Recent Results on Moduli Spaces

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Plan of the Lecture

1. Algebraic Geometric Aspect, with Hao Xu

An explicit recursion formula for the Witten n-point function, the proof of the Faber intersection number conjecture and the derivation of many new recursion formulas.

2. Differential Geometric Aspect, with Xiaofeng Sun and Yau

Proofs of the goodness of the Weil-Petersson metric, the Ricci and the perturbed Ricci metric, and the Kähler-Einstein metric; the dual Nakano negativity of the Weil-Petersson metric and various corollaries about the geometry of moduli spaces.

3. Geometry of Calabi-Yau Moduli, with X. Sun, A. Todorov and Yau

New results about the Teichmüller and moduli spaces of Calabi-Yau manifolds, such as the existence of complete Kähler-Einstein metric on the Hodge metric completion, and the proof of the global Torelli theorem.

Construction of Moduli Spaces of Curves

A point in the moduli space $\mathcal{M}_{g,n}$ is a Riemann surface with n smooth marked points. $\mathcal{M}_{g,n}$ is a quasi-projective orbifold. Its compactification $\overline{\mathcal{M}}_{g,n}$ by adding cusp curves is a projective orbifold.

Constructions $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$

- The quotient of Teichmüller spaces by the action of the mapping class group (Ahlfors, Bers, etc).
- Geometric invariant theory (Mumford, Gieseker, etc).
- Deligne-Mumford compactification.

The compactification is given by a normal crossing divisor.

Stability condition: $2g + n - 2 > 0$.

Boundary of Moduli Spaces of Curves

Singularity of the curves: double points $\{(x, y) \in \mathbb{C}^2 \mid xy = 0\}$

Boundary morphisms by gluing marked points:

$$\overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \longrightarrow \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2}$$
$$\overline{\mathcal{M}}_{g-1,n+2} \longrightarrow \overline{\mathcal{M}}_{g,n}$$

Mumford defined the Chow ring $A^*(\overline{\mathcal{M}}_{g,n})$ in 1983. It is more reasonable to consider the tautological subring

$$R^*(\overline{\mathcal{M}}_{g,n}) \subset A^*(\overline{\mathcal{M}}_{g,n})$$

Since $A^*(\overline{\mathcal{M}}_{g,n})$ is too large and $R^*(\overline{\mathcal{M}}_{g,n})$ contains all geometrically natural classes.

Some tautological classes on $\overline{\mathcal{M}}_{g,n}$

- ψ_i the first Chern class of the line bundle whose fiber over each pointed stable curve is the cotangent line at the ith marked point.
- $\lambda_i = c_i(\mathbb{E})$ the ith Chern class of the Hodge bundle $\mathbb{E}$ with fibre $H^0(C, \omega_C)$.
- κ classes originally defined by Miller-Morita-Mumford on $\overline{\mathcal{M}}_g$ and generalized to $\overline{\mathcal{M}}_{g,n}$ by Arbarello-Cornalba.

We have the morphism forgetting the last marked point

$$\pi : \overline{\mathcal{M}}_{g,n+1} \longrightarrow \overline{\mathcal{M}}_{g,n}.$$

Denote by $\sigma_1, \ldots, \sigma_n$ the canonical sections of π, and by $D_1, \ldots, D_n$ the corresponding divisors in $\overline{\mathcal{M}}_{g,n+1}$. Let ω_π be the relative dualizing sheaf.

$$\psi_i = c_1(\sigma_i^*(\omega_\pi)) \qquad \kappa_i = \pi_*\left(c_1\left(\omega_\pi\left(\sum D_i\right)\right)^{i+1}\right)$$

$$\lambda_\ell = c_\ell(\pi_*(\omega_\pi)), \quad 1 \le \ell \le g.$$

Hodge integrals

$$\langle \tau_{d_1} \cdots \tau_{d_n} \mid \lambda_{b_1} \cdots \lambda_{b_k} \rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \lambda_{b_1} \cdots \lambda_{b_k}.$$

Mariño-Vafa Formula

Consider the total Chern class of the Hodge bundle

$$\Lambda_g^\vee(u) = u^g - \lambda_1 u^{g-1} + \cdots + (-1)^g \lambda_g .$$

For each partition $\mu = (\mu_1 \geq \cdots \geq \mu_{l(\mu)} \geq 0)$, define triple Hodge integral

$$G_{g,\mu}(\tau) = A(\tau) \cdot \int_{\overline{\mathcal{M}}_{g,l(\mu)}} \frac{\Lambda_g^\vee(1)\Lambda_g^\vee(-\tau-1)\Lambda_g^\vee(\tau)}{\prod_{i=1}^{l(\mu)}(1-\mu_i\psi_i)},$$

with

$$A(\tau) = -\frac{\sqrt{-1}^{|\mu|+l(\mu)}}{|\mathrm{Aut}(\mu)|}[\tau(\tau+1)]^{l(\mu)-1} \prod_{i=1}^{l(\mu)} \frac{\prod_{a=1}^{\mu_i-1}(\mu_i\tau+a)}{(\mu_i-1)!}.$$

