On holomorphic sections of a holomorphic family of Riemann surfaces of genus two

Toshihiro Nogi (Osaka City University)

(Joint work with

Yoichi Imayoshi and Yohei Komori)

Riemann Surfaces, Harmonic Maps and Visualization

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Osaka City University

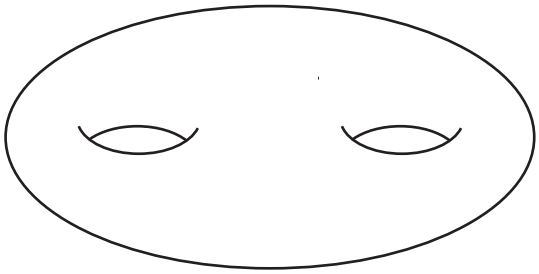
§1. Background of research

§1-1. Mordell's conjecture over a number field

k : a number field

$$C = \{(x, y) \in \mathbb{C}^2 \mid \textstyle\sum_{i+j \leq N} a_{ij} x^i y^j = 0\},$$
$$a_{ij} \in k.$$

genus $g(C) = 2$



Mordell's conjecture over k

$$g(C) \geq 2 \Rightarrow \sharp\{(x, y) \in C \cap k^2\} < \infty.$$

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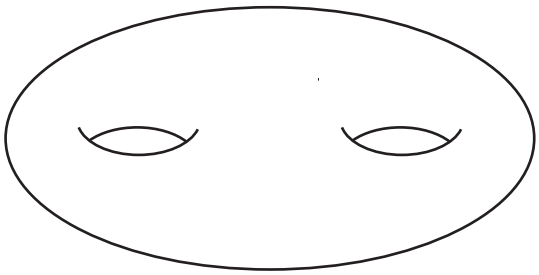


Faltings Theorem('83)

$$g(C) \geq 2 \Rightarrow \sharp\{(x, y) \in C \cap k^2\} < \infty.$$

Example

$k = \mathbb{Q}$

$X^n + Y^n = Z^n$: Fermat equation

$x = X/Z,\ y = Y/Z \Rightarrow$

$$C = \{(x, y) \in \mathbb{C}^2 \mid x^n + y^n - 1 = 0\}.$$

$$g(C) = \frac{(n-1)(n-2)}{2}.$$

$n > 3 \Rightarrow g(C) > 2$

Faltings Theorem $\Rightarrow$

$$\sharp\{(x, y) \in C \cap \mathbb{Q}^2\} < \infty.$$

§1-2. Mordell's conjecture over an algebraic function field

K : algebraic function field over $\mathbb{C}$

$\bar{K}$: algebraic closure of K

$$S = \{(X, Y) \in \bar{K}^2 \mid \textstyle\sum_{i+j \leq n} A_{ij} X^i Y^j = 0\},$$

$$A_{ij} \in K.$$

Mordell's conjecture over K

$g(S) \geq 2 \Rightarrow$

$$\sharp\{(X, Y) \in S \cap K^2\} < \infty.$$

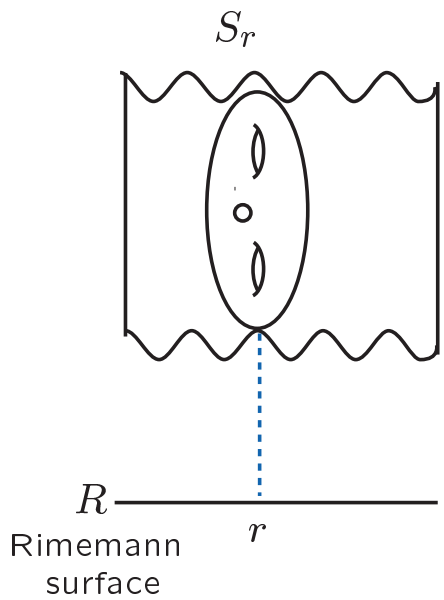
$$\Downarrow$$

<u>Step 1.</u> $^{\exists}$ a closed Riemann surface $\widehat{R}$ such that

$$K \cong \mathrm{Mer}(\widehat{R})$$

$$A_{ij} : \widehat{R} \to \widehat{\mathbb{C}} \text{ :mer}$$
$$r \mapsto A_{ij}(r)$$

<u>Step 2.</u> $^{\exists}$ a finite subset $B \subset \widehat{R}$ such that
$R := \widehat{R} \setminus B$,
for $^{\forall} r \in R$,
$S_r = \{(x, y) \in \mathbb{C}^2 \mid$
$\sum_{i+j \leq n} A_{ij}(r) x^i y^j = 0\}$
is a Riemann surface of genus g.



