Spectral data for Hamiltonian-minimal Lagrangian tori in $\mathbb{C}P^2$

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The 16th Osaka City University International Academic Symposium

2008

Riemann Surfaces, Harmonic Maps and Visualization

- Examples of Hamiltonian-minimal Lagrangian submanifolds in $\mathbb{C}^n$ and $\mathbb{C}P^n$

- Minimal Lagrangian and Hamiltonian-minimal Lagrangian tori in $\mathbb{C}P^2$

1. Integrable deformations of minimal Lagrangian tori under

   Novikov-Veselov hierarchy

2. Spectral data for Hamiltonian-minimal Lagrangian tori in $\mathbb{C}P^2$

P — Kähler manifold,

ds^2 — Hermitian metric on P,

$\omega = \mathrm{Im} ds^2$ —symplectic form on P

def. Vector field W along L is called *Hamiltonian* if 1-form $\omega(W, .)$ on L is exact.

def. L is called *Hamiltonian-minimal (Hamiltonian stationary)* if the variations of the volume along all Hamiltonian vector fields with compact support are zero.

$\mathbb{C}^n$, $\quad \omega = dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n$

At each point $x \in L$ choose an orthonormal tangent frame ξ that agrees with the orientation of L (locally)

$$e^{i\beta(x)} = dz_1 \wedge \cdots \wedge dz_n(\xi)$$

def. $\beta(x)$ — Lagrangian angle on L.

The values of β does not depend on the choice of ξ.

In general the function β is multivalued. Going round a closed curve it can change its value by $2k$, $k \in \mathbb{Z}$.

Mean curvature vector

$$H = J\nabla\beta.$$

$L \subset \mathbb{C}^n$ is Hamiltonian-minimal if and only if Lagrangian angle is harmonic function on L

$$\Delta\beta = 0.$$

Example. Clifford torus is **HML**-submanifold

$$S^1(r_1) \times \cdots \times S^1(r_n) \subset \mathbb{C}^n,$$

where $S^1(r_j)$ is a circle of radius r_j in $\mathbb{C}$.

Let M be a k-dimensional manifold in $\mathbb{R}^n$ defined by

$$e_{11}u_1^2+\cdots+e_{n1}u_n^2=d_1,$$

$$\cdots$$

$$e_{1n-k}u_1^2+\cdots+e_{nn-k}u_n^2=d_{n-k},$$

$d_j \in \mathbb{R},\ e_{ij} \in \mathbb{Z}$

$e_j = (e_{j1},\ldots,e_{jn-k}) \subset \mathbb{Z}^{n-k},\ j=1,\ldots,n$

$\Lambda \subset \mathbb{R}^{n-k}$ — lattice generated by $e_1,\ldots,e_n$

$\Lambda^* = \{\lambda^* \in \mathbb{R}^{n-k} | (\lambda^*,\lambda) \in \mathbb{Z}, \lambda \in \Lambda\}$ — dual lattice

$\Gamma = \Lambda^*/2\Lambda^* \simeq \mathbb{Z}_2^{n-k}.$

Denote by T^{n-k} the $(n-k)$-dimensional torus

$$T^{n-k} = \{(e^{\pi i(e_1,y)}, \dots, e^{\pi i(e_n,y)})\} \subset \mathbb{C}^n,$$

$$y = (y_1, \dots, y_{n-k}) \in \mathbb{R}^{n-k}, (e_j, y) = e_{j1}y_1 + \dots + e_{jn-k}y_{n-k}.$$

Define an action of group Γ on manifold $M \times T^{n-k}$. For $\gamma \in \Gamma$ we set

$$\gamma(u_1, \dots, u_n, y) = (u_1 \cos \pi(e_1, \gamma), \dots, u_n \cos \pi(e_n, \gamma), y + \gamma).$$

Note $\cos \pi(e_j, \gamma) = \pm 1$.

Introduce the map

$$\varphi : M \times T^{n-k}/\Gamma \to \mathbb{C}^n,$$

$$\varphi(u_1, \dots, u_n, y) = (u_1 e^{\pi i(e_1,y)}, \dots, u_n e^{\pi i(e_n,y)}).$$

Theorem (M.) The group Γ acts on $M \times T^{n-k}$ freely. The map φ is **HML**-immersion. If $e_1 + \cdots + e_n = 0$, then φ is **ML**-immersion.

It is possible to construct **HML**-embeddings of generalized Klein bottle K^{2n+1}, $S^{2n+1} \times S^1$, $K^{2n+1} \times S^1$, $S^{2n+1} \times T^2$ and others.

