

Instanton counting (Survey)

at $\beta\bar{\alpha}$ 2008/12/16

origin : $N=2$ supersymmetric Yang-Mills theory
on \mathbb{R}^4

→ physical **quantities** defined via the path
integral over the space of all G -connections on \mathbb{R}^4
(G : cpt Lie group)

Witten (~ 1990)

If $\mathbb{R}^4 \cong X$: 4-mfd, the quantities are

Donaldson invariants! (Topological quantum field theory)

Seiberg-Witten (1994)

Computed the quantities for \mathbb{R}^4 .

→ Discovery of the Seiberg-Witten invariants

But a mathematically rigorous definition of the quantities for \mathbb{R}^4
was given much later

by Nekrasov (2002). It is "instanton counting".

Today $G = U(r) \quad (r \geq 1)$.

$M_0^{\text{reg}}(n, r)$: framed **moduli space** of G -instantons
on \mathbb{R}^4 with $C_2 = n$

$$= \left\{ A : G\text{-connection on } \mathbb{R}^4 \mid \begin{array}{l} *F_A = -F_A \\ \frac{1}{8\pi^2} \int_{\mathbb{R}^4} |F_A|^2 = n < \infty \end{array} \right\} / \text{gauge transf}$$

\times s.t. $\lim_{|\mathbb{R}^4| \rightarrow \infty} \delta(x) = 1$

\mathcal{C}^∞ -mfld & $\dim_{\mathbb{R}} = 4nr$ noncompact

$M_0(n, r)$: **Ohloneck partial compactification**

$$= \coprod_{m=0}^n M_0^{\text{reg}}(n-m, r) \times S^m \mathbb{R}^4$$

$$S^m \mathbb{R}^4 = (\mathbb{R}^4)^m / \mathbb{G}_m$$

Known affine algebraic variety (singular space, but described by polynomials)

In particular, integration of cohomology classes is well-defined.

$\tilde{\tau}: \overline{T}^r \times T^2 \curvearrowright M_0(n,r)$ by $\begin{array}{l} T^r \subset U(r) = G \\ T^2: \text{change of framing} \\ \end{array}$
 $T^2: \text{via the action on } \mathbb{C}^2 = \mathbb{R}^4$

We consider the equivariant cohomology

$$H_{\tilde{\tau}}^*(M_0(n,r)).$$

This is a module over $H_{\tilde{\tau}}^*(\text{point}) = \mathbb{C}[[\varepsilon_1, \varepsilon_2, a_1, \dots, a_r]]$
 $(\text{Lie } T^2 \otimes \mathbb{C} = \mathbb{C}[[\varepsilon_1, \varepsilon_2]], \text{Lie } T^r = \mathbb{C}[[a_1, \dots, a_r]])$

Def (instanton partition function)

$$\sum^{\text{inst}}(\varepsilon_1, \varepsilon_2, \vec{a}, \Lambda) := \sum_{n=0}^{\infty} \Lambda^{2nr} \int_{M_0(n,r)} 1$$

(the generating function of integrals over $M_0(n,r)$)

Definition & $\int_{M_0(n,r)} 1$

Localization theorem in equivariant cohomology (Segal, Atiyah-Bott)

$$H_T^*(M_0(n,r)) \otimes_{\mathbb{C}[[\varepsilon_1, \varepsilon_2, \vec{\alpha}]]} \mathbb{C}(\varepsilon_1, \varepsilon_2, \vec{\alpha}) \cong H_T^*(M_0(n,r)^\sim) \otimes_{\mathbb{C}[[\varepsilon_1, \varepsilon_2, \vec{\alpha}]]} \mathbb{C}(\varepsilon_1, \varepsilon_2, \vec{\alpha})$$

↑
fixed point set

In our case, it is easy to see $M_0(n,r)^\sim = pt$

$$\therefore H_T^*(M_0(n,r)) \otimes_{\mathbb{C}[[\varepsilon_1, \varepsilon_2, \vec{\alpha}]]} \mathbb{C}(\varepsilon_1, \varepsilon_2, \vec{\alpha}) \cong \mathbb{C}(\varepsilon_1, \varepsilon_2, \vec{\alpha})$$

$\int_{M_0(n,r)}$

Therefore $\Sigma^{inst}(\varepsilon_1, \varepsilon_2, \vec{\alpha}, \Lambda) \in \mathbb{C}(\varepsilon_1, \varepsilon_2, \vec{\alpha})[[\Lambda]]$.