Step 3. Set

$$M = \bigsqcup_{r \in R} \{r\} \times S_r,$$

$$\pi : M \to R, \quad (r, q) \mapsto r.$$

$$\Downarrow$$

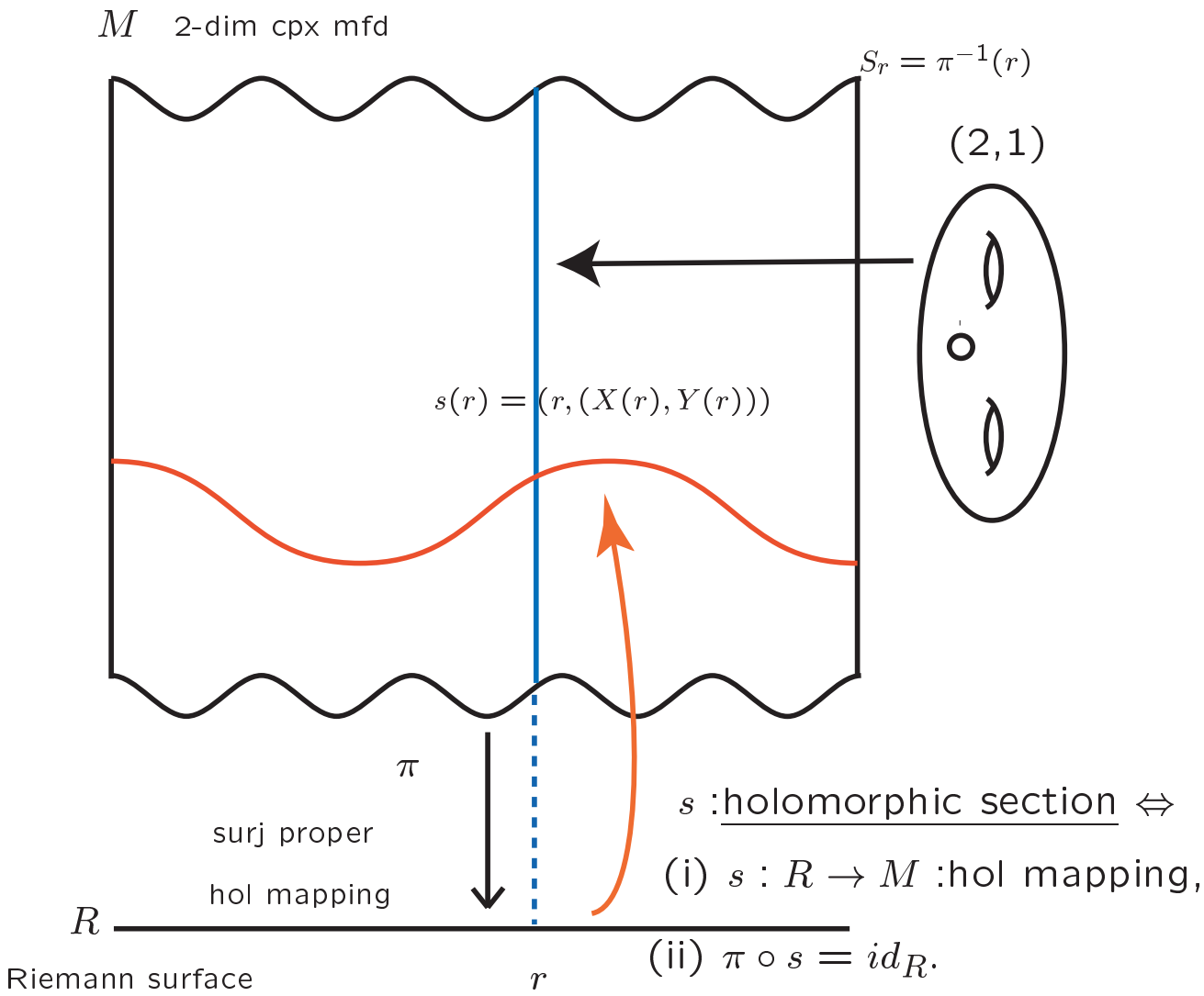
- M : 2-dim complex manifold

- $\pi : M \to R$: surjective proper holomorphic mapping

(M, π, R) satisfies the following conditions:

(i) For ${}^{\forall}m \in M$, rank$J_\pi(m) = 1$.
(ii) For ${}^{\forall}r \in R$,
$S_r = \pi^{-1}(r)$ is a Riemann surface of fixed type (g, n), $g = \sharp\{\text{genus}\}$ and $n = \sharp\{\text{puncture}\}$.

(M, π, R) is called a holomorphic family of Riemann surfaces of type (g, n)


M 2-dim cpx mfd
$S_r = \pi^{-1}(r)$
(2,1)
$s(r) = (r, (X(r), Y(r)))$
π
surj proper
hol mapping
R
Riemann surface
r
s :holomorphic section $\Leftrightarrow$
(i) $s : R \to M$:hol mapping,
(ii) $\pi \circ s = id_R$.

solution $\Leftrightarrow$ holomorphic section

$(X, Y) \in K^2$ $\qquad$ $s(r) = (r, (X(r), Y(r)))$, for $r \in R$

Mordell's conjecture for sections

$g(S_r) \geq 2$, M : locally non-trivial$\Rightarrow$ $\sharp\{$holomorphic sections of $(M, \pi, R)\} < \infty$.

proved by

- Manin ('63) and Coleman ('90)
- Grauert ('65)
- Imayoshi and Shiga ('88)

Problem

How many holomorphic sections does (M, π, R) have ?

Goal

Our family has at most twelve holomorphic sections.

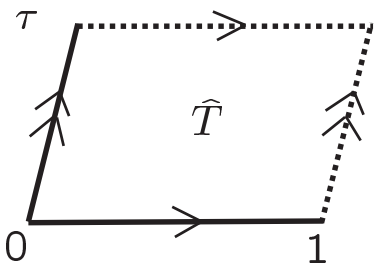
§2. Construction of a holomorphic family due to Riera

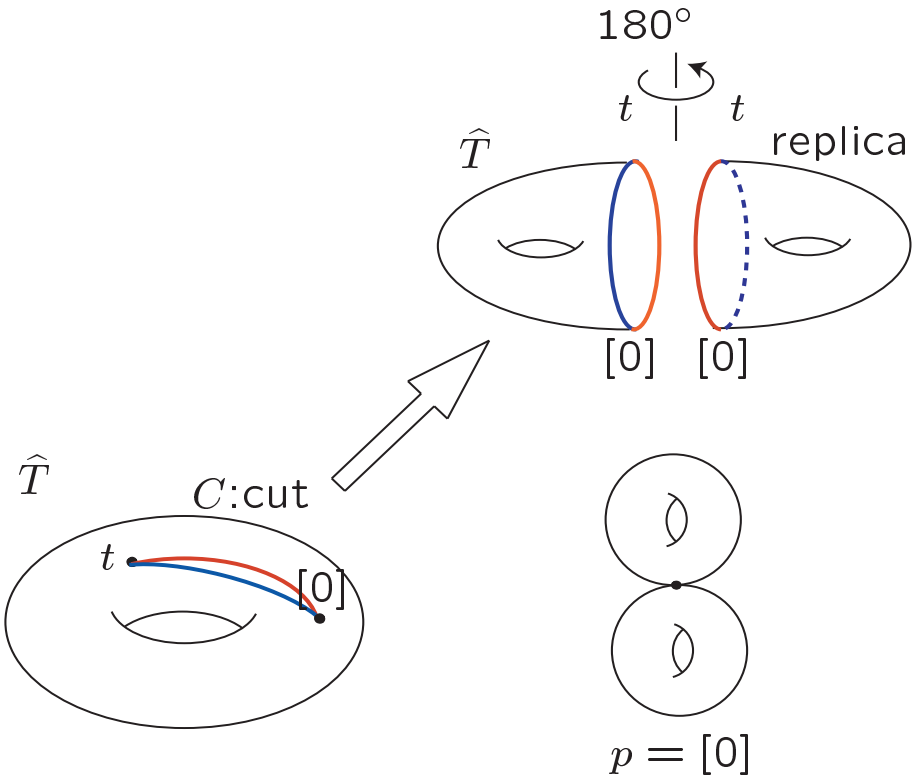
$\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im} z > 0\}$

$\tau \in \mathbb{H}$: fixed

$\hat{T} = \mathbb{C}_z/(\mathbb{Z} + \tau\mathbb{Z})$: torus

$\ni [z]$





$$T = \hat{T} \setminus \{[0]\}$$

<u>Remark</u> S_t depends on the choice of cut $C \Rightarrow S_{t,C}$.