The induced metric on $\varphi(M \times T^{n-k}/\Gamma)$ has block-diagonal form

$$ds^2 = \sum_{i,j=1}^{k} g_{ij}(x) dx_i dx_j + \sum_{i,j=1}^{n-k} \tilde{g}_{ij}(x) dy_i dy_j,$$

$x = (x_1, \ldots, x_k)$,

$$\beta = \alpha_1 y_1 + \cdots + \alpha_{n-k} y_{n-k}, \ \alpha_j \in \mathbb{R},$$

$$\Delta \beta = 0.$$

Example. Let M be the $(n-1)$-dimensional sphere

$$u_1^2 + \cdots + u_n^2 = 1.$$

A non-zero element $\gamma \in \mathbb{Z}_2$ acts on $S^{n-1} \times S^1$ by

$$\gamma(u, y) = (-u, -y).$$

If $n = 2k$, then $S^{n-1} \times S^1/\Gamma \simeq S^{2k-1} \times S^1$.

If $n = 2k+1$, then $S^{n-1} \times S^1/\Gamma \simeq K^{2k+1}$.

If M is cone in $\mathbb{R}^n$

$$e_{1j}u_1^2 + \dots e_{nj}u_n^2 = 0, \; j = 1, \dots, n-k$$

$$M_1 = \varphi(M \times T^{n-k}/\Gamma) \cap S^{2n-1},$$

$\mathcal{H} : S^{2n-1} \to \mathbb{C}P^{n-1}$ — Hopf projection

Theorem(M.) $\mathcal{H}(M_1)$ — **HML** in $\mathbb{C}P^{n-1}$, if $e_1 + \dots + e_n = 0$, then **ML**.

Define Lagrangian surface $\Sigma \subset \mathbb{C}P^2$ as composition

$$\varphi : \mathbb{R}^2 \to S^5 \subset \mathbb{C}^3$$

and Hopf projection

$$\mathcal{H} : S^5 \to \mathbb{C}P^2.$$

Suppose that induced metric on Σ has form

$$ds^2 = 2e^{v(x,y)}(dx^2 + dy^2).$$

From Lagrangiality and the conformality of $\mathcal{H} \circ \varphi$ it is follows

$$< \varphi, \varphi_x >=< \varphi, \varphi_y >=< \varphi_x, \varphi_y >= 0, \ |\varphi_x|^2 = |\varphi_y|^2 = 2e^v,$$

where $< .,. >$ is a Hermitian product in $\mathbb{C}^3$.

$$\tilde{\Phi} = \left(\varphi, \frac{1}{\sqrt{2}} e^{-\frac{v}{2}} \varphi_x, \frac{1}{\sqrt{2}} e^{-\frac{v}{2}} \varphi_y\right)^\top \in U(3)$$

$$e^{i\beta(x,y)} = \det \tilde{\Phi}$$

def. $\beta(x, y)$ — Lagrangian angle

$$H = J \nabla \beta$$

$\Sigma \subset \mathbb{C}P^2$ is Hamiltonian-minimal if and only if Lagrangian angle is harmonic function on Σ

$$\Delta \beta = 0$$

From the definition of a Lagrangian angle we get

$$\Phi = \left(\varphi, \frac{1}{\sqrt{2}}e^{-\frac{v}{2}-i\frac{\beta}{2}}\varphi_x, \frac{1}{\sqrt{2}}e^{-\frac{v}{2}-i\frac{\beta}{2}}\varphi_y\right)^\top \in \mathsf{SU}(3)$$

Matrix Φ satisfies equations

$$\Phi_x = A\Phi,\ \ \Phi_y = B\Phi,$$

where

$$A = \begin{pmatrix} 0 & \sqrt{2}e^{\frac{v}{2}+i\frac{\beta}{2}} & 0 \\ -\sqrt{2}e^{\frac{v}{2}-i\frac{\beta}{2}} & if & -\frac{v_y}{2}+i(h+\frac{\beta_y}{2}) \\ 0 & \frac{v_y}{2}+i(h+\frac{\beta_y}{2}) & -if \end{pmatrix} \in \mathsf{su}(3),$$

$$B = \begin{pmatrix} 0 & 0 & \sqrt{2}e^{\frac{v}{2}+i\frac{\beta}{2}} \\ 0 & ih & \frac{v_x}{2}+i(-f+\frac{\beta_x}{2}) \\ -\sqrt{2}e^{\frac{v}{2}-i\frac{\beta}{2}} & -\frac{v_x}{2}+i(-f+\frac{\beta_x}{2}) & -ih \end{pmatrix} \in \mathsf{su}(3),$$

$f(x,y)$ and $h(x,y)$ being some real functions.

