Warm-up Talk

"Again Riemann Surfaces?

Still Riemann Surfaces?

Yes, Riemann Surfaces forever!"

by Motohico Mulase

- Riemann Surface

  = Oriented Topological Surface

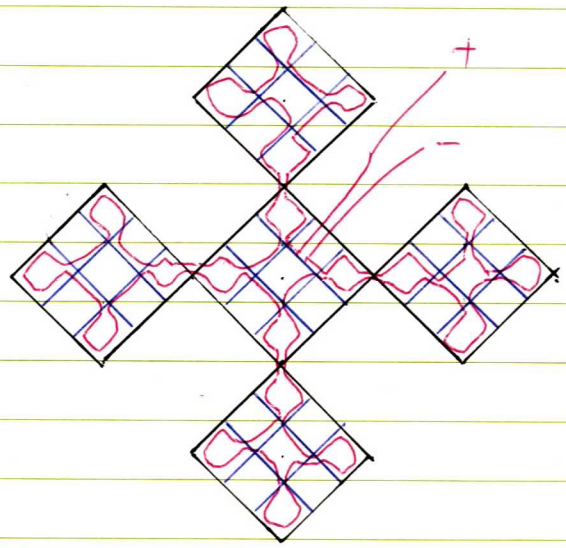
  + Complex Structure

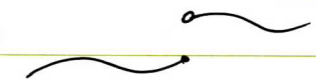
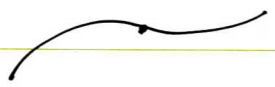
- We find a Riemann Surface in most unusual places where we do not expect them at all, for example,

  <u>In Your Pocket</u> !

- Please take a look at your cell phone. It contains a "Fractal Antenna" invented by Nathan Cohen in 1988.

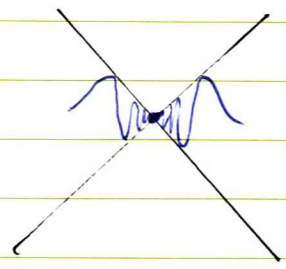
- Fractal Antenna of N. Cohen



- When you learn the ε-δ definition of continuity, you wonder why such a mysterious definition is necessary. After all, isn't it obvious what is discontinuous  and what is continuous  ?

- But when you see examples like

$$y = \begin{cases} x \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases},$$



which is continuous everywhere,

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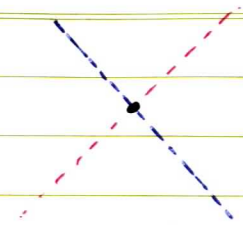
or $y = \begin{cases} x & x \in \mathbb{Q} \\ -x & x \notin \mathbb{Q}, \end{cases}$



which <u>is</u> continuous at $x = 0$,

you notice that our intuition is not so accurate.

- The Weierstrass example of a function, which is everywhere continuous but nowhere differentiable, shows something really interesting is happening beyond our imagination.

- The recent developments in complex dynamics demonstrated that these "fractal" curves should be understood as boundary behavior of Riemann surfaces, not as 1-dimensional objects.

- Thus in your pocket, you carry an application of the modern Riemann surface theory!

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Riemann Surfaces in Finite Groups

- As an example of unexpected appearances of Riemann surfaces in algebra, let us recall the fundamental formula of representation theory of finite groups:

$$|G| = \sum_{\lambda \in \hat{G}} (\dim \lambda)^2 . \qquad (\star)$$

Here
$$\begin{cases} G = \text{a finite group} \\ \hat{G} = \text{the set of irreducible representations} \\ \dim \lambda = \text{the dimension of representation } \lambda . \end{cases}$$

- Do you see a surface hidden behind this formula?
- It is in the exponent $2 = \chi(S^2)$.
- Let Σ be an oriented surface of genus g.

Then we have

$$\sum_{\lambda \in \hat{G}} (\dim \lambda)^{2-2g} = \frac{|\mathrm{Hom}(\pi_1(\Sigma), G)/G|}{|G|^{2g-2}} .$$

When $g = 0$, this formula recovers $(\star)$.

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- The above formula, and its generalizations involving non-orientable surfaces, have an elementary algebraic proof. But it is the matrix integral proof due to Josephine Yu - M.M. that shows the origin of the surface Σ.

Random Matrix Theory

- In one aspect, RMT is a powerful analysis tool, replacing the complex variable in complex analysis by a matrix variable.
- A typical question is to compute

$$Z_N(V) = \int_{\mathcal{H}_{N\times N}} e^{-\frac{N}{2}\operatorname{tr} X^2} e^{N\cdot \operatorname{tr}(V(X))} dX,$$

where the integration is taken on the space of $N\times N$ Hermitian matrices $\mathcal{H}_{N\times N} = \mathbb{R}^{N^2}$, dX is the Lebesgue measure on $\mathcal{H}_{N\times N}$, and V is a function in 1 complex variable.

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- If we also allow the characteristic function of a domain in the integrand, the matrix integral gives the relative probability of finding a Hermitian matrix satisfying given eigenvalue conditions.