It has a very bad singularities at $\varepsilon_1 = \varepsilon_2 = 0$.

Nekrasov Conjecture (Proved by N+Yoshioka, Nekrasov-Okounkov)

1) $\log Z^{\text{inst}} \in \frac{1}{\varepsilon_1 \varepsilon_2} (\mathbb{C}[\varepsilon_1, \varepsilon_2, \vec{\alpha}] \wedge \mathbb{I})$ Braherman-Etingof

2) $\varepsilon_1 \varepsilon_2 \log Z^{\text{inst}} \Big|_{\varepsilon_1 = \varepsilon_2 = 0} = (\text{the instanton part of})$
the Seiberg-Witten prepotential $F^{\text{SW}}(\vec{\alpha}; \Lambda)$.

Rem. The partition function has the so-called perturbative part given by an explicit elementary function.

$$\varepsilon_1 \varepsilon_2 \log Z^{\text{inst}} + \text{pert. part} \Big|_{\varepsilon_1 = \varepsilon_2 = 0} = \text{SW prepotential}$$

The Seiberg-Witten prepotential is defined by a period integral over a hyper-elliptic curve.

The conjecture is a kind of "mirror symmetry".

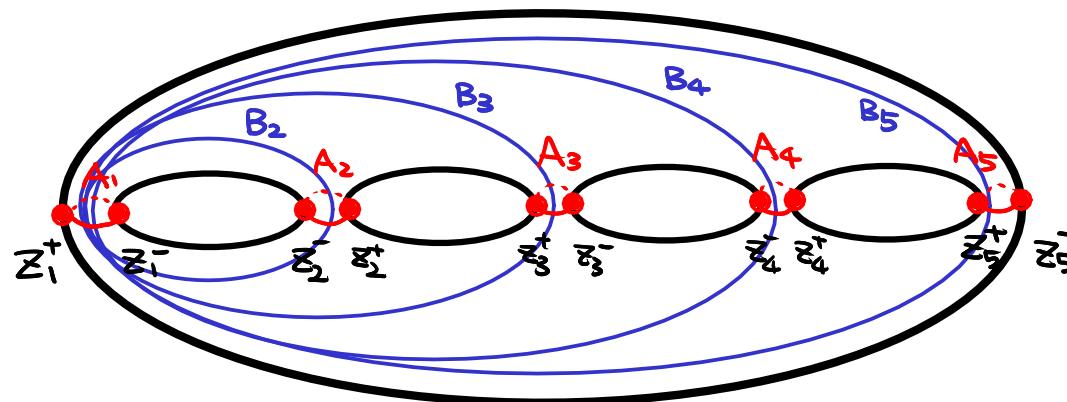
Seiberg-Witten prepotential

$$\vec{u} = (u_2, \dots, u_r) \in \mathbb{C}^{r-1}$$

$$C_{\vec{u}} : y^2 = P(z)^2 - 4\Lambda^{2r}$$

$$P(z) = z^r + u_2 z^{r-2} + \dots + u_r$$

z_α ($\alpha = 1, \dots, r$) : solutions of $P(z) = 0$
 z_α^\pm : solutions of $P(z) = \pm 2\Lambda^r$
 s.t. $z_\alpha^\pm \approx z_\alpha$ Δ : very small