The zero-curvature equation $A_y - B_x + [A, B] = 0$ implies the following

Lemma Lagrangian surfaces are defined by a system of equations

$$2V_y + 2U_x = (\beta_{xx} - \beta_{yy})e^v,$$

$$2U_y - 2V_x = (\beta_y v_x + \beta_x v_y)e^v,$$

$$\Delta v = 4(U^2 + V^2)e^{-2v} - 4e^v - 2(U\beta_x - V\beta_y)e^{-v},$$

where $U = fe^v, V = he^v$.

From $\Phi_x = A\Phi,\ \Phi_y = B\Phi$ we get

Lemma (M.) The components φ^j of the vector function φ satisfy the Schrödinger equation

$$\partial_x^2\varphi^j + \partial_y^2\varphi^j - i(\beta_x\partial_x\varphi^j + \beta_y\partial_y\varphi^j) + 4e^v\varphi^j = 0.$$

If mapping $\mathcal{H}\circ\varphi$ is **HML** doubly periodic, then the Lagrangian angle is the linear function $\beta(x,y)=ax+by+c,\ \ a,b,c\in\mathbb{R}$.

Lemma **HML**-tori in $\mathbb{C}P^2$ are described by the following equations

$$2U_y-2V_x=(bv_x+av_y)e^v,\qquad V_y+U_x=0,$$

$$\Delta v=4(U^2+V^2)e^{-2v}-4e^v-2(Ua-Vb)e^{-v}.$$

Shrödinger operator has the form: $L=\Delta-ia\partial_x-ib\partial_y+4e^v$

In the minimal case we have Tzizéica equation

$$\partial_x^2v+\partial_y^2v=4e^{-2v}-4e^v$$

and potential Shrödinger operator: $L=\Delta+4e^v$

Let $\Sigma \subset \mathbb{C}P^2$ be **ML**-torus defined by the map $\varphi : \mathbb{R}^2 \to S^5$.

Theorem (M.) There is a mapping $\widetilde{\varphi}(t), t = (t_1, t_2, \dots), \widetilde{\varphi}(0) = \varphi$, defining a deformation of torus Σ in the class of minimal Lagrangian tori in $\mathbb{C}P^2$. The map $\tilde{\varphi}$ satisfies the equations

$$L\widetilde{\varphi} = \partial_x^2 \widetilde{\varphi} + \partial_y^2 \widetilde{\varphi} + 4e^{\widetilde{v}} \widetilde{\varphi} = 0, \qquad \partial_{t_n} \widetilde{\varphi} = A_n \widetilde{\varphi},$$

where A_n are operators of order $(2n+1)$ on the variables (x, y). Deform the potential $\widetilde{V} = 4e^{\widetilde{v}}, \widetilde{v}(0) = v$, according to the Novikov–Veselov hierarchy

$$\frac{\partial L}{\partial t_n} = [L, A_n] + B_n L,$$

where B_n are operators of order $(2n - 1)$ on the variables (x, y). The deformations $\widetilde{\varphi}(t)$ preserve the spectrum of torus Σ and its conformal type.

Theorem (M.) Stationary Novikov-Veselov equation

$$[L, A_3] + B_0 L = 0,$$

where

$$L = \partial_z \partial_{\bar{z}} + e^{v(z,\bar{z})},$$

$$A_3 = \partial_z^3 + \partial_{\bar{z}}^3 - (v_z^2 + v_{zz})\partial_z - (v_{\bar{z}}^2 + v_{\bar{z}\bar{z}})\partial_{\bar{z}},$$

$$B_0 = -\partial_z(v_z^2 + v_{zz}) - \partial_{\bar{z}}(v_{\bar{z}}^2 + v_{\bar{z}\bar{z}}),$$

is equivivalent to the system

$$\partial_z(e^{-2v} - e^v - v_{z\bar{z}}) + 2v_z(e^{-2v} - e^v - v_{z\bar{z}}) = 0,$$

$$\partial_{\bar{z}}(e^{-2v} - e^v - v_{z\bar{z}}) + 2v_{\bar{z}}(e^{-2v} - e^v - v_{z\bar{z}}) = 0.$$

Theorem (M.) The first three equations from the Novikov-Veselov hierarhy give symmetries of the Tzizéica equation.