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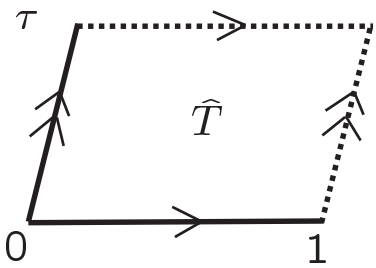

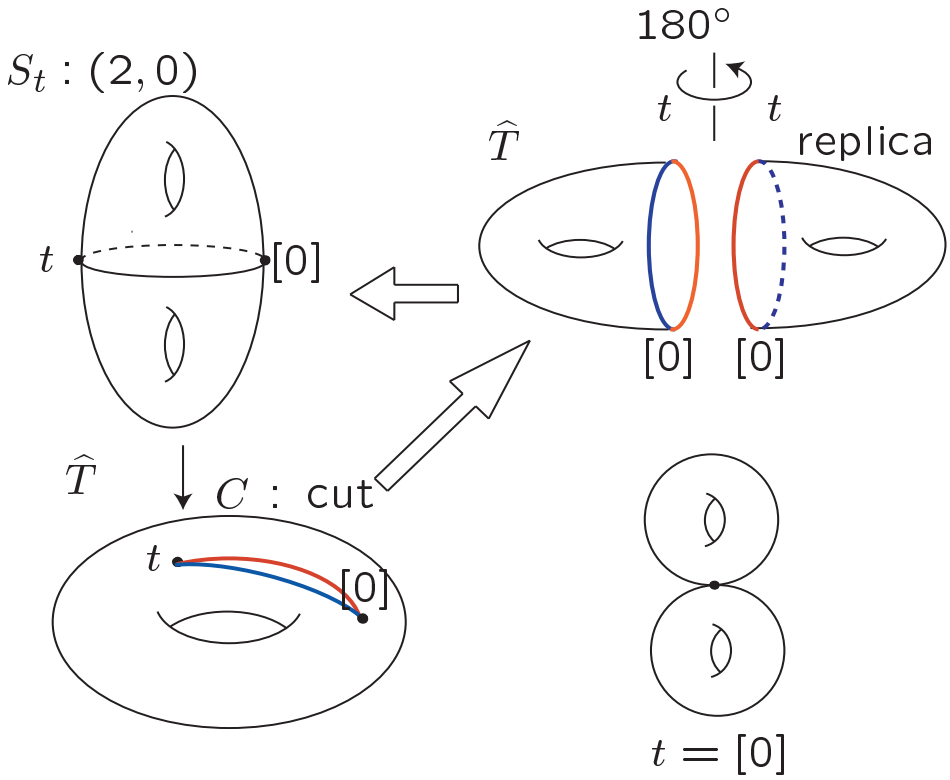

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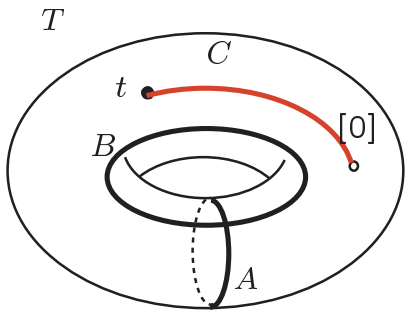
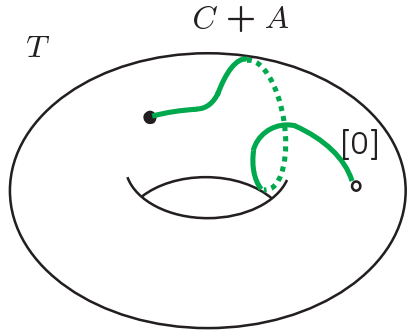
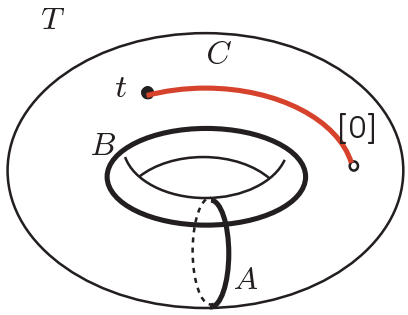


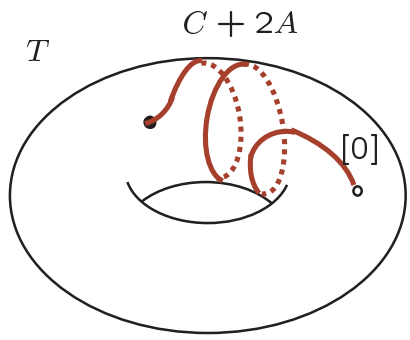


$\downarrow$ $S_{t,C}$ type $(2,0)$

$\downarrow$ $S_{t,C+A}$ type $(2,0)$

$$S_{t,C} \cong S_{t,C+A} \ ?$$





$\downarrow$ $S_{t,C}$

$\downarrow$ $S_{t,C+2A}$ type $(2,0)$

Fact1

$$S_{t,C} \cong S_{t,C+2A}$$

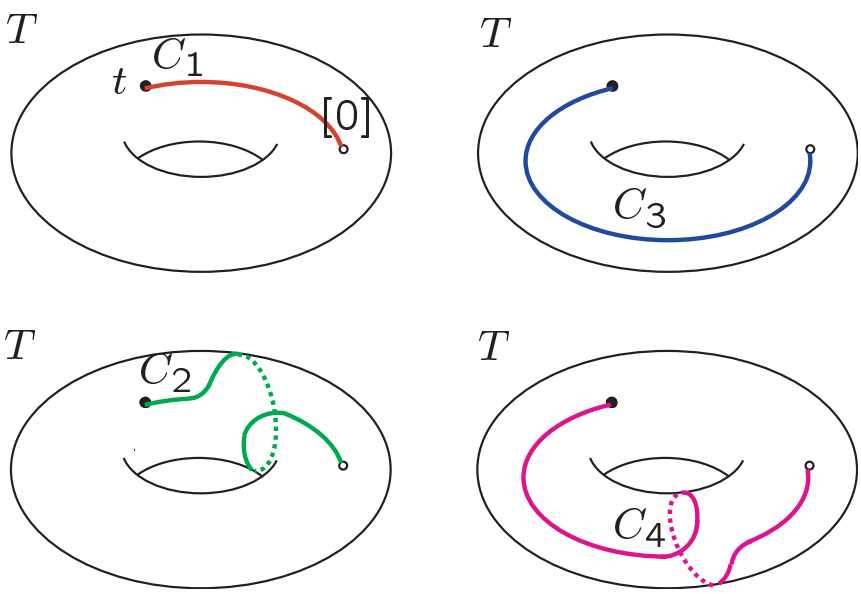
$$S_{t,C} \cong S_{t,C+2B}$$

Fact2

$$S_{t,C} \cong S_{t,C+2A\mathbb{Z}+2B\mathbb{Z}}$$

$$\Downarrow$$

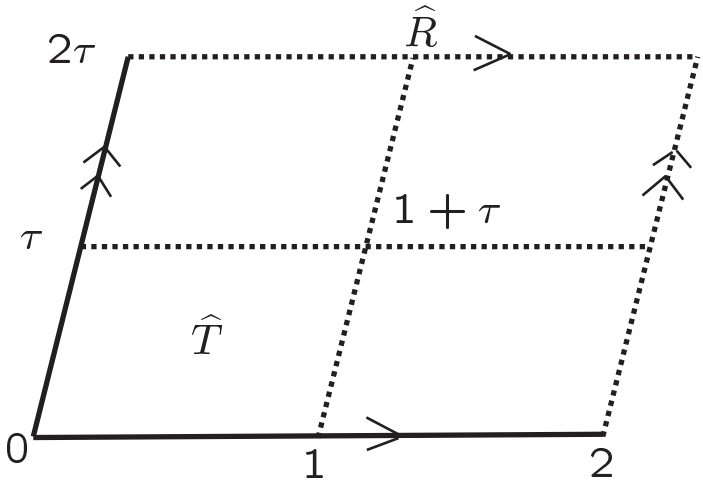
Essentially there are four cuts which determine different complex structures on $S_{t,C}$.