- The question of particular importance is to determine the probability distribution of the largest eigenvalue of a random Hermitian matrix.

- Let (X_{ij}) be a Hermitian matrix, and $\mathrm{Re}\, X_{ij}$ ($\mathrm{Im}\, X_{ij}$) denote its real (imaginary) part. When $\mathrm{Re}\, X_{ij}$, $\mathrm{Im}\, X_{ij}$ $(i>j)$ and X_{ii} are given independently and identically distributed random variables, the question is to determine the probability distribution of the largest eigenvalue of (X_{ij}).

- Under a certain convenient normalization, λ behaves $\lambda \sim \sqrt{2N}$ as $N \to \infty$, and

$$\lambda - \sqrt{2N} \sim \frac{1}{\sqrt{2}} N^{-\frac{1}{6}} \quad (N \to \infty).$$

- The fundamental theorem is the following:

<u>Thm (Tracy-Widom 1992)</u>

Let $$F_2(t) = \lim_{N\to\infty} \mathrm{Prob}\left(\frac{\lambda - \sqrt{2N}}{\frac{1}{\sqrt{2}} N^{-\frac{1}{6}}} \leq t \right).$$

Then this limit exists, and the limiting function is given by

$$F_2(t) = \exp\left(- \int_t^\infty (s-t)\, q(s)^2 \, ds \right)$$

$q(t)$ is the solution of the Painlevé II

$$q'' = 2q^3 + t \cdot q$$

with the boundary condition

$$q(t) \sim -Ai(t) \quad \text{as } t \to +\infty,$$

where Ai = Airy function $\quad Ai'' = t \cdot Ai$.

- The ubiquitous appearance of $F_2(t)$ in most any non-Gaussian probability distribution is now called the <u>Tracy-Widom law</u>. It is a standard statistical tool which can be used, for example, in the recent determination that the longevity of Japanese people is due to genetic reasons, not so much of environmental factor!
- One of the early instances where the Tracy-Widom law was proved is the theory of random permutations.
- Here the question is to study the asymptotic behavior of the length of the longest increasing subsequence of a random permutation:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ \sigma(1) & \sigma(2) & \sigma(3) & \sigma(4) & \cdots & \sigma(n) \end{pmatrix} \in S_n$$

$$\sigma(i_1) < \sigma(i_2) < \cdots < \sigma(i_m) \qquad m = \text{length} = l_n.$$

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- We give the uniform measure to the symmetric group S_n, and denote by $E(l_n)$ = expectation value of l_n. Then

$$\begin{cases} E(l_n) \sim 2\sqrt{n}, & n \to \infty \\ E(l_n) - 2\sqrt{n} \sim \text{const.}\, n^{1/6} \end{cases}$$

- We ask the same question as before:

Does the limit

$$\lim_{n\to\infty} \text{Prob}\left(\frac{E(l_n) - 2\sqrt{n}}{n^{1/6}} \leq t \right) \quad \text{exist?}$$

Baik - Deift - Johansson proved in 1999 that yes it does, and moreover, it is exactly $F_2(t)$!!

- Why such random permutations are important in the context of Riemann surface theory?

- An unexpected answer was provided by Okounkov.

Recall the Richardson - Shensted - Knuth correspondence

$$S_n \ni \sigma \longleftrightarrow (\lambda_1, \lambda_2) \text{ pair of Young Tableaux of the same shape } \lambda.$$

- In this correspondence, a permutation of the shape λ has a measure $(\dim\lambda)^2/n!$.
- Recall that $n! = \sum_{\lambda} (\dim\lambda)^2$.
- This measure on the set of partitions is called the Plancherel measure.

Monodromy Problem

Find the number of degree d coverings of $\mathbb{P}^1$ by a connected Riemann surface ramified at $x_1, \cdots, x_n \in \mathbb{P}^1$ with the ramification profile $\mu^{(1)}, \cdots, \mu^{(n)}$, where $\mu^{(i)} \vdash d$ is a partition of d.

- Let $H_d(\mu^{(1)}, \cdots, \mu^{(n)})$ denote the number (Hurwitz number). Then by an elementary argument, we have

$$H_d(\mu^{(1)}, \cdots, \mu^{(n)}) = \frac{1}{d!} \sum_{\lambda \vdash d} \frac{(\dim\lambda)^2}{d!} \prod_{i=1}^{n} \left(|C_{\mu^{(i)}}| \frac{\chi_\lambda(\mu^{(i)})}{\dim\lambda} \right),$$

where C_μ is the conjugacy class of S_d belonging to μ and χ_λ is the character of irrep λ.

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- Now we have

questions of random permutations

$\Updownarrow$

questions of random ramified coverings of $\mathbb{P}^1$

$\Updownarrow$

Gromov-Witten invariants of $\mathbb{P}^1$

$\Updownarrow$

Intersection theory on $\overline{\mathcal{M}_{g,n}}$ generalizing the Witten-Kontsevich theory.