↗ hyperelliptic involution

Take A, B-cycles as above. ($\sum_{\alpha=1}^r A_\alpha = 0$)

$\{A_\alpha, B_\alpha\}_{\alpha=2}^r$: symplectic base of $H_1(C_{\vec{u}})$

Seiberg-Witten meromorphic differential

$$dS := -\frac{1}{2\pi} \frac{z P'(z) dz}{y} \quad (\text{poles at } \infty_{\pm})$$

Refine

$$\begin{aligned} a_\alpha &\equiv a_\alpha(\vec{u}) = \int_{A_\alpha} dS \\ a_\alpha^D &\equiv a_\alpha^D(\vec{u}) = \int_{B_\alpha} dS \end{aligned} \quad (\alpha=2,\dots,r)$$

$$\frac{\partial a_\alpha}{\partial u_p} = \frac{1}{2\pi} \int_{A_\alpha} \underbrace{\frac{z^{r-p} dz}{y}}_{\curvearrowright} \quad \text{a basis of holomorphic differentials}$$

$\therefore (\tau_{\alpha\beta}) = \left(\frac{\partial a_\alpha}{\partial u_p} \right)^{-1} \left(\frac{\partial a_\beta^D}{\partial u_p} \right)$ is the period matrix of $C_{\vec{u}}$.

$$= \left(\frac{\partial a_\beta^D}{\partial a_\alpha} \right) \leftarrow \text{symmetric!}$$

$$\therefore \exists \phi_i: \text{potential} \quad \text{s.t.} \quad a_\alpha^D = -\frac{1}{2\pi F_i} \frac{\partial \phi_i}{\partial a_\alpha} \quad \text{i.e.} \quad \tau_{\alpha\beta} = -\frac{1}{2\pi F_i} \frac{\partial^2 \phi_i}{\partial a_\alpha \partial a_\beta}$$

This is the SW prepotential.

About the proofs

[NY] : Both $\varepsilon_1 \varepsilon_2 \log Z|_{\varepsilon_1=\varepsilon_2=0}$ & \mathcal{F}_1 satisfy
the same recursion (in Λ).

$\varepsilon_1 \varepsilon_2 \log Z|_{\varepsilon_1=\varepsilon_2=0}$: analysis of $\mathbb{C}^2 \leftarrow \hat{\mathbb{C}}^2$ blow-up
 \mathcal{F}_1 : Fay's trisecant identity.
(originally found by

Losev, Neft业sov, Shatashvili
Gorsky, Marshakov, Mironov, Morozov)

[NO] : Express $\varepsilon_1 \varepsilon_2 \log Z|_{\varepsilon_1=\varepsilon_2=0}$ in terms of Young diagrams
→ analysis of asymptotic behaviour
when the size of Young diagram is large.

Question What are the meaning of higher terms in

$$\log \mathcal{Z}(\varepsilon_1, \varepsilon_2, \vec{\alpha}, \Lambda) = \frac{1}{\varepsilon_1 \varepsilon_2} (F(\vec{\alpha}, \Lambda) + (\varepsilon_1 + \varepsilon_2) H(\vec{\alpha}, \Lambda) + \varepsilon_1 \varepsilon_2 A(\vec{\alpha}, \Lambda) + \frac{\varepsilon_1^2 + \varepsilon_2^2}{3} B(\vec{\alpha}, \Lambda) + \dots) ?$$

Conjecture All terms can be expressed in terms of the Seiberg-Witten curve.