$\widehat{R} = \mathbb{C}_w/(2\mathbb{Z} + 2\tau\mathbb{Z})$: torus

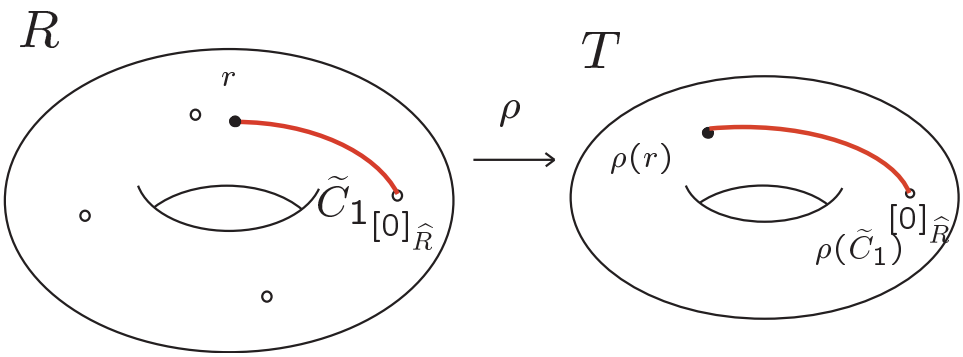
$\ni [w]_{\widehat{R}}$

$\widehat{\rho} : \widehat{R} \to \widehat{T}, [w]_{\widehat{R}} \mapsto [w]$: four-sheeted covering



$R = \widehat{R} \setminus \{[0]_{\widehat{R}}, [1]_{\widehat{R}}, [\tau]_{\widehat{R}}, [1+\tau]_{\widehat{R}}\}$

$\rho = \widehat{\rho}|R : R \to T$: four-sheeted covering



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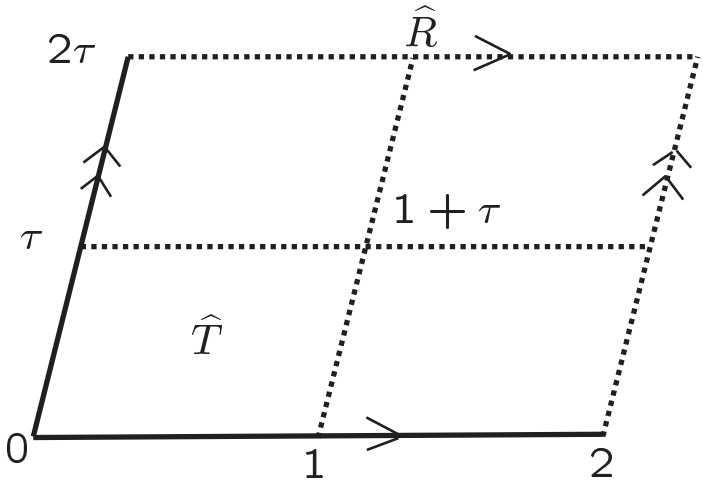

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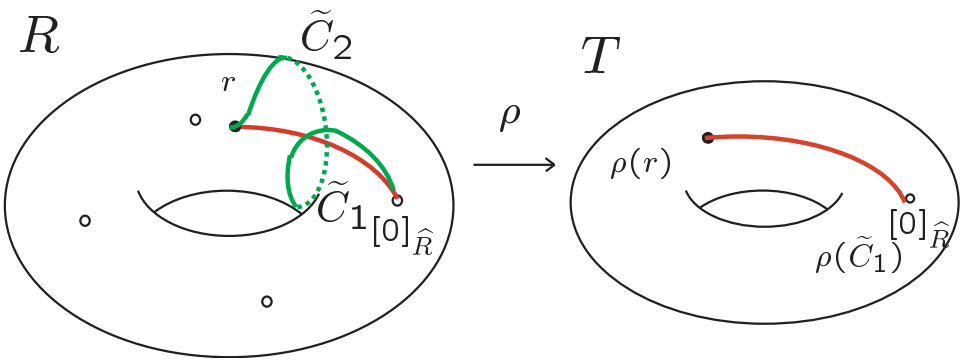

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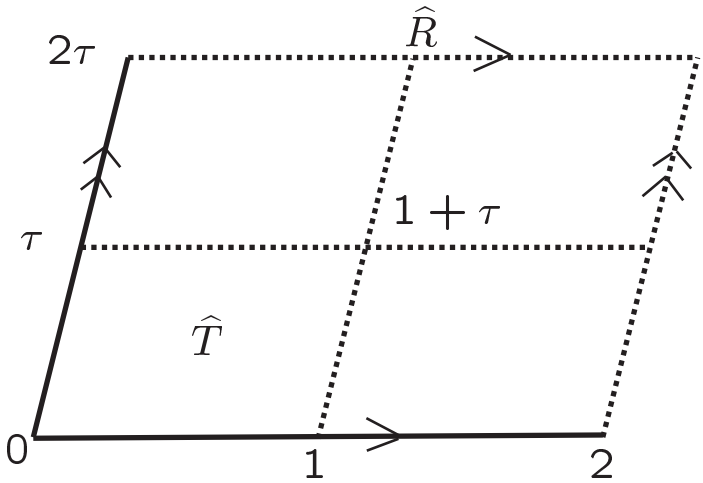

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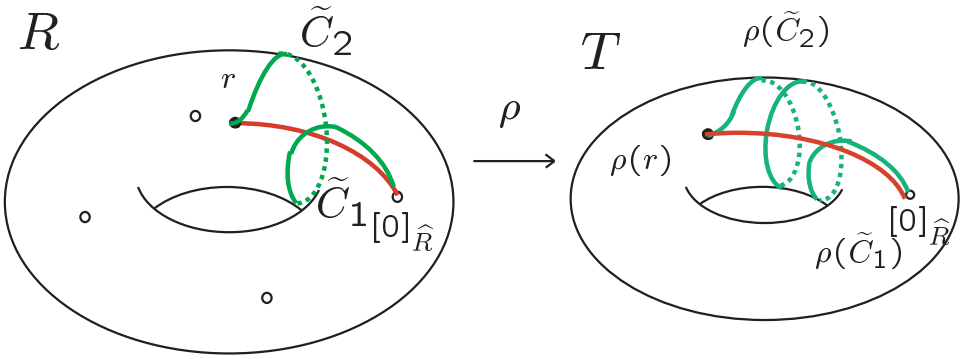