Introduce the generating series

$$G_\mu(\lambda;\tau) = \sum_{g\geq 0} \lambda^{2g-2+l(\mu)} G_{g,\mu}(\tau).$$

Mariño-Vafa Conjecture

Each $G_\mu(\lambda,\tau)$ is given by a finite and closed expression in terms of Chern-Simons knot invariants, proved by C. Liu-Liu-Zhou in 2003:

$$G_\mu(\lambda,\tau)=\sum_{n\geq 1}\frac{(-1)^{n-1}}{n}\sum_{\cup_{i=1}^n\mu^i=\mu}\prod_{i=1}^{n}\sum_{|\nu^i|=|\mu^i|}\frac{\chi_{\nu^i}(C(\mu^i))}{z_{\mu^i}}e^{\sqrt{-1}(\tau+\frac{1}{2})\kappa_{\nu^i}\lambda/2}\mathcal{W}_{\nu^i}(\lambda).$$

Here the quantum dimension

$$\mathcal{W}_\mu(q)=q^{\kappa_\mu/4}\prod_{1\leq i<j\leq \ell(\mu)}\frac{[\mu_i-\mu_j+j-i]}{[j-i]}\prod_{i=1}^{\ell(\mu)}\frac{1}{\prod_{v=1}^{\mu_i}[v-i+\ell(\mu)]}$$

$$\kappa_\mu=|\mu|+\sum_i(\mu_i^2-2i\mu_i),\quad [m]=q^{m/2}-q^{-m/2}$$

$$q=e^{\sqrt{-1}\lambda},\quad z_\mu=\prod_j\mu_j!j^{\mu_j}.$$

χ_μ: the character of the representation of symmetric group $S_{|\mu|}$ indexed by μ with $|\mu|=\sum_j\mu_j$; $C(\mu)$: the conjugacy class of $S_{|\mu|}$ indexed by μ.

String Duality

String duality has produced many remarkable mathematical formulas related to moduli spaces.

Topological vertex theory was first developed in string theory by Aganagic-Klemm-Mariño-Vafa in 2003. The mathematical theory of topological vertex was developed by Li-C. Liu-Liu-Zhou in 2004. The theory gives close formulas for generating series of Gromov-Witten invariants of all degree and all genera for all (formal) toric Calabi-Yau manifolds, in terms of Chern-Simons knot invariants.

A remarkable conjecture of Labastilda-Mariño-Ooguri-Vafa about the new algebraic and integrality structures of the generating series of the $SU(N)$ quantum knot invariants was proved by Liu-Pan Peng.

String theorists made these conjectures based on the Chern-Simons and string duality, through the 't Hooft large N expansion.

Witten Conjecture

Correlation functions of 2D gravity

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n}, \qquad \sum_{j=1}^{n} d_j = 3g - 3 + n.$$

The Witten Conjecture, proved by Kontsevich, states that the generating function

$$F(t_0, t_1, \ldots) = \sum_g \sum_{\mathbf{n}} \langle \prod_{i=0}^{\infty} \tau_i^{n_i} \rangle_g \prod_{i=0}^{\infty} \frac{t_i^{n_i}}{n_i!}$$

is a τ-function for the KdV hierarchy.

Witten's pioneering work revolutionized this field and motivated a surge of subsequent developments: Gromov-Witten theory, Faber's conjecture, Virasoro conjecture. . . .

DVV Recursion Formula

The following equivalent formulation of Witten's conjecture is due to Dijkgraaf-E.Verlinde-H.Verlinde.

$$
\begin{aligned}
\langle \tau_{k+1} \prod_{j=1}^{n} \tau_{d_j} \rangle_g = \frac{1}{(2k+3)!!} \Bigg[& \sum_{j=1}^{n} \frac{(2k+2d_j+1)!!}{(2d_j-1)!!} \langle \tau_{d_1} \cdots \tau_{d_j+k} \cdots \tau_{d_n} \rangle_g \\
& + \frac{1}{2} \sum_{r+s=k-1} (2r+1)!!(2s+1)!! \langle \tau_r \tau_s \tau_{d_1} \cdots \tau_{d_n} \rangle_{g-1} \\
& + \frac{1}{2} \sum_{r+s=k-1} (2r+1)!!(2s+1)!! \sum_{\underline{n}=I \coprod J} \langle \tau_r \prod_{i\in I} \tau_{d_i} \rangle_{g'} \langle \tau_s \prod_{i \in J} \tau_{d_i} \rangle_{g-g'} \Bigg]
\end{aligned}
$$

where $\underline{n} = \{1, 2, \ldots, n\}$.