"Proof": Obvious from the mirror symmetry.

e.g. $\bullet H = -\pi \sum_i \langle \vec{\alpha}_i, \rho \rangle \quad \rho = \text{half-sum of positive roots}$
 $= -\frac{\pi \sqrt{r}}{2} \sum_{\alpha < \rho} (\alpha_\alpha - \alpha_\beta)$

$\bullet A = \log \det \left(\frac{\partial U_p}{\partial \alpha} \right) \underset{\substack{\text{up to} \\ \text{const}}}{,} \quad B = \log \Delta \underset{\substack{\text{up to} \\ \text{const}}}{\text{discriminant}}$

(proved only for $r=2$ so far)

Geometric Engineering

$\Gamma \subset \mathrm{SL}_2(\mathbb{C})$ finite

(Katz-Klemm-Vafa)

subgroup \longleftrightarrow ADE Dynkin diagram
e.g. $\mathbb{Z}_r \leftrightarrow A_{r-1}$

$$X = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \hookrightarrow \Gamma \\ \downarrow \\ \mathbb{P}^1$$

$$X_\Gamma = \text{crepant resolution of } X/\Gamma \\ \downarrow \\ \mathbb{P}^1$$

fiber = minimal resolution of \mathbb{C}^2/Γ



$$H_2(X_\Gamma, \mathbb{Z}) = H_2(\mathbb{P}^1; \mathbb{Z}) \oplus H_2(\text{fiber}; \mathbb{Z}) \\ \downarrow \quad \downarrow \\ [\mathbb{P}^1] \quad [C_\alpha]$$

(toric if Γ : type A)

Consider **Gromov-Witten** invariants for X_Γ (local Calabi-Yau 3-fold).

$$\sum_{i=1, \dots, r-1} Z^{\mathrm{GW}}(h, g_b, g_i) = \exp \left[\sum_{g=0}^{\infty} h^{2g-2} \sum_{d_b=1}^{\infty} g_b^{d_b} \sum_{d_i=0}^{\infty} g_i^{d_i} \int \frac{1}{[M_{g,0}(X_\Gamma, d_b, d_i)]^{\mathrm{vir.}}} \right]$$

degree

$$\mathcal{Z}^{\text{inst}}(\varepsilon_1 = \hbar, \varepsilon_2 = -\hbar, \vec{a}, \Lambda) = \text{a certain limit of } \mathcal{Z}^{\text{GW}}(\hbar, g_b, g_i)$$

with a substitution $\begin{cases} \Lambda = g_b \\ g_i = e^{a_{i+1} - a_i} \end{cases}$ (up to const)

Remark, ① If we consider the K-theoretic integration in the instanton side, we do **not** need to take a limit.

(i.e. The parameter corresponds to the parameter for $K \xrightarrow{\text{gr}} H^*$)

② For Γ : type A, $=$ is proved in a mathematically rigorous way by Zhou.

Compute both sides.

a) $M_{0,n,r}$ has a resolution of singularities $M(n,r)$

& $M(n,r)^T = r$ -tuples of Young diagrams (Y_1, \dots, Y_r) $\sum |Y_\alpha| = n$
 \leadsto purely combinatorial expression of $\mathcal{Z}^{\text{inst}}$

b) GW side : Use the topological vertex.

\leadsto Get the same combinatorial expression.

③ perturbative part of the instanton partition function
= up to constant map contribution GW invariants for $db = 0$

Note that the parameters $\varepsilon_1, \varepsilon_2$ ($\in \text{Lie } T^2$) are specialized to the line $\varepsilon_1 + \varepsilon_2 = 0$, ($\hbar = \varepsilon_1 = -\varepsilon_2$)

So a question remains

What is the meaning of $\varepsilon_1 + \varepsilon_2$?

An explanation was given by Vafa, Gukov ---
Related to Khovanov link homology.

But it is not precise enough, and a further study is required.

Go back to GW invariants for $X_P = \left(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) / P \right)^{\sim}$.

How these are computed? (I mentioned
the "topological vertex".)

I do not go to the detail.