- In this way, we see hidden Riemann surfaces in the theory of random permutations.
- From the random matrix theory, the relation to the moduli spaces $\mathcal{M}_{g,n}$ of Riemann surfaces of genus g with n marked points is straightforward.
- This is due to the Feynman diagram expansion of matrix integrals.

Feynman Diagram Expansion

$$\log \int_{\mathcal{H}_{N\times N}} e^{-\frac{N}{2}\operatorname{tr} X^2} e^{N\sum_j \frac{t_j}{j}\operatorname{tr} X^j} \, dX$$ ← normalized measure

$$= \sum_{\substack{\Gamma \text{ connected} \\ \text{ribbon graph}}} \frac{1}{|\operatorname{Aut}\Gamma|} N^{\chi(\Sigma_\Gamma)} \prod_j t_j^{v_j(\Gamma)},$$

- Consider a closed oriented surface Σ of genus g.
- A "ribbon graph" Γ is a cell-decomposition of Σ consisting of v vertices, e edges and n faces.
- $v_j(\Gamma)$ = # of vertices of degree j.

Then $\sum_j v_j = v$ and $\frac{1}{2}\sum_j j v_j = e$.

Of course we have $\chi(\Sigma) = 2-2g = v-e+n$.

- The topological type of cell-decompositions of Σ (i.e., ribbon graphs) define a cell-decomposition of the moduli space $\mathcal{M}_{g,n}$. The orbifold structure of $\mathcal{M}_{g,n}$ is thus encoded in matrix integrals.

Let us remark that the above expansion formula immediately imply the formula

$$\sum_{\lambda \in \hat{G}} (\dim \lambda)^{2-2g} = |\mathrm{Hom}(\pi_1(\Sigma_g), G)| \Big/ |G|^{2g-1}.$$

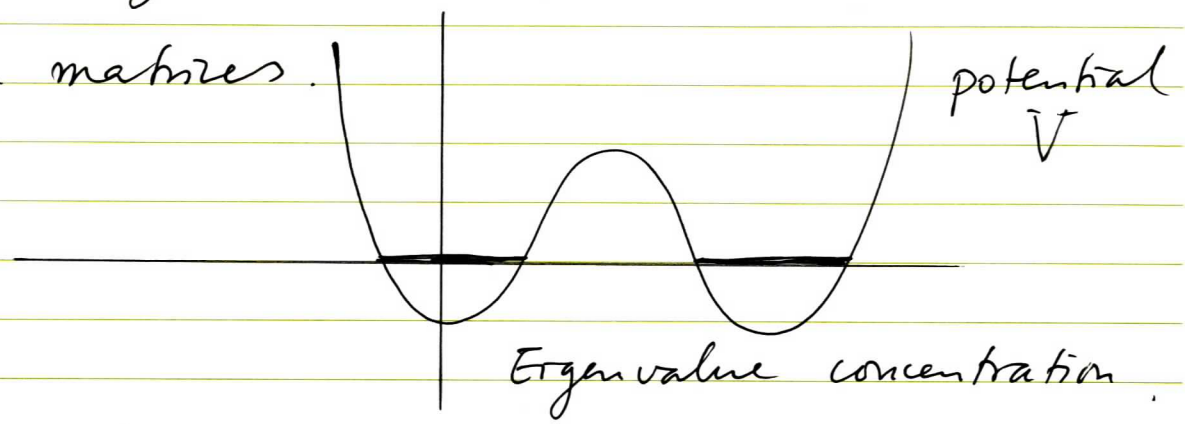
- It follows from the application of Feynman diagram expansion to each irreducible factor of the algebra decomposition of the group algebra $\mathbb{C}[G]$ of the finite group G:

$$\mathbb{C}[G] = \bigoplus_{\lambda \in \hat{G}} \mathrm{End}(\lambda).$$

Since $\mathrm{End}(\lambda) = \mathrm{Mat}_{\dim \lambda}(\mathbb{C})$, we apply the integration formula on the self-adjoint elements of $\mathbb{C}[G]$, using the above decomposition.

- The result is the formula for $\mathrm{Hom}(\pi_1(\Sigma_g), G)$, that comes out quite nicely.

- Another Riemann surface hidden in the random matrix theory is the complex resolvent set of random matrices.



- As the size $N \to \infty$, the eigenvalues of random Hermitian matrices are concentrated in the intervals determined by the potential function. The resolvent

$$\langle \mathrm{tr}(z-X)^{-1} \rangle = \int_{\mathcal{H}_{N\times N}} e^{-N\,\mathrm{tr}\,V(X)} \mathrm{tr}(z-X)^{-1}\, dX$$

is holomorphic outside of these intervals. In this case, the Riemann surface of the holomorphic function $\langle \mathrm{tr}(z-X)^{-1} \rangle$ is a hyperelliptic curve.

- A fascinating new theory based on these resolvent curves (called "spectral curves") is being developed by Eynard and his collaborators now.

- Many of these topics will be discussed in the Conference.