Different Proofs of Witten's conjecture

- Kontsevich's combinatorial model for the intersection numbers and matrix integral.
- Via ELSV formula that relates intersection numbers to Hurwitz numbers (Okounkov-Pandharipande, Kazarian-Lando, Chen-Li-Liu).

$$H_{g,\mu} = \frac{(2g-2+|\mu|+l)!}{|\mathrm{Aut}(\mu)|} \prod_{i=1}^{l} \frac{\mu_i^{\mu_i}}{\mu_i!} \int_{\overline{\mathcal{M}}_{g,l}} \frac{\Lambda_g^\vee(-1)}{\prod_{i=1}^{l}(1-\mu_i\psi_i)}.$$

- Mirzakhani's recursion formula of Weil-Petersson volumes.
- Kim-Liu's proof by localization on moduli spaces of relative stable morphisms and asymptotic analysis.

Taking various limits of the Mariño-Vafa formula gives ELSV formula, the Witten conjecture, the λ_g conjecture and many other Hodge integral identities. Mariño-Vafa formula is like a K-theory version of the Witten conjecture.

n-Point Functions of Intersection Numbers

Definition

We call the following generating function

$$F(x_1, \ldots, x_n) = \sum_{g=0}^{\infty} \sum_{\sum d_j = 3g-3+n} \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g \prod_{j=1}^{n} x_j^{d_j}$$

the n-point function.

Note that n-point functions encoded all information of intersection numbers on moduli spaces of curves.

Two and Three-Point Functions

Let $G(x_1,\ldots,x_n) = \exp\left(\frac{-\sum_{j=1}^n x_j^3}{24}\right)\cdot F(x_1,\ldots,x_n)$.

(1) Witten's One-point function $G(x) = \frac{1}{x^2}$
(2) R. Dijkgraaf's two-point function (1993)

$$G(x,y) = \frac{1}{x+y}\sum_{k=0}^{\infty}\frac{k!}{(2k+1)!}\left(\frac{1}{2}xy(x+y)\right)^k.$$

(3) Don Zagier's three-point function (1996)

$$G(x,y,z) = \sum_{r,s\geq 0}\frac{r!S_r(x,y,z)}{4^r(2r+1)!!}\frac{(\Delta/8)^s}{(r+s+1)!},$$

where S_r and Δ are the homogeneous symmetric polynomials

$$S_r(x,y,z) = \frac{(xy)^r(x+y)^{r+1}+(yz)^r(y+z)^{r+1}+(zx)^r(z+x)^{r+1}}{2(x+y+z)},$$

$$\Delta(x,y,z) = (x+y)(y+z)(z+x) = \frac{(x+y+z)^3}{3}-\frac{x^3+y^3+z^3}{3}.$$

Explicit Recursive Formula of n-Point Functions

Let $n \geq 2$, we have

$$G(x_1, \ldots, x_n) = \sum_{r,s \geq 0} \frac{(2r+n-3)!! P_r(x_1, \ldots, x_n) \Delta(x_1, \ldots, x_n)^s}{4^s (2r+2s+n-1)!!}$$

where P_r and Δ are homogeneous symmetric polynomials

$$\Delta = \frac{(\sum_{j=1}^n x_j)^3 - \sum_{j=1}^n x_j^3}{3},$$

$$P_r = \left(\frac{1}{2 \sum_{j=1}^n x_j} \sum_{\underline{n} = I \coprod J} (\sum_{i \in I} x_i)^2 (\sum_{i \in J} x_i)^2 G(x_I) G(x_J) \right)_{3r+n-3}$$

$$= \frac{1}{2 \sum_{j=1}^n x_j} \sum_{\underline{n} = I \coprod J} (\sum_{i \in I} x_i)^2 (\sum_{i \in J} x_i)^2 \sum_{r'=0}^{r} G_{r'}(x_I) G_{r-r'}(x_J).$$

Proof of n-point Function Formula

We have two proofs of the theorem. The first proof is to check that $G(x_1, \ldots, x_n)$ satisfies an ODE coming from Witten's KdV equation with initial condition given by string equation.

