Integrable aspects of global surface geometry

Franz Pedit

joint projects with A. Gerding, S. Heller, N. Schmitt¹, Y. Shen

University of Tübingen and UMass Amherst

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¹Images by Nicholas Schmitt



Figure: Twizzled Torus



Figure: Wente Torus



Figure: Delaunay Surfaces



Figure: Tetranoid



Figure: Sprungnoid



Figure: Symmetric 4-noids



Figure: Experimental 4-noids

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- More generally, consider (constrained) Willmore surfaces ↔ critical for ∫ H² under (conformal) variations of the surface.
- ► Minimal surfaces in space forms ~→ Willmore surfaces.
- ► CMC surfaces in space forms ~→ constrained Willmore surfaces.

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$$egin{aligned} \mathcal{M} imes \mathbb{R}^3 &= \mathcal{T}\mathcal{M} \oplus \mathcal{N}\mathcal{M}\,, \ & \mathcal{d} &= egin{pmatrix}
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abla^\perp \end{pmatrix}\,. \end{aligned}$$

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With g = e^{2u}|dz|², B = Hg + Q + Q̄ where H is the mean curvature, Q = qdz² the Hopf differential:

$$\bigtriangleup u + e^{2u}H^2 - e^{-2u}|q|^2 = 0 \qquad \text{and} \qquad H_z e^{2u} = q_{\bar{z}} \,.$$

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Period closing means holonomy of d (rotation periods) is trivial and df has no translation periods. Above recipe is impossible to carry out in general, but works surprisingly well for CMC, due to an observation (insertion of a spectral parameter) coming from mathematical physics. For B_λ = Hg + λQ + λ⁻¹Q
 the holomorphic family in λ ∈ C_{*} of SL(2, C)-connections with simple poles at λ = 0,∞

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Holomorphic family of holonomy representations

$$\rho^p_{\boldsymbol{\lambda}} \colon \pi_1(M,p) \to \mathsf{SL}(2,\mathbb{C})$$

based at $p \in M$ with essential (exponential) singularities at $\lambda = 0, \infty$ and unitary for $|\lambda| = 1$.

Basic paradigm: use λ -dependence to reconstruct holomorphic family ρ_{λ}^{p} of holonomy representations for each $p \in M \rightsquigarrow$ gives holomorphic family of flat connections $d_{\lambda} \rightsquigarrow$ constructs f at $\lambda = 1$.

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Period closing: at $\lambda = 1$

$$ho^{
ho}_{oldsymbol{\lambda}}=1$$
 and $rac{d}{d\lambda}
ho^{
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to close rotational and translational periods of f.

• $M = T^2 = \mathbb{R}^2 / \Lambda$ and thus ρ_{λ}^p diagonalizes for generic $\lambda \in \mathbb{C}_*$;

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- eigenline bundle flow

$$T^2
ightarrow \mathsf{Pic}(\Sigma) \, : \, p \mapsto \mathcal{E}^p$$

is linear tangent to Abel map of Σ in Pic(Σ). explain this some more in real time...



Figure: Delaunay cylinders, the embedded unduloids and immersed nodoids, have spectral genus 1.



Figure: Equivariant CMC tori in \mathbb{S}^3 with increasing numbers of lobes have spectral genus 1. The surfaces are stereographically projected to \mathbb{R}^3 .



Figure: The Wente and Dobriner CMC tori have respective spectral genera 2 and 3.



Figure: Spectral genus 3 CMC tori (Nick Schmitt 2008).



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Figure: Spectral genus 4 CMC tori (Matthias Heil Ph D thesis, 1995).



Figure: Spectral genus 5 CMC torus (Matthias Heil Ph D thesis, 1995).

Conjecture: To any closed space curve γ there is a CMC torus inside a small circular tube around γ .

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 associate to a CMC torus in space holomorphic data (hyperelliptic curve and holomorphic line bundle) which in turn reconstruct the surface from linear line bundle flow:

$$T^2 \xrightarrow{\text{linear}} \operatorname{Pic}(\Sigma) \xrightarrow{\Theta} \mathbb{R}^3$$

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- ► This description is central to understand the moduli space of all CMC cylinders of finite spectral genus, and hence all CMC tori in ℝ³ and also S³ → Lawson conjecture (Kilian and Schmidt).
- Similar result for constrained Willmore tori (Bohle, Schmidt): SL(2, C) → SL(4, C) and have 4-fold cover of P¹ → Willmore conjecture?

