Equilibria for anisotropic energies

Osaka City University, December 17, 2008

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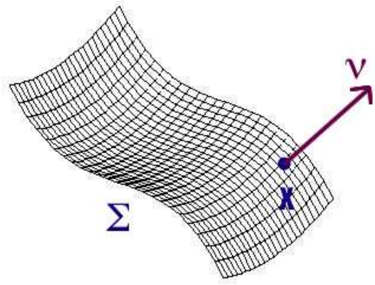
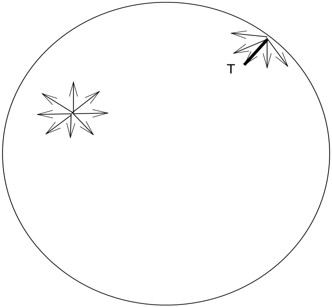
An anisotropic surface energy assigns to a surface a value which depends on the direction of the surface at each point. e.g.

$$\mathcal{F} = \int_{\Sigma} \gamma(X, \nu)\, d\Sigma\,,$$

Here $X : \Sigma \to \mathbf{R}^3$ is an immersed, oriented surface with unit normal ν.



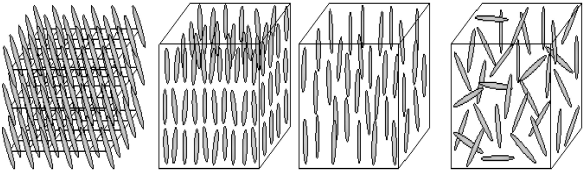
Small liquid drop



thermotropic liquid crystals

temperature

crystal	**liquid crystal (*mesophases*)**		liquid



• 3-D lattice • orientation • solid	• 1- (2-)D lattice • orientation • fluid	• no lattice • orientation • fluid	• no lattice • no orientation • fluid
↳ ***anisotropic***	↳ ***anisotropic***	↳ ***anisotropic***	↳ ***isotropic***

$$\text{Free} \quad \text{energy} = \mathcal{F} = \int_\Sigma \gamma(\nu)\, d\Sigma$$

We will assume that a *convexity condition* holds:

$$\chi := D\gamma + \gamma\nu : S^2 \to \mathbf{R}^3$$

defines a smooth, convex surface $W := \chi(S^2)$ called the **Wulff shape.**

For example, if $\gamma = ||\cdot||$ is a smooth norm on $\mathbf{R}^3$, with dual norm $||\,.\,||_*$, then

$$\mathcal{F} := \int_\Sigma ||\nu||\, d\Sigma\,,$$

satisfies the convexity condition and $W = \{\, ||\chi||_* = 1\,\}$.

First variation

$$\text{Free} \quad \text{energy} = \mathcal{F} = \int_\Sigma \gamma(\nu)\, d\Sigma$$

Variation: $X_\epsilon = X + \epsilon(\delta X) + ...$

First variation

$$\delta\mathcal{F} = -\int_\Sigma \Lambda\, \delta X \cdot \nu\, d\Sigma$$

$\Lambda =$ anisotropic mean curvature.

$\Lambda \equiv$ constant characterizes critical points of $\mathcal{F}$ subject to a volume constraint.

Local expressions for Λ.

$D\gamma =$ gradient of γ on S^2

$D^2\gamma =$ Hessian of γ on S^2

$$-\Lambda = \mathrm{Div}_\Sigma(D\gamma) - 2H\gamma = \mathrm{Trace}_\Sigma[(D^2\gamma + \gamma I)\cdot d\nu]\,.$$

If W and Σ are both surfaces of revolution, then

$$\Lambda = \frac{k_1}{\mu_1} + \frac{k_2}{\mu_2}\,.$$

$0 < \mu_i =$ principal curvatures of W.

Constant anisotropic mean curvature surfaces have much in common with constant mean curvature surfaces. For example we have:

Theorem

The Wulff shape W is the unique minimizer of the free energy $\mathcal{F}$ among all closed surfaces enclosing the same volume as W.

Theorem

The only closed, stable surface with constant anisotropic mean curvature is the Wulff shape.

Recently, Yijun He, Haizhong Li, Hui Ma, Jianquan Ge generalized the Alexandrov Theorem:

Theorem

The only embedded closed surface with $\Lambda \equiv$constant is the Wulff shape.