The second proof is to re-write the Witten-Kontsevich theorem as the recursion which determines the n-point function F uniquely:

$$\sum_{g=0}^{\infty}(2g+n-1)\left(\sum_{i=1}^{n} x_i\right) F_g(x_1, \ldots, x_n) = \frac{1}{12}\left(\sum_{i=1}^{n} x_i\right)^4 F(x_1, \ldots, x_n)$$

$$+\frac{1}{2}\sum_{\underline{n}=I\coprod J}\left(\sum_{i\in I} x_i\right)^2\left(\sum_{i\in J} x_i\right)^2 F(x_I)F(x_J).$$

Effective Recursion Formula

By taking the coefficients of the n-point function we get many new recursion formulas. The following recursion formula may be used to compute intersection numbers more effectively.

$$
\begin{aligned}
&(2g+n-1)(2g+n-2)\langle\prod_{j=1}^{n}\tau_{d_j}\rangle_g \\
&\quad = \frac{2d_1+3}{12}\langle\tau_0^4\tau_{d_1+1}\prod_{j=2}^{n}\tau_{d_j}\rangle_{g-1} - \frac{2g+n-1}{6}\langle\tau_0^3\prod_{j=1}^{n}\tau_{d_j}\rangle_{g-1} \\
&\quad + \sum_{\{2,\ldots,n\}=I\coprod J}(2d_1+3)\langle\tau_{d_1+1}\tau_0^2\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\tau_0^2\prod_{i\in J}\tau_{d_i}\rangle_{g-g'} \\
&\quad - \sum_{\{2,\ldots,n\}=I\coprod J}(2g+n-1)\langle\tau_{d_1}\tau_0\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\tau_0^2\prod_{i\in J}\tau_{d_i}\rangle_{g-g'}.
\end{aligned}
$$

When indices $d_j \geq 1$, all non-zero intersection indices on the right hands have genera strictly less than g.

Faber's Conjecture on Tautological Rings

In 1993, Carel Faber proposed his remarkable conjectures about the structure of tautological ring $\mathcal{R}^*(\mathcal{M}_g)$, which is generated by $\kappa_1, \ldots, \kappa_{g-2}$. Faber's conjecture motivated a tremendous progress toward understanding of the topology of moduli spaces of curves.

Faber's conjecture is a fundamental question about moduli space. For example, see the discussions in the monographs *Moduli of Curves* (Harris, Morrison, pp. 68-70) and *Intersection Theory over Moduli Spaces of Curves* (L. Gatto, pp. 148-155).

Many experts have made important contributions to Faber's conjecture and its generalizations.

Faber Intersection Number Conjecture

An important part, which is the only quantitative part, of Faber's conjectures is the famous Faber intersection number conjecture, which can be formulated as recursion:

$$\frac{(2g-3+n)!}{2^{2g-1}(2g-1)!\prod_{j=1}^{n}(2d_j-1)!!} = \langle \tau_{2g} \prod_{j=1}^{n} \tau_{d_j} \rangle_g$$

$$- \sum_{j=1}^{n} \langle \tau_{d_j+2g-1} \prod_{i \neq j} \tau_{d_i} \rangle_g + \frac{1}{2} \sum_{j=0}^{2g-2} (-1)^j \langle \tau_{2g-2-j} \tau_j \prod_{i=1}^{n} \tau_{d_i} \rangle_{g-1}$$

$$+ \frac{1}{2} \sum_{\underline{n}=I \coprod J} \sum_{j=0}^{2g-2} (-1)^j \langle \tau_j \prod_{i \in I} \tau_{d_i} \rangle_{g'} \langle \tau_{2g-2-j} \prod_{i \in J} \tau_{d_i} \rangle_{g-g'},$$

Importance of the Conjecture

The Faber intersection number conjecture computes all top intersections in the tautological ring $\mathcal{R}^*(\mathcal{M}_g)$ and determines its ring structure:

$$\kappa_1^{a_1} \cdots \kappa_{g-2}^{a_1} = C(a_1, \ldots, a_{g-2}) \cdot \kappa_{g-2}$$

where C are some constants with explicit formulae.

In March 2008, the Banff Math Institute and the Clay Math Institute held a joint workshop devoted to the understanding of the Faber intersection number conjecture.

$\lambda_g\lambda_{g-1}$ Integral

By using the Mumford's formula for the Chern character of Hodge bundles, one can show that the Faber intersection number conjecture is equivalent to the $\lambda_g\lambda_{g-1}$ integral formula derived by Getzler and Pandharipande (1998) from the degree 0 Virasoro conjecture for $\mathbb{P}^2$.

