of pairs* (A, B).

The multiplier spectral curve

Let Spec $:= \{h \mid \exists \alpha \in \ker D : \gamma^* \alpha = \alpha h_\gamma, \gamma \in \Gamma\}$, and Σ its normalization.



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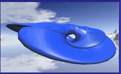
(see Bohle-L-Pedit-Pinkall for general conformal tori).



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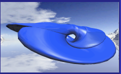

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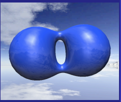
- *In particular, the multiplier spectral curve of a Hamiltonian stationary torus can be compactified and has geometric genus 0.*



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$$\hat{f} = f + TH^{-1}$$

of a HSL torus f where $T = \alpha\beta^{-1}$, $d\alpha = -dfH\beta$ and $H = \pi g^{-1}\bar{\beta}_0 e^{\frac{j\beta}{2}} k$.

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Theorem (L-Romon)

- *If $\alpha = \alpha_\delta$ is a monochromatic holomorphic section then $\hat{f}$ is HSL with (after reparametrization) Lagrangian angle β.*

Darboux transforms

Again, we can use holomorphic sections with multiplier to define a Darboux transform

$$\hat{f} = f + TH^{-1}$$

of a HSL torus f where $T = \alpha\beta^{-1}$, $d\alpha = -dfH\beta$ and $H = \pi g^{-1}\bar{\beta}_0 e^{\frac{j\beta}{2}} k$.

Theorem (L-Romon)

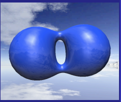
- *If $\alpha = \alpha_\delta$ is a monochromatic holomorphic section then $\hat{f}$ is HSL with (after reparametrization) Lagrangian angle β.*
- *f is obtained as limit of Darboux transforms with multiplier $\sigma \to \sigma_\infty \in \bar{\Sigma}$.*

Darboux transforms

The Darboux transformation is a further generalization of the μ-Darboux transformation:

Theorem (L-Romon)

The monochromatic holomorphic sections are exactly the d^μ-parallel sections for some $\mu \in \mathbb{C}_*$.



Darboux transforms

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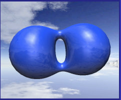
Theorem (L-Romon)

The monochromatic holomorphic sections are exactly the d^μ-parallel sections for some $\mu \in \mathbb{C}_*$.

In other words, the monochromatic Darboux transforms are exactly the μ-Darboux transforms.



Darboux transforms

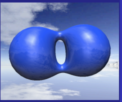
Theorem (L-Romon)

- *If* $|\Gamma^*_{0,0}| = 4$ *then all monochromatic Darboux transforms are after reparametrization* f.

Darboux transforms

Theorem (L-Romon)

- *If* $|\Gamma^*_{0,0}| = 4$ *then all monochromatic Darboux transforms are after reparametrization* f.
- *However, there exist examples where* $|\Gamma^*_{0,0}| > 4$, *and the resulting monochromatic Darboux transforms are not Möbius transformations of* f.



Darboux transforms

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Darboux transforms

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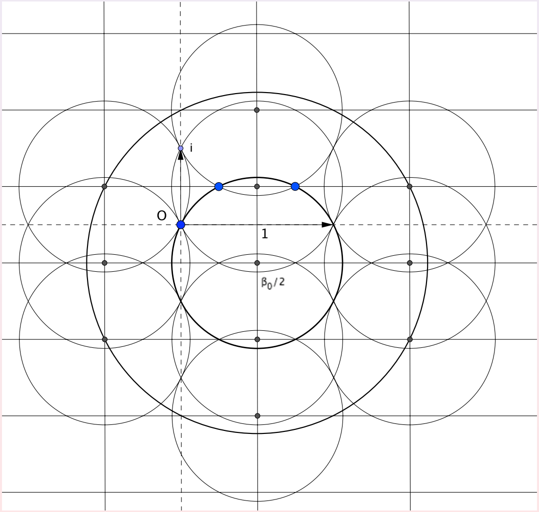
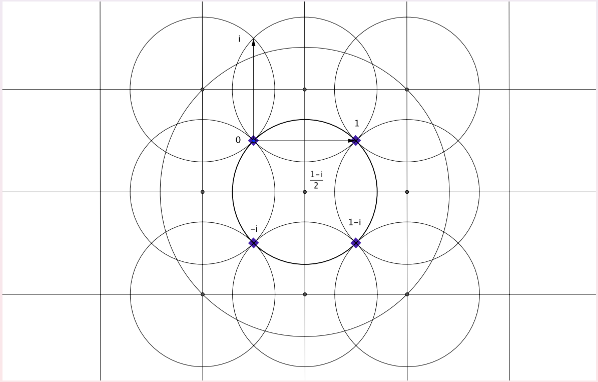
- *However, there exist examples where* $|\Gamma^*_{0,0}| > 4$, *and the resulting monochromatic Darboux transforms are not Möbius transformations of* f.
- *Moreover, there exists HSL tori with polychromatic holomorphic sections* α *so that the corresponding Darboux transforms are not Lagrangian in* $\mathbb{C}^2$.

Homogeneous torus



Clifford torus



Thanks!