All problem witch are solvable by the finite-gap theory are divided by two part.

I PART: there is direct and inverse problem.

Example.

$$L_1 = \frac{d^n}{dx^n} + u_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + u_0(x),$$

$$L_2 = \frac{d^m}{dx^m} + v_{m-1}(x)\frac{d^{m-1}}{dx^{m-1}} + \cdots + v_0(x).$$

Lemma (Burchnall, Chaundy, 1923)

If $L_1L_2 = L_2L_1$, then there exist a non-trivial polynomial Q of two commuting variables such that $Q(L_1, L_2) = 0$.

Spectral curve: $\Gamma = \{(\lambda, \mu) \in \mathbb{C}^2 : Q(\lambda, \mu) = 0\}$

II PART: there is only inverse problem.

Example. n-orthogonal curvilinear coordinate systems in $\mathbb{R}^n$

$$\frac{\partial x^1}{\partial u^i}\frac{\partial x^1}{\partial u^j} + \cdots + \frac{\partial x^n}{\partial u^i}\frac{\partial x^n}{\partial u^j} = 0, \ i \neq j$$

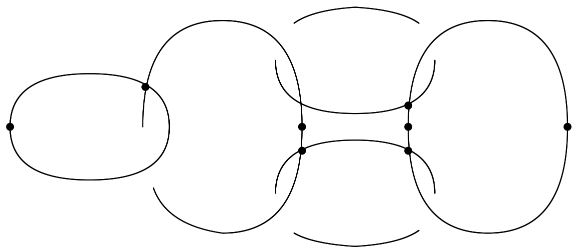
$$x^j = \psi(u^1, \ldots, u^n, Q_j), \ Q_j \in \Gamma,$$

Γ-spectral curve, $\psi(u, P)$ — n-point Baker-Akhiezer function.

Example. (M.-Taimanov). Polar coordinate system

$$x = \rho \cos \varphi, \;\; y = \rho \sin \varphi$$

Spectral curve



The equation

$$< \varphi_x, \varphi_y >= \varphi_x^1 \bar{\varphi}_y^1 + \varphi_x^2 \bar{\varphi}_y^2 + \varphi_x^3 \bar{\varphi}_y^3 = 0,$$

is "similar" to equation

$$\frac{\partial x^1}{\partial u^1}\frac{\partial x^1}{\partial u^2} + \frac{\partial x^2}{\partial u^1}\frac{\partial x^2}{\partial u^2} + \frac{\partial x^3}{\partial u^1}\frac{\partial x^3}{\partial u^2} = 0.$$

The *two-points Baker-Akhiezer function*, corresponding to *spectral data*

$$\{\Gamma, P_1, P_2, k_1^{-1}, k_2^{-1}, \gamma_1 + \dots + \gamma_{g+l}, r + r_1 + \dots + r_l\},$$

is called a function $\psi(x, y, P), P \in \Gamma$, with the following characteristics:

1) in the neighborhood of P_1 and P_2 function ψ has following form

$$\psi = e^{k_1 x}\left(f_1(x,y) + \frac{g_1(x,y)}{k_1} + \frac{h_1(x,y)}{k_1^2} + \dots\right),$$

$$\psi = e^{k_2 y}\left(f_2(x,y) + \frac{g_2(x,y)}{k_2} + \frac{h_2(x,y)}{k_2^2} + \dots\right).$$

2) on $\Gamma\backslash\{P_1, P_2\}$, function ψ is meromorphic with simple poles on γ.

3) $\psi(x, y, r) = d$, $\psi(x, y, r_i) = 0$, $i = 1, \dots, l$, where d is a non-zero constant.

Let $\varphi^1, \varphi^2, \varphi^3$ denote the following functions

$$\varphi^i = \alpha_i \psi(x, y, Q_i),$$

$Q_1, Q_2, Q_3 \in \Gamma$ is an additional set of points, α_i are some constants.