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \lambda_g \lambda_{g-1} = \frac{(2g-3+n)!|B_{2g}|}{2^{2g-1}(2g)! \prod_{j=1}^n (2d_j-1)!!}.$$

Givental announced a proof of Virasoro conjecture for $\mathbb{P}^n$ in 2001. Y.-P. Lee and R. Pandharipande are still writing a book giving details.

Tautological Relations in $\mathcal{R}^*(\mathcal{M}_g)$

The most useful form of the Faber intersection number conjecture is given by the following tautological relation in $\mathcal{R}^*(\mathcal{M}_g)$

$$\pi_*(\psi_1^{d_1+1}\cdots\psi_n^{d_n+1}) = \frac{(2g-3+n)!(2g-1)!!}{(2g-1)!\prod_{j=1}^n(2d_j+1)!!}\kappa_{g-2},$$

where $\pi : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_g$ is the morphism of forgetting marked points.

In 2006, Goulden, Jackson and Vakil gave a proof of the above relation when $n \le 3$ and arbitrary g. Their remarkable proof uses relative virtual localization and a combinatorialization of the Hodge integrals.

Proof of Faber Intersection Number Conjecture I

Our approach to Faber intersection number conjecture is to prove the following three identities. In fact, we proved much more.

$$(1)\qquad \langle\prod_{j=1}^{n}\tau_{d_j}\tau_{2g}\rangle_g=\sum_{j=1}^{n}\langle\tau_{d_j+2g-1}\prod_{i\neq j}\tau_{d_i}\rangle_g$$

$$-\frac{1}{2}\sum_{\underline{n}=I\coprod J}\sum_{j=0}^{2g-2}(-1)^j\langle\tau_j\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\tau_{2g-2-j}\prod_{i\in J}\tau_{d_i}\rangle_{g-g'};$$

$$(2)\qquad \sum_{j=0}^{2g}(-1)^j\langle\tau_{2g-j}\tau_j\prod_{i=1}^{n}\tau_{d_i}\rangle_g = \frac{(2g+n-1)!}{4^g(2g+1)!\prod_{j=1}^{n}(2d_j-1)!!};$$

(3) Let $k>g$. Then

$$\sum_{j=0}^{2k}(-1)^j\langle\tau_{2k-j}\tau_j\prod_{i=1}^{n}\tau_{d_i}\rangle_g=0.$$

Proof of Faber Intersection Number Conjecture II

Relation between n-point function and the Faber conjecture:

$$\frac{1}{2}\sum_{\underline{n}=I\coprod J}\sum_{j=0}^{2g-2}(-1)^j\langle\tau_j\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\tau_{2g-2-j}\prod_{i\in J}\tau_{d_i}\rangle_{g-g'}+\langle\prod_{j=1}^{n}\tau_{d_j}\tau_{2g}\rangle_g$$

$$-\sum_{j=1}^{n}\langle\tau_{d_j+2g-1}\prod_{i\neq j}\tau_{d_i}\rangle_g$$

$$=\frac{1}{2}\sum_{\underline{n}=I\coprod J}\sum_{j\in\mathbb{Z}}(-1)^j\langle\tau_j\prod_{i\in I}\tau_{d_i}\rangle_{g'}\langle\tau_{2g-2-j}\prod_{i\in J}\tau_{d_i}\rangle_{g-g'}$$

$$=\left[\sum_{g'=0}^{g}\sum_{\underline{n}=I\coprod J}G_{g'}(y,x_I)G_{g-g'}(-y,x_J)\right]_{y^{2g-2}\prod_{i=1}^{n}x_i^{d_i}}=0.$$

Corollaries

Our methods and results have the following corollaries and applications:

- Proofs of new explicit recursion formulas.
- Finding new effective recursion formula for the Witten r-spin intersection numbers.
- Proving recursion formulas for general higher Weil-Petersson volumes, generalizing Mizkharni-Mulase-Safnuk.
- Finding effective recursion formulas of all intersection numbers involving $\lambda_g \lambda_{g-1}$.
- Proof of the crepant resolution conjecture of all genera for type A surface singularity (Zhou).
- Finding new vanishing identity for Gromov-Witten invariants, (our conjecture proved by Xiaobo Liu and Pandharipande).

We expect much more to come.

Geometric Aspects of $\mathcal{M}_g$

Let $\mathcal{M}_g$ be the moduli space of genus g curves with $g \geq 2$. There are many famous Kähler and Finsler metrics on $\mathcal{M}_g$ and Teichmüller spaces which are very useful in understanding the geometry and topology of the moduli spaces and the structure of mapping class groups. We only mention those Kähler metrics:

- The Weil-Petersson metric $\omega_{_{WP}}$.
- The Kähler-Einstein metric $\omega_{_{KE}}$.
- The Ricci metric ω_{τ}.
- The perturbed Ricci metric $\omega_{\tilde{\tau}}$.
- The McMullen metric.
- The Bergman metric.