Variation of the anisotropic mean curvature

$$X_\epsilon = X + \epsilon(\delta X) + ...$$

$$\delta X = \psi\nu + \xi$$

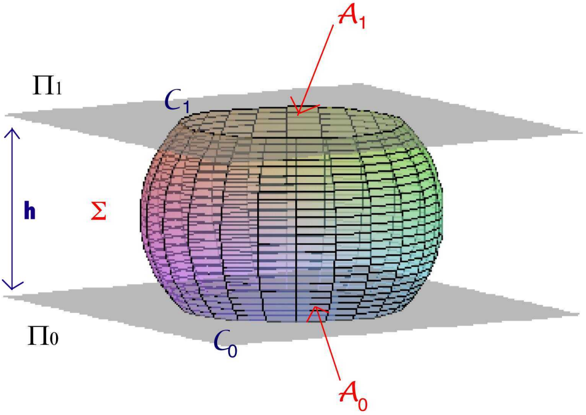
$\delta\Lambda = J[\psi] + \nabla\Lambda \cdot \xi$ where

$$J[\psi] = \mathrm{Div}_\Sigma[(D^2\gamma + \gamma I)\nabla\psi] + \langle (D^2\gamma + \gamma I)\cdot d\nu, d\nu\rangle\psi \ ,$$

Because of the convexity condition, the operator J is elliptic for any sufficiently smooth surface. This implies that the equation for constant, or more generally prescribed, anisotropic mean curvature has a Maximum Principle.

Capillary problem

Consider a fixed volume of material trapped between two horizontal planes.



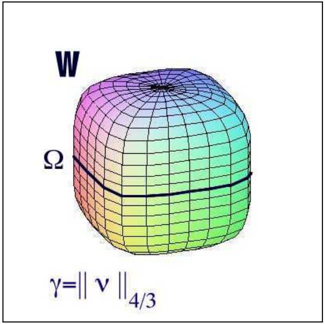
Consider the volume as the body of a physical drop. We will assign an energy to each part of the boundary of the drop.

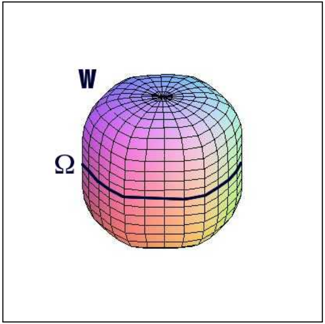
- $\Sigma \longrightarrow \mathcal{F}[\Sigma]$.
- $\mathcal{A}_i \longrightarrow \omega_i \cdot \text{Area}\ (\mathcal{A}_i)$ where ω_i are constants. This is called the *wetting energy*. $\omega_i > 0$ is called lyophobic wetting, $\omega_i < 0$ is called lyophilic wetting.
- $C_i \longrightarrow \tau_i \mathcal{L}[C_i]$ where τ_i are constants and $\mathcal{L}$ is a one dimensional anisotropic energy. This term is called the *line tension*.

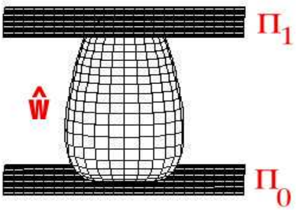
We consider a Wulff shape *W* of *product form*:

$$\chi(\sigma,\tau) = (u(\sigma)[\alpha(t),\beta(t)], v(\sigma)) \quad 0 \le \sigma \le \bar{\sigma}\,, 0 \le t \le \bar{t}\,.$$

It is assumed that (u, v) and (α,β) are smooth, convex, closed curves. The curve parameterized by (α,β) will be denoted by Ω.







Winterbottom's Theorem: The part of the Wulff shape $\hat{W}$ between two planes is the absolute minimizer of the energy

$$\mathcal{E} = \mathcal{F} + \omega_0 \cdot \mathcal{A}_0 + \omega_1 \cdot \mathcal{A}_1 \ ,$$

among all surfaces enclosing the same volume and having free boundary on the two planes. Here, $\omega = (\omega_0, \omega_1)$ are the values of the wetting constants for which $\hat{W}$ is in equilibrium.

We now consider an anisotropic line tension given by the one dimensional parametric functional whose Wulff shape is the curve Ω given by (α, β). To do this we define:

$$\mathcal{L}_\Omega[C] = \int_C \Gamma[N]\, dL\,,$$

Here N is the unit normal to the curve C.
With this definition and appropriate constants (ω_i, τ_i), the parts of the Wulff shape between horizontal planes is still in equilibrium when the line tension is included in the total energy:

$$\mathcal{E} := \mathcal{F} + \omega \cdot \mathcal{A} + \tau \cdot \mathcal{L}\,,\ \ \omega, \alpha \in \mathbf{R}\,.$$

Here: $\omega \cdot \mathcal{A} = \omega_0 \mathcal{A}_0 + \omega_1 \mathcal{A}_1$, etc.