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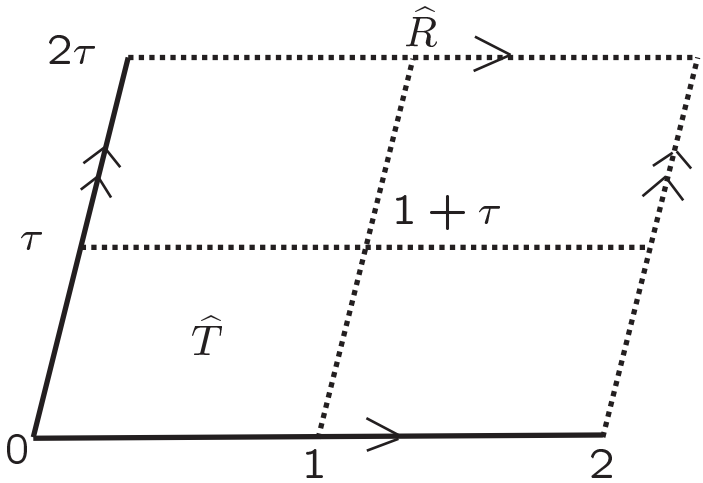

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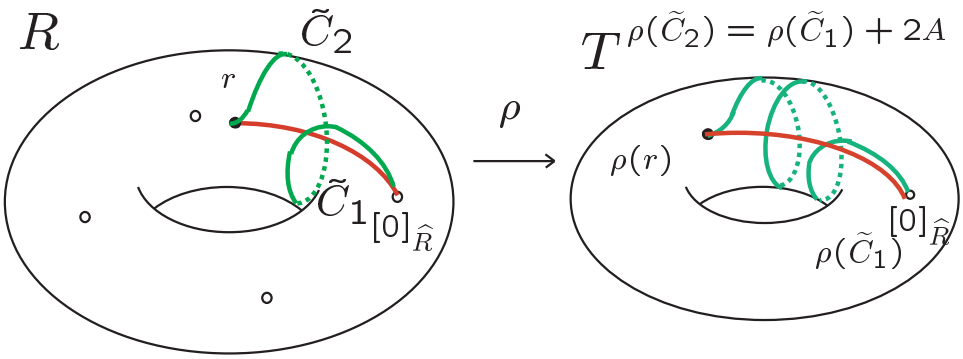

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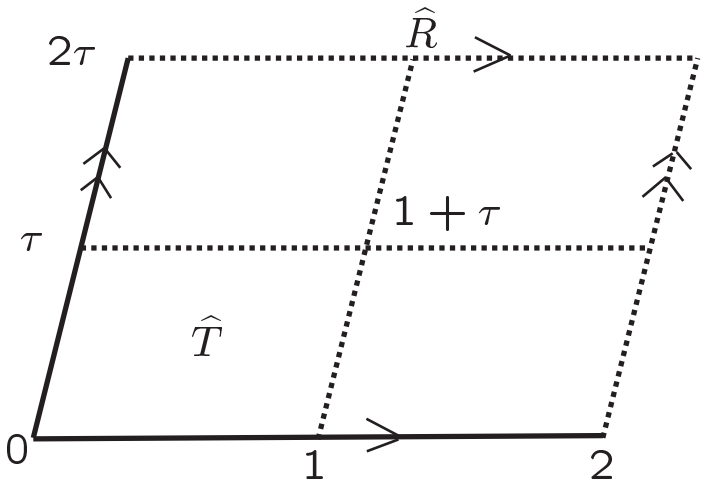

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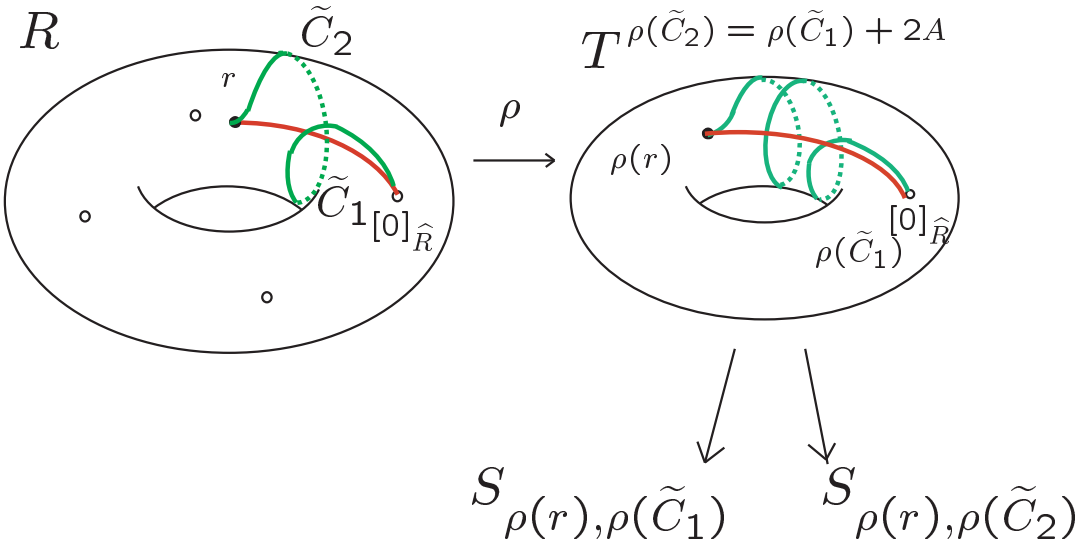

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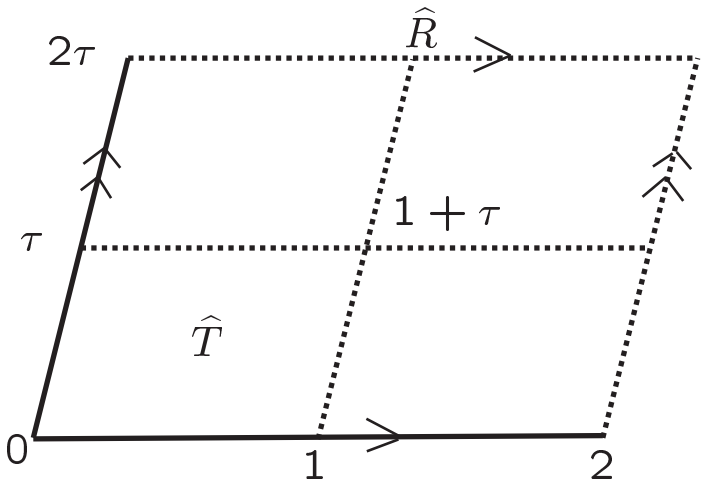