By the Kodaira-Spencer theory and the Hodge theory, for any point $X \in \mathcal{M}_g$,

$$T_X\mathcal{M}_g \cong H^1(X, T_X) = HB(X)$$

where $HB(X)$ is the space of harmonic Beltrami differentials on X. By the Serre duality we have

$$T_X^*\mathcal{M}_g \cong Q(X)$$

where $Q(X) = H^0(X, K_X^2)$ is the space of holomorphic quadratic differentials on X.

Pick $s \in \mathcal{M}_g$, let $\pi^{-1}(s) = X_s$. Let $s_1, \cdots, s_n$ be local holomorphic coordinates on $\mathcal{M}_g$ and let z be local holomorphic coordinate on X_s. The Kodaira-Spencer map is

$$\frac{\partial}{\partial s_i} \mapsto A_i \frac{\partial}{\partial z} \otimes d\bar{z} \in HB(X_s).$$

The Weil-Petersson metric is the natural L^2 metric on $\mathcal{M}_g$.

$$h_{i\bar{j}} = \int_{X_s} A_i \overline{A}_j \, dv$$

where $dv = \frac{\sqrt{-1}}{2} \lambda dz \wedge d\bar{z}$ is the volume form of the Poincaré metric λ on X_s.

Ahlfors and Wolpert proved that the Ricci curvature of the Weil-Petersson metric has explicit negative upper bound. The Ricci metric is the negative of the Ricci curvature of the Weil-Petersson metric

$$\omega_{\tau} = -Ric(\omega_{WP}).$$

We introduced the perturbed Ricci metric as a combination of the Ricci metric and the Weil-Petersson metric

$$\omega_{\tilde{\tau}} = \omega_{\tau} + C\omega_{WP}$$

where C is a positive constant.

We have proved the following results in 2004:

Theorem

- *The Ricci and perturbed Ricci metrics are complete, have bounded geometry and Poincaré growth.*
- *The holomorphic sectional and Ricci curvatures of the perturbed Ricci metric are negatively pinched.*
- *The holomorphic sectional curvature of the Ricci metric is asymptotically negative in the degeneration directions.*
- *The Kähler-Einstein metric has strongly bounded geometry and Poincaré growth.*

Corollary

The known complete metrics on $\mathcal{M}_g$, the Teichmüller-Kobayashi metric, the Carathéodory metric, the induced Bergman metric, the Kähler-Einstein metric and the McMullen metric are equivalent to the Ricci and perturbed Ricci metrics.

Mumford introduced the notion of good metric to extend the Chern-Weil theory to quasi-projective manifolds. Essentially goodness controls the singularities of a metric, its connection and curvature so that the Chern classes can be defined as closed currents. The following theorem is due to Mumford in 1970.

Theorem

Let $X = \overline{X} \setminus D$ be a quasi-projective manifold with D normal crossing divisor. Then

- *There is at most one extension of E to $\overline{X}$ such that h is good.*
- *If h is a good metric and $\overline{E}$ is the unique extension, then the Chern forms $c_k(E, h)$ are good. Furthermore, as currents,*

$$[c_k(E, h)] = c_k(\overline{E}).$$

For tangent and cotangent bundle we call the unique extension under good metric the log extension.

Good Metrics on $\mathcal{M}_g$

We first generalized the notion of Mumford goodness and introduced new weaker integral notions of goodness, and by a very careful analysis, we proved the following results in 2006-2007:

Theorem

- *The metrics on log extension $T_{\overline{\mathcal{M}}_g}(-\log D)$ induced by the Weil-Petersson, Ricci and perturbed Ricci metrics are good in the sense of Mumford.*
- *The metric induced by the Kähler-Einstein metric is (intrinsic) good.*

The goodness of the Weil-Petersson metric is a long-standing problem. The above good metrics make moduli spaces of curves the first nontrivial examples of manifolds with good metrics, other than the symmetric spaces of noncompact type studied by Mumford.

The proof of the goodness of Kähler-Einstein metric used Ricci flow and subtle estimates of Monge-Amperé equation with unbounded potential.

The goodness of the Kähler-Einstein metric gives important algebraic geometric consequences:

Theorem

The log tangent bundle of the moduli space $\overline{E} = T_{\overline{\mathcal{M}}_g}(-\log D)$ *is Mumford stable with respect to the canonical polarization. (2004) Furthermore, we have the strong Chern number inequality (2007)*

$$c_1(\overline{E})^2 \leq \frac{6g-4}{3g-3} c_2(\overline{E}).$$

This Chern number inequality is obtained by using the Kähler-Einstein metric and is stronger than the inequality obtained by the stability.