Line tension was introduced by Gibbs in 1878. It is known play an important role in determining the geometry of drops on a very small scale (microns). It is insignificant for larger drops.

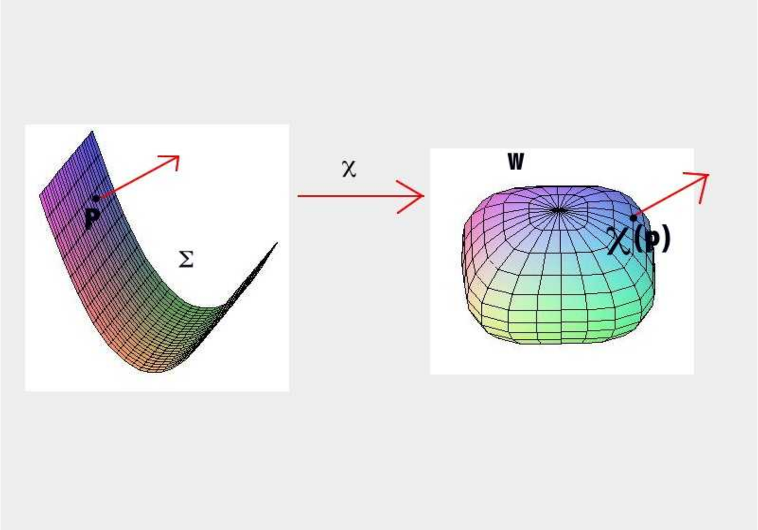
Proposition

- *Assume that* $\omega = (\omega_0, \omega_1)$ *and* $\tau = (\tau_0, \tau_1)$*, with* $\tau_i < 0$ *are chosen so that* $\hat{W}$ *is in equilibrium. Then,* $\hat{W}$ *is the absolute minimizer of the energy*

$$\mathcal{E} = \mathcal{F} + \omega \cdot \mathcal{A} + \tau \cdot \mathcal{L} \,,$$

among all symmetric surfaces enclosing the same volume and having free boundary on the two planes.

For some choices of ω and $\tau < 0$, the problem has no minimizer.



p
Σ
χ
W
χ(p)

Equilibrium conditions

Define $\tilde{\kappa} =$ anisotropic curvature of C_i,

$$\delta\mathcal{L} := \int_C \tilde{\kappa}\, \delta C \cdot N\, ds$$

Critical points of $\mathcal{E}$ are characterized by:

$$(*) \quad \Lambda \equiv \text{contant}\,, \;\; \text{in} \;\; \Sigma\,,$$

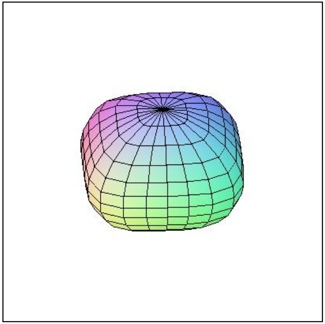
$$(**) \qquad \chi \cdot E_3 + (-1)^i(\omega_i + \tau_i\tilde{\kappa}) \equiv 0 \qquad \text{on} \qquad C_i,\; i = 0, 1$$

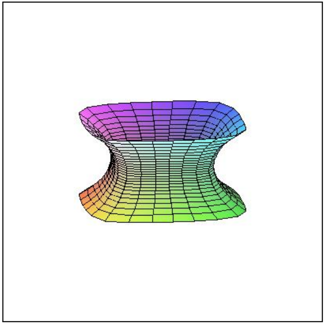
Solutions of (*), (**) will be called *capillary surfaces*.

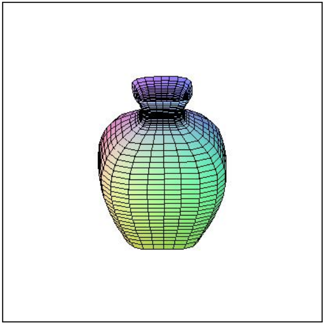
For rotationally symmetric functionals and $\tau \geq 0$, it is possible to show using the Maximum Principle that a capillary surface for this configuration is, a priori, a surface of revolution.