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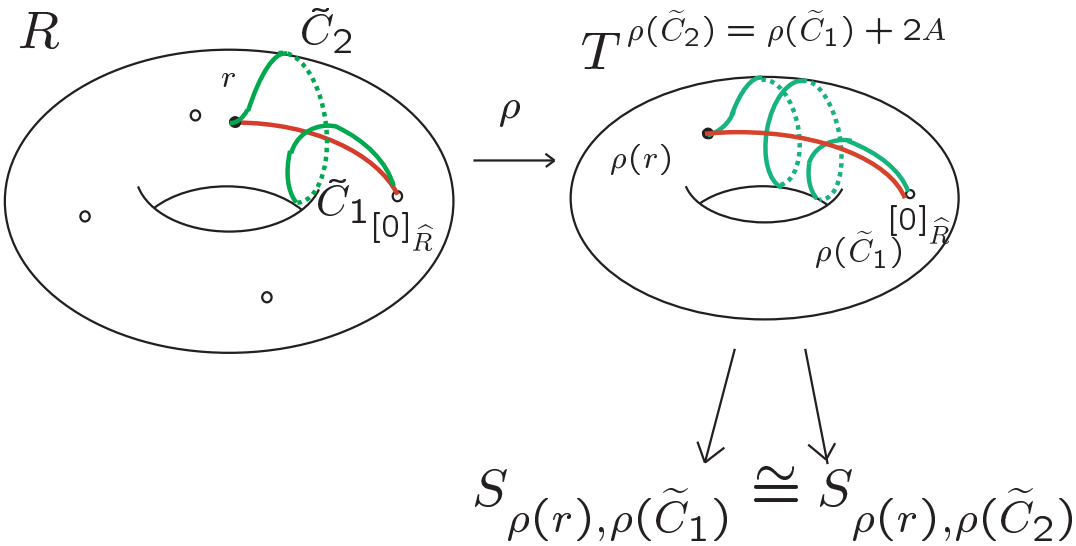
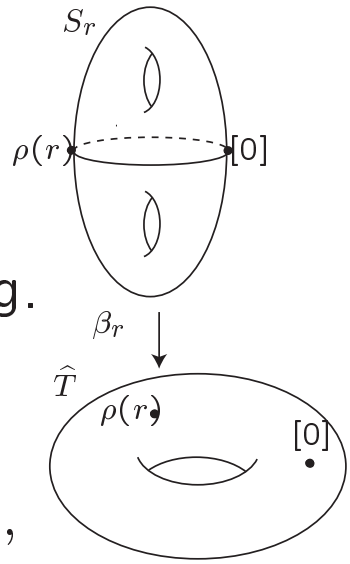


$S_{\rho(r),\rho(\widetilde{C}_1)} \Rightarrow S_r$.



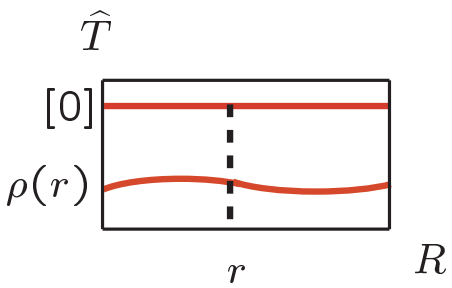
$\beta_r : S_r \to \widehat{T}$: two-sheeted branched covering.

Set

$$M = \bigsqcup_{r \in R} \{r\} \times S_r,$$

$$\begin{aligned} P : M \quad &\to \quad R \times \widehat{T}, \\ (r, q) &\mapsto (r, \beta_r(q)) \end{aligned}$$

M : two-sheeted branched covering surface branched over $\{(r, [0]) | r \in R\}$ and $\{(r, \rho(r)) | r \in R\}$.



$$\pi = Pr_1 \circ P : M \to R \times \widehat{T} \to R,$$
$$(r, q) \mapsto (r, \beta_r(q)) \mapsto r$$

Theorem (Riera('77)),Kodaira('67))

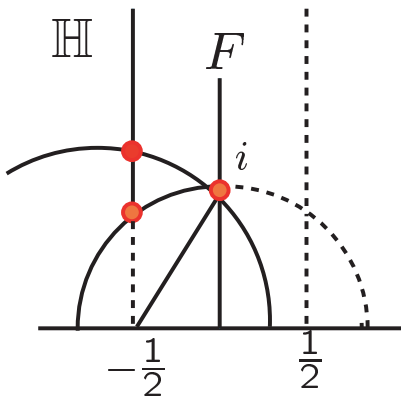
(M, π, R) is a holomorphic family of Riemann surfaces of type $(2, 0)$.

§3.Main Theorem

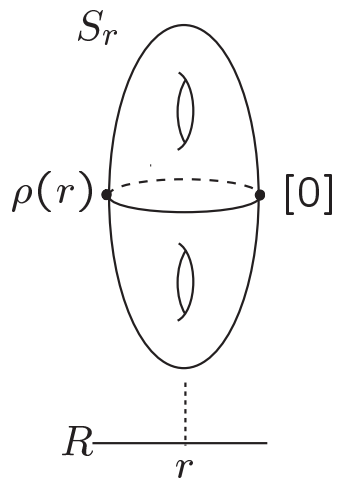
Main Theorem

N : the number of holomorphic sections of (M, π, R).

(i) $2 \leq N \leq 4$, if $\tau \neq i, e^{2\pi i/3}, \frac{-1+\sqrt{7}i}{2}$,
(ii) $2 \leq N \leq 8$, if $\tau = i, \frac{-1+\sqrt{7}i}{2}$,
(iii) $2 \leq N \leq 12$, if $\tau = e^{2\pi i/3}$,
where τ satisfies the following conditions : (i) $\mathrm{Im}\tau > 0$ (ii) $-1/2 \leq \mathrm{Re}\tau < 1/2$, (iii) $|\tau| \geq 1$, (iv) $\mathrm{Re}\tau \leq 0$ if $|\tau| = 1$.



Remark (M, π, R) has two holomorphic sections $s_0(r) = (r, [0])$ and $s_\rho(r) = (r, \rho(r))$ for $r \in R$.