Dual Nakano Negativity

Let (E^m, h) be a holomorphic vector bundle with a Hermitian metric over a complex manifold M. The curvature of E is given by

$$P_{i\bar{j}\alpha\bar{\beta}} = -\partial_\alpha \partial_{\overline{\beta}} h_{i\bar{j}} + h^{p\bar{q}} \partial_\alpha h_{i\bar{q}} \partial_{\overline{\beta}} h_{p\bar{j}}.$$

(E, h) is Nakano positive if the curvature P defines a positive form on the bundle $E \otimes T_M$.

E is dual Nakano negative if the dual bundle (E^*, h^*) is Nakano positive. We proved in 2005 the following

Theorem

The Weil-Petersson metric on the tangent bundle $T_{\mathcal{M}_g}$ and on the log tangent bundle $T_{\overline{\mathcal{M}}_g}(-\log D)$ are dual Nakano negative.

This is the strongest negativity property of the Weil-Petersson metric.

In 2006 we proved the following identification of the L^2-cohomology of $\mathcal{M}_g$ and the Dolbeault cohomology of $\overline{\mathcal{M}}_g$ with bundle twist

Theorem

$$H^*_{(2)}\left((\mathcal{M}_g, \omega_\tau), (T_{\mathcal{M}_g}, \omega_{WP})\right) \cong H^*(\overline{\mathcal{M}}_g, T_{\overline{\mathcal{M}}_g}(-\log D)).$$

Combining with the dual Nakano negativity of the Weil-Petersson metric we have

Theorem

The Chern numbers of the log cotangent bundle $T^*_{\overline{\mathcal{M}}_g}(\log D)$ of the moduli spaces of Riemann surfaces are positive.

As another corollary we have

Theorem

When $q \neq 3g-3$, the L^2-cohomology groups vanish

$$H_{(2)}^{0,q}\left((\mathcal{M}_g, \omega_\tau), \left(T_{\overline{\mathcal{M}}_g}(-\log D), \omega_{WP}\right)\right) = 0.$$

This means the complex structure of the moduli space $\mathcal{M}_g$ is infinitesimally rigid: no holomorphic deformation.
The proof depends on the Kodaira-Nakano identity and the use of cut-off functions.

An interesting application of the properties of the Ricci, perturbed Ricci and Kähler-Einstein metrics is the Gauss-Bonnet theorem on the noncompact moduli space we proved in 2007

Theorem

(Ji-Liu-Sun-Yau) The Gauss-Bonnet theorem holds on the moduli space equipped with the Ricci, perturbed Ricci or Kähler-Einstein metrics:

$$\int_{\mathcal{M}_g} c_n(\omega_\tau) = \int_{\mathcal{M}_g} c_n(\omega_{\tilde{\tau}}) = \int_{\mathcal{M}_g} c_n(\omega_{KE}) = \chi(\mathcal{M}_g) = \frac{B_{2g}}{4g(g-1)}.$$

Here $\chi(\mathcal{M}_g)$ is the orbifold Euler characteristic of $\mathcal{M}_g$ and $n = 3g - 3$.

The explicit topological computation of the Euler characteristic of the moduli space is due to Harer-Zagier.

As an application of the Mumford goodness of the Weil-Petersson metric and the Ricci metric we have

Theorem

$$\chi(T_{\overline{\mathcal{M}}_g}(-\log D)) = \int_{\mathcal{M}_g} c_n(\omega_\tau) = \int_{\mathcal{M}_g} c_n(\omega_{_{WP}}) = \frac{B_{2g}}{4g(g-1)}$$

where $n = 3g - 3$.

The Gauss-Bonnet theorem for the Weil-Petersson metric cannot be proved directly. It based substantially on the Mumford goodness.

This theorem also gives an explicit expression of the top log Chern number:

$$\chi(\overline{\mathcal{M}}_g, T_{\overline{\mathcal{M}}_g}(-\log D)) = \chi(\mathcal{M}_g) = \frac{B_{2g}}{4g(g-1)}.$$

Our understanding of the geometry of the moduli spaces of curves has revealed new geometric properties of the moduli spaces. These new geometric results may be used to understand the topology and algebraic geometric structures, in particular to compute the Hodge integrals more effectively, and to understand the cohomology ring of the moduli spaces of curves.

The good geometry of the metrics opens a way to apply index theory and L^2-fixed point formula to moduli and Teichmüller spaces to understand their geometric and topological properties, and the mapping class groups.

