Superconformal surfaces in the Euclidean four space in terms of null complex holomorphic curves

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Overview

$$\{\text{Superconformal surfaces in } \mathbb{R}^4\}$$
$$\updownarrow \text{ bijective}$$
$$\{\text{Null complex holomorphic curves in } \mathbb{C}^4\}$$

- explicit bijection by simple calculation

Reference:

[BFLPPP] Burstall, Ferus, Leschke, Pedit, and Pinkall, Lecture Notes in Mathematics 1772 (2002)

Surface

- (M, J): Riemann surface
- $F : M \to \mathbb{R}^4$: conformal immersion $=$ surface
- dA: the volume form
- K: Gaussian curvature
- $K^\perp$: normal curvature
- $\mathbb{I}$: 2nd fundamental form
- $\mathcal{H}$: mean curvature vector
- $\Phi_F = (\phi_F, \psi_F) : M \to S^2 \times S^2$: Gauss map

Superconformal surface

- $F \colon M \to \mathbb{R}^4$: surface
- $X \in \Gamma(T_pM)$, $X \neq 0$
- $c(t) = (\cos t)X + (\sin t)JX$: circle in T_pM
- $\mathrm{I\!I}(c(t), c(t))$: ellipse centered at $\mathcal{H}|df(X)|^2$ in $(T_pM)^\perp$ (curvature ellipse)

F is said to be **superconformal** if its curvature ellipse is a circle. (ex. 2-sphere in $\mathbb{R}^4$)

- $\int_M (|\mathcal{H}|^2 - K - K^\perp)dA = 0$

Construction

$$M \xrightarrow{h} \mathbb{C}P^3 \xrightarrow{tw} S^4 \xrightarrow{st_p} \mathbb{R}^4$$

- h: holomorphic map
- tw: twistor projection
- st_p: stereographic projection from $p \notin tw(h(M))$

$st_p \circ tw \circ h \colon M \to \mathbb{R}^4$: superconformal

Null curve

- $h = (h_1, h_2, h_3, h_4) \colon M \to \mathbb{C}^4$: holomorphic map
- h is said to be [redacted] if $\sum_{m=1}^{4} \partial h_m \otimes \partial h_m = 0$

Fact. • h is null holomorphic curve
$\Leftrightarrow$ Re h and Im h are (branched) minimal surfaces in $\mathbb{R}^4$ conjugate each other.

- h is a null holomorphic curve $\Rightarrow \Phi_{\mathrm{Re}\, h} = \Phi_{\mathrm{Im}\, h}$.

Characterization

$S^2 \times S^2 \cong \mathbb{C}P^1 \times \mathbb{C}P^1$

Fact. • F is superconformal

$\Leftrightarrow$ ϕ_F or ψ_F is anti-holomorphic

• F is minimal

$\Leftrightarrow$ ϕ_F and ψ_F are holomorphic

Result

- $\iota \colon S^2 \to \mathbb{R}^3$: inclusion, $\tilde{\phi}_F = \iota \circ \phi_F$

Theorem (Dajczer and Tojeiro (local), M–).

$$\{F \,|\, \text{superconformal, } \phi_F \text{ is anti-holomorphic, } \tilde{\phi}_F \text{ is an immersion}\}$$

$$\Updownarrow \text{ bijective}$$

$$\{G = G_0 + iG_1 \,|\, \text{null holomorphic curve, } \tilde{\phi}_{G_0} \text{ is extended smoothly at the branch points of } G_0,\ \tilde{\phi}_{G_0} \text{ is an immersion, } G_0^{-1}(0) = \emptyset\ \}$$

Surfaces in terms quaternions

$F\colon M \to \mathbb{R} \cong \mathbb{H}$: surface

$\Leftrightarrow$

$*dF := dF \circ J = N\,dF = -dF\,R$

$N, R\colon M \to S^2 \subset \operatorname{Im}\mathbb{H}$

Then F, $G_0 + iG_1$ in Theorem have the relations

$$F = NG_0 - G_1, \quad dF = dN\,G_0$$

$$*dG_0 = -N\,dG_0$$

Characterization

Fact. • $(N, R) = (\tilde{\phi}_F, \tilde{\psi}_F)$

- F is superconformal
  $\Leftrightarrow *dN = -dN\,N$ or $*dR = -dR\,R$
- F is minimal
  $\Leftrightarrow *dN = dN\,N$ and $*dR = dR\,R$
- $G = G_0 + iG_1$ is a null holomorphic curve
  $\Leftrightarrow *dG_0 = -dG_1$, G_0 and G_1 are (branched) minimal
- $*dN = N\,dN \Leftrightarrow *d(-N) = -(-N)\,d(-N)$

Key Lemma

Lemma. M: 2-manifold

$f, g: M \to \mathbb{H}$: immersions

$df \wedge dg = 0$

$\Leftrightarrow$

$f, g: (M, {}^{\exists}J) \to \mathbb{H}$: surface

$*df = N_f\, df = -df\, R_f$,

$*dg = N_g\, dg = -dg\, R_g$,

$-R_f = N_g$

From F to G

- ∘ $*dF = N\,dF$, $*dN = -dN\,N$
- $dF = dN\,G_0$, (def. of G_0)
- $dN \wedge dG_0 = 0$, (exterior derivative)
- $*dN = -dN\,N$, (superconfomality of F)
- $*dG_0 = -N\,dG_0$, (key lemma)
- $*d(-N) = d(-N)(-N)$, ((branched) minimality of G_0)
- $G_1 := NG_0 - F$, (def. of G_1)
- ∘ $dG_1 = dN\,G_0 + N\,dG_0 - dF = N\,dG_0 = -*dG_0$

From G to F

- ∘ $*dG_0 = -N\,dG_0 = -dG_1$, $G^{-1}(0) = \emptyset$, $*d(-N) = d(-N)\,(-N)$,
- $F := NG_0 - G_1$, (def. of F)
- $dF = dN\,G_0 + N\,dG_0 - dG_1 = dN\,G_0$, (exterior derivative)
- $*dN = -dN\,N = N\,dN$, ($N^2 = -1$)
- $*dF = *dN\,G_0 = N\,dN\,G_0 = N\,dF$ (F is a surface)
- ∘ $*dN = -dN\,N$, (superconformality of F)

Remarks

- If $\tilde{\phi}_F$ is branched, then G_0 is not defined at the branch point.
- If $\tilde{\phi}_{G_0}$ is branched, then F is branched exactly at the same branch points.
- If $\tilde{\phi}_{G_0}$ is not extendable at a branch point of G_0, then F is not defined at the branch point.
- If G_0 vanishes at a point, then F is branched at the point.

Thank you very much   Vielen Dank   謝謝

Б о л ь ш о е с п а с и б

Merci beaucoup   Gracias mucho   Grazie molto

obrigado muito   Dank u zeer   ありがとう