$2 \leq N$

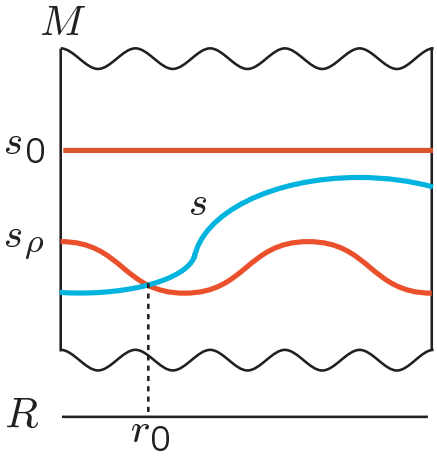
§4. Idea of a proof of main theorem

Key Theorem

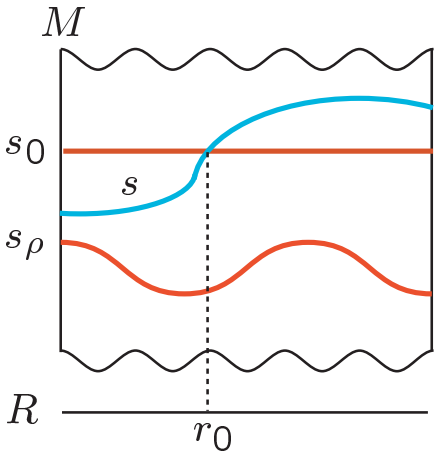
Let $s : R \to M$ be any holomorphic section and $\beta : M \to \widehat{T}$ a holomorphic mapping defined by $\beta(r, q) = \beta_r(q)$. Then composite mapping $\beta \circ s : R \to \widehat{T}$ has a holomorphic extension $\widehat{R} \to \widehat{T}$.

$$
\begin{tikzcd}
 & M \arrow[dr, "\beta", "(r,q) \mapsto \beta_r(q)"'] & \\
R \arrow[ur, "s"] \arrow[rr, "g = \beta \circ s"] \arrow[d, phantom, "\cap"] & & \widehat{T} \arrow[d, phantom, "||"] \\
\widehat{R} \arrow[rr, "\widehat{g}"] & & \widehat{T} \\
\mathbb{C}_z \arrow[u] \arrow[rr, "\widetilde{g}", "||"', "Az + B"] & & \mathbb{C}_w \arrow[u]
\end{tikzcd}
$$

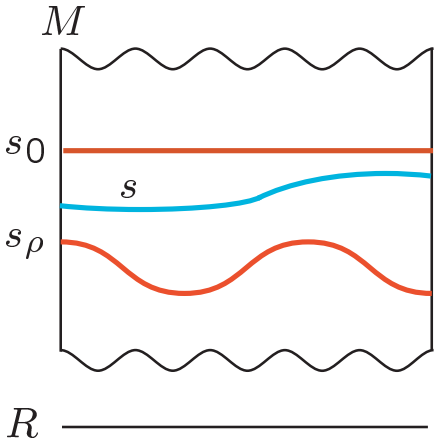
Case1



Case2



Case3



Assume $s \neq s_0$ and $s \neq s_\rho$.

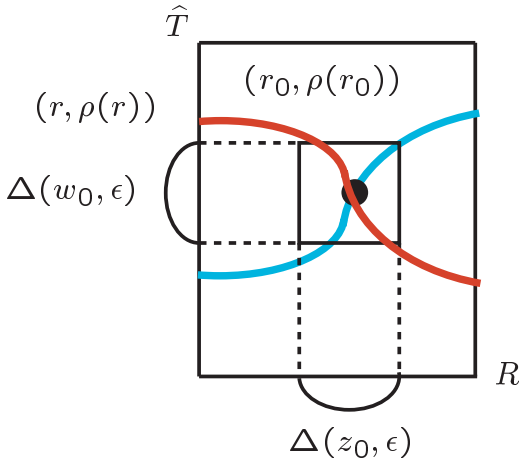
<u>**Case 1**</u> $^\exists r_0 \in R,\ s(r_0) = s_\rho(r_0)$.

$$
\begin{tikzcd}
 & M \arrow[dr, "\beta"] & & \\
r_0 \in R \arrow[ur, "s,\, s_\rho"] \arrow[rr, "g = \beta \circ s"', "\rho = \beta \circ s_\rho"] & & \widehat{T} & \beta \circ s(r_0) = \beta \circ s_\rho(r_0) \\
\cap & & \| & \\
\widehat{R} \arrow[rr, "\widehat{g},\, \widehat{\rho}"] & & \widehat{T} & \widehat{g}(r_0) = \widehat{\rho}(r_0) \\
\uparrow & & \uparrow & \\
z_0 \in \mathbb{C}_z \arrow[rr, "\tilde{g}(z) = Az + B", "\tilde{\rho}(z) = z"'] & & \mathbb{C}_w & Az_0 + B = z_0
\end{tikzcd}
$$

M is locally written as

$$M : u^2 = w - z$$

in $\mathbb{C}_u \times \Delta(z_0, \epsilon) \times \Delta(w_0, \epsilon)$.



Set $z = z_0 + \epsilon' e^{i\theta},\ \epsilon' < \epsilon$.

$$
\begin{aligned}
u^2 &= Az + B - z \\
&= A(z_0 + \epsilon' e^{i\theta}) + B - (z_0 + \epsilon' e^{i\theta}) \\
&= (A - 1)\epsilon' e^{i\theta}
\end{aligned}
$$

s is two-valued. $\Rightarrow$ contradiction.

Case 2 $^{\exists}r_0 \in R,\ s(r_0) = s_0(r_0)$.

By the same argument as in Case 1, we have a contradiction.

Case 3 $^{\forall}r \in R,\ s(r) \neq s_0(r)$ and $s(r) \neq s_0(r)$

$$\widetilde{g}(z) = Az + B$$

Lemma 1

$B = 0$.

$\widetilde{g}(2\mathbb{Z} + 2\tau\mathbb{Z}) \subset \mathbb{Z} + \tau\mathbb{Z},\ \Rightarrow {}^{\exists}p, q, r, s \in \mathbb{Z}$

$$2A = p + q\tau, \tag{1}$$

$$2A\tau = r + s\tau. \tag{2}$$

Lemma 2

$\deg(g) \leq 4$.

$$ps - qr \leq 4. \tag{3}$$

Necessary condition for $\tau \in F$ is

$$(p + s)^2 < 4(ps - qr). \tag{4}$$

Case (i) Fix$(\widetilde{g}) \notin L(1,\tau) = \{m + n\tau\}$.

The same argument as in Case 1
$\Rightarrow$ contradiction.

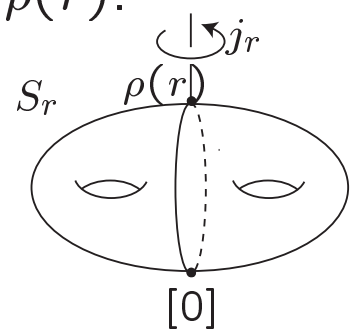
Case (ii) Fix$(\widetilde{g}) \in L(1,\tau)$.

$\widetilde{g}$ may be induced from a holomorphic section s. That is,
there exists a holomorphic section s
with $s \neq s_0$ and $s \neq s_\rho$.

Remark s is a holomorphic section
$\Rightarrow$ $j_r \circ s$ is also a holomorphic section,

where j_r is the involution of S_r
with two fixed points [0] and $\rho(r)$.