Suppose that surface Γ has a holomorphic involution σ and an anti-holomorphic involution μ

$$\sigma : \Gamma \to \Gamma, \;\; \mu : \Gamma \to \Gamma,$$

for which points P_1, P_2 and r are fixed, and

$$k_i(\sigma(P)) = -k_i(P), \;\; k_i(\mu(P)) = \bar{k}_i(P).$$

Let τ denote involution $\sigma\mu$. Involution τ acts on local parameters as follows

$$k_i(\tau(P)) = -\bar{k}_i(P).$$

Theorem Let Q_i be fixed points of the antiholomorphic involution τ. Suppose that on Γ there exists a meromorphic 1-form Ω with the following set of divisors of zeros and a pole

$$(\Omega)_0 = \gamma + \tau\gamma + P_1 + P_2, \ (\Omega)_\infty = Q_1 + Q_2 + Q_3 + r + R + \tau R,$$

where $R = r_1 + \dots + r_l$ and $\mathrm{Res}_{\mathrm{Q_i}}\Omega > 0, \ \mathrm{Res}_{\mathrm{r}}\Omega < 0$. Then, for

$$\alpha_i = \sqrt{\mathrm{Res}_{\mathrm{Q_i}}\Omega}, \ d = \sqrt{\frac{-1}{\mathrm{Res}_{\mathrm{r}}\Omega}},$$

following equality holds:

$$< \varphi, \varphi >= 1, \ < \varphi, \varphi_x >=< \varphi, \varphi_y >=< \varphi_x, \varphi_y >= 0,$$

i.e. the mapping $\mathcal{H} \circ \varphi : \mathbb{R}^2 \to \mathbb{C}P^2$ is Lagrangian, with the induced metric on Σ having a diagonal form

$$ds^2 = |f_1|^2 |c_1| dx^2 + |f_2|^2 |c_2| dy^2.$$

Consider the 1-form $\Omega_1 = \psi(P)\overline{\psi(\tau P)}\Omega$. By the definition of involution τ, function $\overline{\psi(\tau P)}$ has the following form near P_1 and P_2

$$\overline{\psi(\tau P)} = e^{-k_1 x}\left(\bar{f}_1(x,y) - \frac{\bar{g}_1(x,y)}{k_1} + \frac{\bar{h}_1(x,y)}{k_1^2} + \dots\right),$$

$$\overline{\psi(\tau P)} = e^{-k_2 y}\left(\bar{f}_2(x,y) - \frac{\bar{g}_2(x,y)}{k_2} + \frac{\bar{h}_2(x,y)}{k_2^2} + \dots\right).$$

Consequently, form Ω_1 has no essential singularities in points P_1 and P_2. The simple poles $\gamma+\tau\gamma$ of function $\psi(P)\overline{\psi(\tau P)}$ reduce with the zeros in these points of form Ω. The zeros $R+\tau R$ of function $\psi(P)\overline{\psi(\tau P)}$ reduce with the simple poles of form Ω. The form Ω_1 has only simple poles in points Q_1, Q_2, Q_3 and r with residues

$$\varphi^1\bar{\varphi}^1,\ \varphi^2\bar{\varphi}^2,\ \varphi^3\bar{\varphi}^3,\ -1.$$

Let form Ω near points P_1 and P_2 has form:

$$\Omega = (c_1 w_1 + i a w_1^2 + \dots) dw_1, \;\; w_1 = 1/k_1,$$

$$\Omega = (c_2 w_2 + i b w_2^2 + \dots) dw_2, \;\; w_2 = 1/k_2.$$

Consider function

$$F(x,y,P) = \partial_x^2 \psi + \partial_y^2 \psi + A(x,y)\partial_x \psi + B(x,y)\partial_y \psi + C(x,y)\psi.$$

Chose functions $A(x,y), B(x,y)$ and $C(x,y)$ such that $F(x,y,Q_i) = 0$, $i = 1,2,3$. We need to find spectral data such that the metric on surface Σ has a conformal form, and the coefficients A, B are constants.

Lemma 1. *The following equality holds:*

$$A(x,y) = \frac{1}{c_1|f_1|^2}(-c_1\bar{f}_1(g_1+2f_{1x}) + f_1(-ia\bar{f}_1 + c_1(\bar{g}_1 + \bar{f}_{1x})) + c_2 f_2 \bar{f}_{2x}),$$

$$B(x,y) = \frac{1}{c_2|f_2|^2}(-c_2\bar{f}_2(g_2+2f_{2y}) + f_2(-ib\bar{f}_2 + c_2(\bar{g}_2 + \bar{f}_{2y})) + c_1 f_1 \bar{f}_{1y}).$$

Theorem Suppose that on surface Γ there exists a meromorphic form ω with the following divisors of zeros and poles

$$(\omega)_0 = \gamma + \sigma\gamma, \ (\omega)_\infty = P_1 + P_2 + R + \sigma R,$$

where $\mathrm{Res}_{P_1}\omega + \mathrm{Res}_{P_2}\omega = 0$. And suppose that

$$\mu(\gamma) = \gamma, \ \mu(R) = R,$$

then induced metric on surface Σ has the form

$$ds^2 = f_1^2(dx^2 + dy^2).$$

and Lagrangian angle looks as follows

$$\beta = ax + by + c,$$

where c is some real constant, i.e. surface Σ is Hamiltonian-minimal.