The Calabi-Yau (CY) manifolds and their Teichmüller and moduli spaces are central objects in many subjects of mathematics and string theory. A CY n-fold is a complex manifold X of dimension n such that the canonical bundle K_X is trivial and the Hodge numbers $h^{p,0}(X) = 0$ for $p < n$.

We fix a CY manifold X, an ample line bundle L over X and a basis B of $H_n(X, \mathbb{Z})/Tor$ (marking). Let $\mathcal{T} = \mathcal{T}_L(X)$ and $\mathcal{M} = \mathcal{M}_L(X)$ be the Teichmüller and moduli spaces of X with respect to the polarization L which leaves the marking B invariant.

There are two natural metrics on the moduli and Teichmüller spaces: the Weil-Petersson metric which is the natural induced L^2 metric and the Hodge metric which is the pull-back of the Griffiths-Schmid metric on the classifying space by the period map.

Flat Coordinates

Fix $p \in \mathcal{T}$ and let M_p be the corresponding CY n-fold. Let $N = h^{n-1,1}(M_p)$ be the dimension of the Teichmüller and the moduli spaces. Let $\phi_1, \cdots, \phi_N \in \mathbb{H}^{0,1}(M_p, T^{1,0}_{M_p})$ be a harmonic basis with respect to the polarized CY metric. Then there is a unique power series

$$\phi(\tau) = \sum_{i=1}^{N} \tau_i \phi_i + \sum_{|I| \geq 2} \tau^I \phi_I$$

which converges for $|\tau| < \epsilon$ such that

$$\overline{\partial}\phi(\tau) + \frac{1}{2}[\phi(\tau), \phi(\tau)] = 0$$

$$\overline{\partial}^* \phi(\tau) = 0$$

$$\phi_I \lrcorner \Omega_0 = \partial \psi_I$$

for $|I| \geq 2$ where we use the polarized CY metric and Ω_0 is a nowhere vanishing holomorphic $(n,0)$-form on M_p. The coordinates $\tau = (\tau_1, \cdots, \tau_N)$ is the flat coordinates at p.

Canonical Family and Local Torelli Theorem

Fix Ω_0 on M_p and pick local coordinates z on M_p such that

$$\Omega_0 = dz_1 \wedge \cdots \wedge dz_n.$$

For $|\tau| < \epsilon$ let

$$\Omega_\tau = \bigwedge_{i=1}^{n} (dz_i + \phi(\tau)(dz_i)).$$

The family Ω_τ is a canonical holomorphic family of nowhere vanishing holomorphic $(n,0)$-forms. In the cohomology level, it has the expansion

$$[\Omega_\tau] = [\Omega_0] - \sum_{i=1}^{N} \tau_i [\phi_i \lrcorner \Omega_0] + \Xi$$

where $\Xi \in \bigoplus_{k=2}^{n} H^{n-k,k}(M_p)$.

Corollary

(Local Torelli) For $|\tau_1|, |\tau_2| < \epsilon$ and $\tau_1 \neq \tau_2$, we have $p(\tau_1) \neq p(\tau_2)$ where p is the period map.

Global Torelli Theorem

Let $G_{\mathbb{R}}/K_1$ be the classifying space of polarized Hodge structures with data from a Calabi-Yau M. Let $p : \mathcal{T} \to G_{\mathbb{R}}/K_1$ be the period map. Then we proved the following global Torelli theorem

Theorem

The period map

$$p : \mathcal{T} \to G_{\mathbb{R}}/K_1$$

is injective.

The proof relies on tracing the cohomology classes of the holomorphic $(n, 0)$-forms by using the canonical expansion. We extend the flat coordinate lines by realizing them as geodesics of a holomorphic flat connection on $\mathcal{T}$. This connection tied to the "Frobineous" structure on $\mathcal{T}$ closely. Another important ingredient is to extend the canonical family across singularities in $\mathcal{T}$ with finite Hodge distance.

Existence of Kähler-Einstein Metric

Theorem

For any smooth point $p \in \mathcal{T}$, the Teichmüller space $\mathcal{T}$ can be embedded into $H^{n-1,1}(M_p)$ as a domain of holomorphy. Thus there exists a unique Kähler-Einstein metric on $\tilde{\mathcal{T}}$, the Hodge metric completion of the Teichmüller space.

In order to prove this, we construct a potential function of the Hodge metric on $\mathcal{T}$ which is a nice exhaustion function. The existence of Kähler-Einstein metric follows from the general construction of S.-Y. Cheng and Yau.

Theorem

The mapping class group of a polarized CY manifold is an arithmetic group.

This follows from the global Torelli theorem and Sullivan's theorem.

Thank You Very Much!