$\tau \neq i, e^{2\pi i/3}, \frac{-1+\sqrt{7}i}{2}$,



$\Downarrow$

Fix$(\widetilde{g}) \in L(1,\tau)$ if and only if $2A = 1$.

$\Rightarrow \widetilde{g}(z) = z/2$ may be induced from a holomorphic section s.

$$\begin{aligned} N &\leq \sharp\{s_0, s_\rho\} + \sharp\{s, j_r \circ s\} \\ &= 4. \end{aligned}$$

τ	$2A = p + q\tau$	fixed point
i	i	$(4+2i)/5$
$\sqrt{2}i$	$\sqrt{2}i$	$(2+\sqrt{2}i)/3$
$\sqrt{3}i$	$\sqrt{3}i$	$(2+\sqrt{3}i)/7$
$2i$	$2i$	$(1+i)/2$
$e^{2\pi i/3}$	$e^{2\pi i/3}$	$(5+\sqrt{3}i)/7$
$(-1+\sqrt{7}i)/2$	$(-1+\sqrt{7}i)/2$	$(5+\sqrt{7}i)/8$
$(-1+\sqrt{11}i)/2$	$(-1+\sqrt{11}i)/2$	$(5+\sqrt{11}i)/9$
$(-1+\sqrt{15}i)/2$	$(-1+\sqrt{15}i)/2$	$(5+\sqrt{15}i)/10$
i	$-i$	$(2+4i)/5$
$\sqrt{2}i$	$-\sqrt{2}i$	$(2+2\sqrt{2}i)/3$
$\sqrt{3}i$	$-\sqrt{3}i$	$(6+4\sqrt{3}i)/7$
$2i$	$-2i$	$(1+\sqrt{3}i)/2$
$e^{2\pi i/3}$	$-e^{2\pi i/3}$	$(3-\sqrt{3}i)/3$
$(-1+\sqrt{7}i)/2$	$(1-\sqrt{7}i)/2$	$(5+\sqrt{7}i)/4$
$(-1+\sqrt{11}i)/2$	$(1-\sqrt{11}i)/2$	$(3-\sqrt{11}i)/5$
$(-1+\sqrt{15}i)/2$	$(1-\sqrt{15}i)/2$	$(3-\sqrt{15}i)/6$
i	$2i$	$(1+i)/2$
$(-1+\sqrt{15}i)/4$	$(-1+\sqrt{15}i)/2$	$(5+\sqrt{15}i)/10$
$e^{2\pi i/3}$	$2e^{2\pi i/3}$	$\sqrt{3}i/3$
i	$-2i$	$(1+i)/2$
$(-1+\sqrt{15}i)/4$	$(-1+\sqrt{15}i)/2$	$(3-\sqrt{15}i)/6$
$e^{2\pi i/3}$	$-2e^{2\pi i/3}$	lattice point

τ	$2A = p + q\tau$	fixed point
any	1	lattice point
$e^{2\pi i/3}$	$1 + e^{2\pi i/3}$	$(3 + \sqrt{3}i)/3$
$(-1 + \sqrt{7}i)/2$	$(1 + \sqrt{7}i)/2$	$(3 + \sqrt{7}i)/4$
$(-1 + \sqrt{11}i)/2$	$(1 + \sqrt{11}i)/2$	$(3 + \sqrt{11}i)/5$
$(-1 + \sqrt{15}i)/2$	$(1 + \sqrt{15}i)/2$	$(3 + \sqrt{15}i)/6$
i	$1 + i$	lattice point
$\sqrt{2}i$	$1 + \sqrt{2}i$	$(1 + \sqrt{2}i)/3$
$\sqrt{3}i$	$1 + \sqrt{3}i$	$(1 + \sqrt{3}i)/2$
i	$1 - i$	lattice point
$\sqrt{2}i$	$1 - \sqrt{2}i$	$2(1 - \sqrt{2}i)/3$
$\sqrt{3}i$	$1 - \sqrt{3}i$	$(1 - \sqrt{3}i)/2$
$e^{2\pi i/3}$	$1 - e^{2\pi i/3}$	lattice point
$(-1 + \sqrt{7}i)/2$	$(3 - \sqrt{7}i)/2$	lattice point
$e^{2\pi i/3}$	$1 + 2e^{2\pi i/3}$	$2(2 + \sqrt{3}i)/7$
$(-1 + \sqrt{15}i)/4$	$(1 + \sqrt{15}i)/2$	$(3 + \sqrt{15}i)/6$

τ	$2A = p + q\tau$	fixed point
any	-1	$(2+2\tau)/3$
$e^{2\pi i/3}$	$-1+e^{2\pi i/3}$	$(7+\sqrt{3}i)/13$
$(-1+\sqrt{7}i)/2$	$(-3+\sqrt{7}i)/2$	$(7+\sqrt{7}i)/14$
i	$-1+i$	$(3+i)/5$
$\sqrt{2}i$	$-1+\sqrt{2}i$	$2(3+2\sqrt{2}i)/11$
$\sqrt{3}i$	$-1+\sqrt{3}i$	$(3+\sqrt{3}i)/6$
i	$-1-i$	$(4+2i)/5$
$\sqrt{2}i$	$-1-\sqrt{2}i$	$2(2+3\sqrt{2}i)/11$
$\sqrt{3}i$	$-1-\sqrt{3}i$	$(1+\sqrt{3}i)/2$
$e^{2\pi i/3}$	$-1-e^{2\pi i/3}$	$(5-\sqrt{3}i)/7$
$(-1+\sqrt{7}i)/2$	$-(1+\sqrt{7}i)/2$	$(5-\sqrt{7}i)/8$
$(-1+\sqrt{11}i)/2$	$-(1+\sqrt{11}i)/2$	$(5-\sqrt{11}i)/9$
$(-1+\sqrt{15}i)/2$	$-(1+\sqrt{15}i)/2$	$(5-\sqrt{15}i)/10$
$(-1+\sqrt{15}i)/4$	$-(1+\sqrt{15}i)/2$	$(5-\sqrt{15}i)/10$
$e^{2\pi i/3}$	$-1-e^{2\pi i/3}$	$2(2-\sqrt{3}i)/7$
$e^{2\pi i/3}$	$2+e^{2\pi i/3}$	lattice point
$(-1+\sqrt{7}i)/2$	$(1+\sqrt{7}i)/2$	lattice point
$e^{2\pi i/3}$	$2+2e^{2\pi i/3}$	lattice point
$e^{2\pi i/3}$	$-2-e^{2\pi i/3}$	$(7-\sqrt{3}i)/13$
$(-1+\sqrt{7}i)/2$	$-(3+\sqrt{7}i)/2$	$(7-\sqrt{7}i)/14$
$e^{2\pi i/3}$	$-2-2e^{2\pi i/3}$	$(3-\sqrt{3}i)/6$