Example. Suppose that $\Gamma = \mathbb{C}P^1, P_1 = \infty, P_2 = 0$.

$$\sigma(z) = -z, \quad \mu(z) = \bar{z}, \quad \tau(z) = -\bar{z}.$$

Let $l = 0$, then the Baker-Akhiezer function has the form

$$\psi(x, y, z) = e^{xz + \frac{y}{z}} f(x, y).$$

Let points Q_1, Q_2, Q_3 and r have the coordinates $2i, -2i, \frac{i}{2}$ and $\frac{i}{2}$.

$$\Omega = \frac{zdz}{(z - 2i)(z + 2i)(z - \frac{i}{2})(z + \frac{i}{2})}$$

We have

$$\varphi^1 = \frac{3}{\sqrt{14}} e^{\frac{1}{2}i(2x+y)}, \ \varphi^2 = \frac{1}{\sqrt{6}} e^{\frac{3}{2}i(-2x+y)}, \ \varphi^3 = \frac{2}{\sqrt{21}} e^{-3i(\frac{x}{4}+y)}.$$

The induced metric looks as follows:

$$ds^2 = \frac{7}{4}(dx^2 + dy^2), \ \Delta\varphi^j + i\frac{11}{4}\partial_x\varphi^j + i\partial_y\varphi^j + \frac{9}{2}\varphi^j = 0,$$

$$\beta = -\left(\frac{11}{4}x + y - \pi\right).$$

Example. Let Γ_0 be a hyperelliptic surface of genus g given by

$$y^2 = P(x),$$

where $P(x)$ is a polynomial of degree $2g+2$ with real coefficients without multiple roots. Let f denote a meromorphic function on Γ_0

$$f = \frac{x-\beta}{x-\alpha},$$

where α, β are some real numbers such that $P(\alpha) \neq 0, P(\beta) \neq 0$. Let Γ denote the Riemannian surface of function $\sqrt{f}$. The affine part of surface Γ is given in $\mathbb{C}^3$ with coordinates x, y, z by the system

$$y^2 = P(x), \ \ z^2 = \frac{x-\beta}{x-\alpha}.$$

Surface Γ allows a holomorphic and antiholomorphic involution

$$\sigma(x,y,z) = (x,y,-z), \ \ \mu(x,y,z) = (\bar{x},\bar{y},\bar{z}).$$

Function f has two simple zeros in points

$$\tilde{P}_1 = (\beta, \sqrt{P(\beta)}),\ \ \tilde{P}_2 = (\beta, -\sqrt{P(\beta)})$$

and two simple poles in points

$$\tilde{r} = (\alpha, \sqrt{P(\alpha)}),\ \ \tilde{Q}_1 = (\alpha, -\sqrt{P(\alpha)}).$$

P_1, P_2, Q_1, r — the inverse images of points $\tilde{P}_1, \tilde{P}_2, \tilde{Q}_1, \tilde{r}$ under the projection $\Gamma \to \Gamma_0$.

$$Q_2 = \left(\frac{\beta - c^2\alpha}{1 - c^2}, \sqrt{P\left(\frac{\beta - c^2\alpha}{1 - c^2}\right)}, c\right),$$

$$Q_3 = \left(\frac{\beta - c^2\alpha}{1 - c^2}, -\sqrt{P\left(\frac{\beta - c^2\alpha}{1 - c^2}\right)}, c\right),$$

where c some constant.

Open problems:

- Classify topological types of $M \times T^{n-k}/\Gamma$
- Hamiltonian stability of **HML**-submanifolds in $\mathbb{C}^n$ and $\mathbb{C}P^n$
- Integrable deformations of **HML**-tori in $\mathbb{C}P^2$
- Symmetries of the system

$$2U_y - 2V_x = (bv_x + av_y)e^v, \quad V_y + U_x = 0,$$

$$\Delta v = 4(U^2 + V^2)e^{-2v} - 4e^v - 2(Ua - Vb)e^{-v}.$$

- Problem of the periodicity for the **HML**-immersions of $\mathbb{R}^2$ in $\mathbb{C}P^2$

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