A flat connection determines and is determined by its holonomy representation of the fundamental group.

If the fundamental group is complicated (higher genus surfaces) this representation is complicated.

If the surface is not CMC we do not even have holomorphic families of flat connections and thus no hope to obtain classifying holomorphic data from flat connections.

So what shall we do?

Hope that the holonomy for a CMC surface is abelian even in higher genus?

Theorem(Gerding 2009):

Let $f: M \to \mathbb{R}^3$ be a genus g compact CMC surface with holonomy $\rho_{\lambda}: \pi_1(M) \to \mathbf{SL}(2, \mathbb{C})$ abelian for all $\lambda \in \mathbb{C}_*$. Then fhas a spectral curve Σ and factors via

In other words, $f(M) \subset \mathbb{R}^3$ is a CMC torus onto which M maps with branch points.

Different paradigm: work with "half connections", or holomorphic structures, which are not determined by their monodromy, so that we have freedom to choose the type of monodromy we want (e.g., abelian ones).

This works for all surfaces of any genus...

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Skew-hermitian spin pairing

$$(\,,\,)\colon L \times L \to TM^* \otimes \mathbb{H} \quad \rightsquigarrow \quad (\varphi,\varphi) \in \Omega^1(M,\mathbb{R}^3)$$
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• f gives canonical section $\psi \in \Gamma(L)$ with

$$(\psi,\psi)=df\colon M\to\mathbb{R}^3$$
.

• $D: \Gamma(L) \to \Gamma(\overline{K} \otimes L)$, a Dirac operator

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Weierstraß representation

$$\varphi \in H^0(L) = \ker D \quad \rightsquigarrow \quad g = \int_M (\varphi, \varphi) \colon \tilde{M} \to \mathbb{R}^3$$

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► Potential H = 0 ~→ classical Weierstraß representation of minimal surfaces.

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The conformal immersions $\int_{M}(\varphi, \varphi)$ for various $h \in \Sigma$ all have rotational periods around same axis and translational periods (analog of associated family in CMC case). Original surface f recovered at the trivial representation h = 1.

Theorem (Heller, Shen 2009): The spectral variety Σ of a conformal immersion $f: M \to \mathbb{R}^3$ of a compact surface M of genus $g \ge 1$ is an analytic hypersurface

$$\Sigma^{2g-1} \subset \operatorname{Hom}(H_1(M,\mathbb{Z}),\mathbb{C}_*) \cong \mathbb{C}^{2g}/\mathbb{Z}^{2g}$$

which is asymptotic, for $\ln |h| \to \pm \infty$, to the vacuum spectrum Σ_0 corresponding to the operator D with zero potential H = 0.

 Rewrite existence of sections with monodromy for *D* as the non-invertibility in a holomorphic family of Dirac type operators

$$D_{\omega} = egin{pmatrix} ar{\partial} + \omega'' & H \ -H & \partial + \omega' \end{pmatrix}$$

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 $\blacktriangleright \Sigma_0 = \Theta \times H^0(K) \cup \overline{H^0(K)} \times \overline{\Theta} + \operatorname{Harm}^1(M, 2\pi\mathbb{Z}).$

ω ∈ Σ → dim_ℂ ker D_ω = 1 generically → holomorphic line bundle

$$\mathcal{E}^p \to \Sigma$$

where $\varphi_{\omega} \in \ker D_{\omega} \subset \Gamma(L)$, $\mathcal{E}_{\omega}^{p} = \varphi_{\omega}(p)\mathbb{C} \subset L_{p} \cong \mathbb{C}^{2}$.

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▶ Is it true that $M \to \mathsf{Pic}(\Sigma)$: $p \mapsto \mathcal{E}^p$ factorizes via

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► $f: M \to \mathbb{R}^3$ CMC (Willmore), both elliptic problems, Σ can be compactified? \rightsquigarrow algebro-geometric data.

Main evidence from g = 1 case, where all of above has been proven (Bohle, P., Pinkall):

 $\Sigma \subset \mathbb{C}^2/\mathbb{Z}^2$ analytic curve, vacuum $\Sigma_0 = \{0\} \times \mathbb{C} \cup \mathbb{C} \times \{0\} + \mathbb{Z}^2$ and *f* factorizes via



For CMC (and Willmore) the holonomy and monodromy constructions coincide in this case (Bohle; Carberry, Leschke, P.)

How does one detect CMC (or Willmore) in the spectral variety Σ ? Can one obtain Lawson's genus 2 minimal surface this way