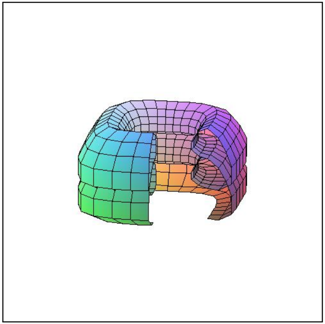
A surface with $\Lambda \equiv$ constant having the form $X(s,t) = (x(s)\alpha(t), x(s)\beta(t), z(s))$ will be called an (anisotropic) Delaunay surfaces.

They can be classified into basic types, (Wulff shape,cylinder over Ω, plane, anisotropic catenoid, anisotropic unduloid, anisotropic nodoids) which are analogous to the classical Delaunay surface.









For a Delaunay surface, the equilibrium boundary condition becomes

$$\chi \cdot E_3 + (-1)^i(\omega_i + \tau_i/x_i) = 0 \;,\; \text{on } C_i \;.$$

where x_i are the 'radii' of the boundary curves. Note that there is a whole continuum of pairs (ω_i, τ_i), with

$$(\sharp) \qquad \omega_i + \tau_i/x_i =: \omega_i^* \;.$$

for which the surface is in equilibrium.

Note that if we cut *any* Delaunay surface by two horizontal planes, we can always define

$$\omega_i^* := \chi|_{C_i} \cdot E_3 \;,$$

then surface $\hat{\Sigma}$ is an equilibrium for $\mathcal{E} = \mathcal{F} + \omega \cdot \mathcal{A} + \tau \cdot \mathcal{L}$ whenever (ω_i, τ_i) satisfy ($\sharp$).

Second variation

If Σ is a capillary surface,

$$\delta^2\mathcal{E} = -\int_\Sigma \psi J[\psi]\, d\Sigma + \oint_{\partial\Sigma} \psi B[\psi]\, dL\,,$$

where

$$\psi := \delta X \cdot \nu \qquad \cot\phi := \nu_3/n_3$$

$$J[\psi] = \delta\Lambda = \mathrm{Div}_\Sigma[(D^2\gamma + \gamma I)\nabla\psi] + \langle (D^2\gamma + \gamma I)\cdot d\nu, d\nu\rangle\psi\,,$$

The boundary operator B is defined as follows: First define B_1 by

$$B_1[\psi] = -\delta\chi \cdot n = A(\nabla\psi + \frac{\nu_3}{n_3}\psi d\nu(n)) \cdot n\,.$$

$A := (D^2\gamma + \gamma I)_\nu$. Then

$$B[\psi] = B_1[\psi] - \frac{\tau}{n_3}\left[(\frac{1}{m}(\frac{\psi}{n_3})_L)_L + \frac{\kappa^2}{mn_3}\psi\right]\,.$$

A capillary surface is called stable if $\delta^2\mathcal{E} \geq 0$ holds for all ψ such that

$$\int_\Sigma \psi \, d\Sigma = 0$$

holds. Consider the spectral problem:

$$(*) \quad J[\psi] + \lambda\psi = 0 \,, \qquad B[\psi] = 0 \,, \qquad \text{on} \quad \partial\Sigma \,.$$

Proposition

Assume that $\lambda_1 < 0 \leq \lambda_2$ holds. If there exists a solution of

$$(*) \;\; J[\phi] = 1 \,, \text{in} \;\; \Sigma, \qquad B[\phi] = 0 \;\; \text{on} \; \partial\Sigma \,.$$

Then, the surface is stable if and only

$$\int_\Sigma \phi \, d\Sigma \geq 0 \,,$$

holds.

If no solution of (*) exists, the surface is unstable.

Schwarz symmetrization

Theorem

Let S be an embedded surface having no horizontal tangent planes. Define the 'symmetrized' surface $\tilde{S}$ by replacing each cross section $z =$ constant of S by the curve homothetic to Ω which encloses the same area. Then the volume within the surface is preserved and the free energy of S is diminished:

$$(\mathcal{F} + \omega \cdot \mathcal{A})[\tilde{S}] \leq (\mathcal{F} + \omega \cdot \mathcal{A})[S] \,.$$

In addition, if $\tau_i \geq 0$ holds, then

$$\mathcal{E}[\tilde{S}] \leq \mathcal{E}[S] \,.$$

Corollary

Let $\hat{\Sigma}$ be a capillary Delaunay surface with no horizontal tangent planes. Assume τ_i, $i = 0, 1 \geq 0$ holds. Then, $\hat{\Sigma}$ is stable if and only if it is stable with respect to symmetric variations.

Note that, except for the Wulff shape, the parts of the Delaunay surfaces which are embedded between horizontal planes do not have any horizontal tangent planes.