..... But I want to mention that
the invariants are related to
Chern-Simons link invariants,
(Jones-Witten)

X is a crepant resolution of the conifold.

$$\textcircled{1} \quad X = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \ni ([z_0 : z_1], \varsigma_1, \varsigma_2)$$

$\uparrow \quad \uparrow \quad \uparrow$
 $\mathbb{P}^1 \quad \text{fibers}$

$$\mapsto (x = z_0 \varsigma_1, y = z_1 \varsigma_2, z = z_0 \varsigma_2, w = z_1 \varsigma_1) \in \{xy = zw \subset \mathbb{C}^4\}$$

conifold

There is another way to get a smooth manifold from the conifold. \rightarrow smoothing.

$xy = zw + t$ is smooth if $t \neq 0$
 $(t \in \mathbb{C} : \text{parameter})$

T^*S^3

diffeomorphic

Large N duality (Gopakumar-Vafa, Ooguri-Vafa, ...)

Closed string theory on $X_P =$ open string theory on T^*S^3/Γ

GW invariants

open GW invariants
with lagrangian $= S^3/\Gamma$

+ Witten SU(N)- Chern-Simons theory on M^3 (real 3-mfd)
 $\vdots = \text{open string theory on } T^*M \quad (\text{But make } N \text{ also as a variable})$

This is "computable".

- $\Gamma = 114$ for simplicity ($\rightarrow \text{No } \vec{\alpha}$)

Dictionary for variables

instanton counting	closed GW for X	open GW for T^*S^3	CS for S^3
specialised $\varepsilon_1 = -\varepsilon_2, \Lambda$ at $\varepsilon_1 + \varepsilon_2 = 0$	h, g_b (genus) (degree)	h, t (genus)' (# of holes)	N, k (rank) (level)

$\varepsilon_1 + \varepsilon_2$

?

{ 1 additional
variable

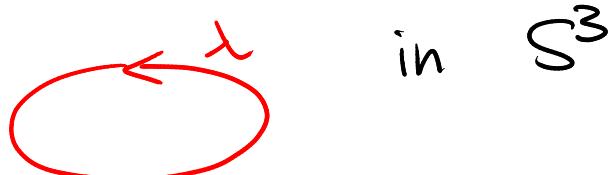
?

link inv.
= Euler char. of
Khovanov homology group
 \therefore Poincaré pol.
gives one more variable!

Unfortunately this is very speculative , as Khovanov homology
is defined only for very restricted link invariants
i.e., $M = S^3, SU(N)$ decorated by minuscule representations.

But still , we can make nontrivial checks.

Consider the unknot



in S^3

decorated by a representation λ of $SU(N)$
(\vdots
Young diagram)

Chern-Simons
Jones witten inv. = "quantum" dimension of λ
(deformation of the actual dim.)

In open or closed GW invariants , it is expected that
it corresponds to a Lagrangian subvariety
in X or T^*S^3
 \cup conformal b'dle of the link
+ information of λ

In the instanton counting , it is expected that it corresponds to a natural vector b'dle over the resolution of the moduli spaces :

$$\begin{aligned} \text{U(1)-case: resolution} &= \text{Hilbert scheme of points on } \mathbb{C}^2 \\ &= \{ I \subset \mathbb{C}[x,y] \mid \text{ideals, } \dim \mathbb{C}[x,y]/I = n \} \\ M(1,n) &\rightarrow M_0(1,n) = S^n \mathbb{R}^4 \end{aligned}$$

Let E be a vector b'dle over $M(1,n)$, whose fiber at I
 $= \mathbb{C}[x,y]/I$ (tautological line b'dle)

$$\sum_{n=0}^{\infty} \sum_{\lambda} (-1)^{\lambda} \text{ch}_{T^2} H^{\lambda}(M(1,n), S^{\lambda} E) \Delta^{2n} \quad S^{\lambda} : \text{Schur functor} \\ \text{e.g. } \Lambda^p E, S^p E \text{ etc} \\ \left(\text{K-theoretic correlation function} \right) \\ \text{for U(1)-gauge theory}$$

This is essentially equals to the Poincaré polynomial of Khovanov homology of the unknot.
 (not yet defined beyond $\Lambda^p E$)