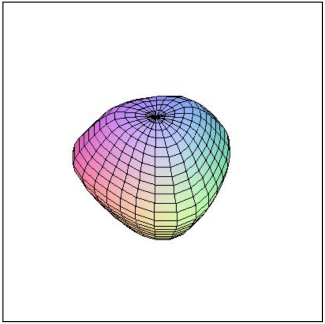
Proposition

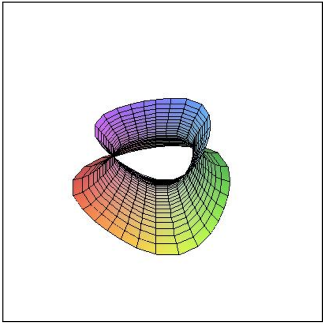
Let $X(s,t) = (x(s)\alpha(t), x(s)\beta(t), z(s))$ be an immersion of a capillary surface for an anisotropic energy with Wulff shape $\chi(\sigma, t) = (u(\sigma)\alpha(t), u(\sigma)\beta(t), v(\sigma))$. Let

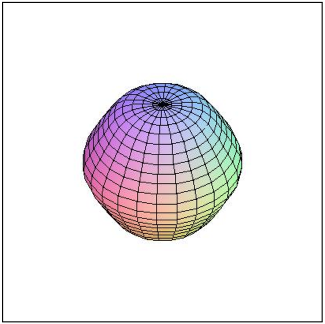
$$\bar{X}(s,t) = (x(s)\cos(t), x(s)\sin(t), z(s)),$$

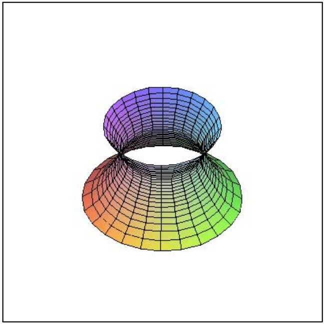
$$\bar{\chi}(\sigma, t) = (u(\sigma)\cos(t), u(\sigma)\sin(t), v(\sigma)) .$$

Then X is stable with respect to symmetric variations, if and only if $\bar{X}$ is stable with respect to symmetric variations for the free energy with Wulff shape given by $\bar{\chi}$.









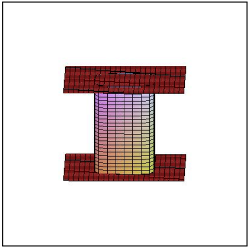
- Assume $\tau_i \geq 0$ holds. For surfaces with no horizontal tangent planes, we need only consider symmetric variations.
- For symmetric variations, stability of capillary Delaunay surface depends on the numbers $\omega_i^* = \chi \cdot E_3$ which are determined from the surface, not on the values ω_i, τ_i.
- For symmetric variations, the surface can be replaced by the surface having circular cross sections if the functional is replaced by the Wulff shape having circular cross sections.
- One or two vertical supporting planes can be included if their wetting constants are all zero and they cut the Wulff shape at a right angle.

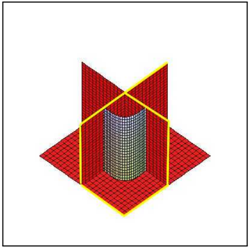
As long as $\tau_i \geq 0$ holds, the stability of any anisotropic Delaunay surface can be determined numerically.

Theorem

Assume $\tau_i \geq 0$ holds. Suppose $\hat{\Sigma}$ is a capillary Delaunay surface with $\omega_0^* = 0 = \omega_1^*$. Then $\hat{\Sigma}$ is stable if and only if $\hat{\Sigma}$ is a cylinder over Ω whose height is small compared to its volume.

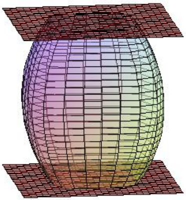
(The isotropic case $\gamma \equiv 1$ without line tension is due to Athanassenas and Vogel. (1987))

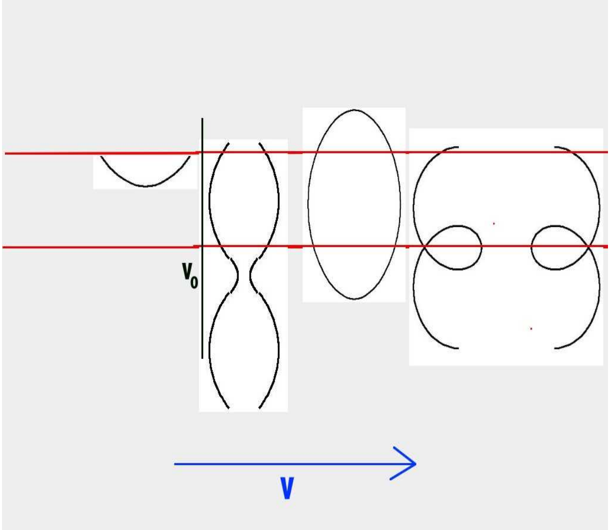




Theorem

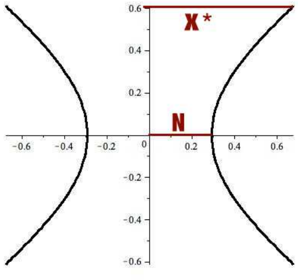
Assume that $\tau_i \geq 0$*. Suppose* $\hat{\Sigma}$ *is a capillary Delaunay surface with* $\omega_0^*, \omega_1^* > 0$*. Then* $\hat{\Sigma}$ *is stable if and only if the generating curve* $(x(s), z(s))$ *of* $\hat{\Sigma}$ *has no interior inflection points.*




V_0
V

An anisotropic unduloid (with lyophilic wetting) is stable if it is ‘close to being a cylinder".

If the neck size N satisfies $\sqrt{3}N \geq x^*$, where x^* is the radius of the circle at an inflection point, then for any T, with $0 \leq T \leq -\omega$, the part of the unduloid with $T \leq v \leq -T$ is stable.

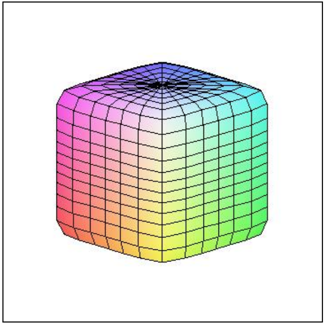


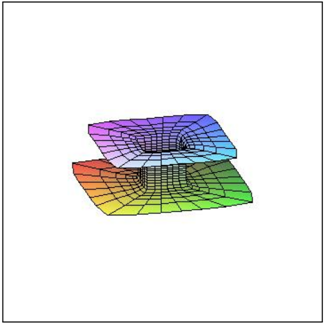
Theorem

The part of an anisotropic catenoid $z_0 \leq z \leq z_1$ is stable for the anisotropic energy with wetting and line tenision if and only if the integral

$$\int_\Sigma \phi \, d\Sigma = \frac{\pi}{6}\left\{9\left(\int_{z_0}^{z_1} x^2 \, dz\right)^2 - 5(z_1 - z_0)\int_{z_0}^{z_1} x^4 \, dz\right\}$$

is non negative.





Lyophobic wetting

Theorem

Let Σ be a capillary surface with free boundary on two horizontal planes for the functional $\mathcal{F}$ with $\omega_0 = \omega_1 = \omega \geq 0$ and with the Wulff shape for the functional satisfying the conditions (W1) through (W3) stated above.
(i) If $\omega = 0$, then Σ is stable if and only if the surface is either homothetic to a half of the Wulff shape or a cylinder which is perpendicular to $\Pi_0 \cup \Pi_1$ which satisfies

$$\frac{\mu_1(0)}{\mu_2(0)}\, h^2 \leq (\pi R)^2 \,.$$

(ii) If $\omega > 0$ holds, then Σ is stable if and only if Σ is a portion of an anisotropic Delaunay surface whose generating curve has no inflection points in its interior.

Theorem

We assume (W1) through (W3) stated above.
(I) Assume $0 < \omega < \bar{\omega}$ *:=maximum height on* W*. Then, there exist constants* $0 < V_0 < V_1$ *such that*

(i) For volumes $V_0 \leq V < V_1$*, there exists a unique stable spanning capillary surface with volume* V*, height* h *and wetting constant* ω*, and the surface is an anisotropic unduloid. This surface has inflection points on the boundary exactly when* $V = V_0$*.*

(ii) For $V = V_1$*, there exists a unique stable capillary surface with volume* V*, height* h *and wetting constant* ω*, and the surface is homothetic to a part of the Wulff shape.*

(iii) For $V_1 < V$*, there exists a unique stable capillary surface with volume* V*, height* h *and wetting constant* ω*, and the surface is an anisotropic nodoid.*

(II) Assume $\omega = \bar{\omega}$*. Then, for* $V_0 < V$*, there exists a unique stable capillary surface with volume* V*, height* h *and wetting constant* ω*, and the surface is an anisotropic nodoid.*

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