Period Condition of Algebraic Minimal Surfaces and Nevanlinna Theory

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Period Condition

Let M be an open Riemann surface and ϕ_i's ($i = 1, 2, 3$) three holomorphic 1-forms on M. Consider the equation

$$\partial x = \frac{1}{2}(\phi_1, \phi_2, \phi_3) \ .$$

Suppose that the conditions

$$\begin{array}{rl} [C] & \sum_{i=1}^{3} \phi_i^2 = 0 \text{ conformality} \\ [R] & \sum_{i=1}^{3} |\phi_i|^2 > 0 \text{ regularity} \\ [P] & \forall \gamma \in H_1(M, \mathbb{Z}) \ , \int_\gamma \phi_i \text{ is pure imaginary } (i = 1, 2, 3) \end{array}$$

are satisfied. The condition [P] is called the **period condition**.

Enneper-Weieretrass Representation. Minimal Surface

Then the **Enneper-Weierstrass representation**

$$x = \Re \int_{z_0}^{z} (\phi_1, \phi_2, \phi_3)$$

defines a regular **minimal surface** $x : M \to \mathbb{R}^3$ and all regular minimal surfaces in $\mathbb{R}^3$ are obtained this way.
The **Weieretrass data** (hdz, g) is defined by $hdz = \phi_1 - i\phi_2$ and $g = \frac{\phi_3}{\phi_1 - i\phi_2}$.
Conversely $\phi_1 = \frac{h}{2}(1 - g^2)dz$, $\phi_2 = \frac{ih}{2}(1 + g^2)dz$ and $\phi_3 = hgdz$ hold.
The meromorphic function $g : M \to \mathbb{P}^1$ coincides with the **Gauss map** of the minimal surface $x : M \to \mathbb{R}^3$.

Algebraic Minimal Surface

Definition 1. The total curvature of a minimal surface $x : M \to \mathbb{R}^3$ is defined by $\tau(M) = \int_M K dA \in \mathbb{R}_{\leq 0} \cup \{-\infty\}$ where K is the Gaussian curvature.

Definition 2. An **algebraic minimal surface** means a complete minimal surface in $\mathbb{R}^3$ with finite total curvature.

Theorem 3 (Huber/Osserman). Let $x : M \to \mathbb{R}^3$ be an algebraic minimal surface. Then :

(1) M is conformally equivalent to a compact Riemann surface finitely many points removed, i.e., $\overline{M} \backslash \{p_1, \ldots, p_k\}$ where $\overline{M}$ is a compact Riemann surface and $\{p_i\}_{i=1}^k$ are finitely many points on $\overline{M}$.

(2) the Weierstrass data (hdz, g) extends meromorphically to $\overline{M}$.

Basic Setup in Nevanlinna Theory

Let $f : \mathbb{C} \to \mathbb{P}^1$ be a holomorphic function. Problem : What can we know by studying how f approximates a divisor $D = \{p_1, \ldots, p_n\}$ on $\mathbb{P}^1$?

The approximation of f to D is measure by the asymptotic behavior of the **proximity function** $m_{f,D}(r) = \int_0^{2\pi} \log \frac{1}{|\sigma(f(re^{i\theta}))|} \frac{d\theta}{2\pi}$, where σ is the canonical section, i.e., $(\sigma)_0 = D$ and $|\cdot|$ is a smooth Hermitian norm on the line bundle $\mathcal{O}_X(D)$.

How often f intersects D is counted by the **counting function** $N_{f,D}(r) = \int_0^r \frac{dt}{t}\{n_{f,D}(t) - n_{f,D}(0)\} + n_{f,D}(0) \log r$ where $n_{f,D}(t) = \sum_{|z|<t} \nu_z(f^*D)$ and $n_{f,D}(0) = \nu_0(f^*D)$.

The total complexity of f is measured by the **height function** $T_{f,D}(r) = \int_0^r \frac{dt}{t} \int_{|z|<t} f^* c_1(\mathcal{O}_X(D), |\cdot|)$.

Then we have the **First Main Theorem** $T_{f,D}(r) = m_{f,D}(r) + N_{f,D}(r) + O(1)$.

Lemma on Logarithmic Derivative (Analytic Version)

Let $x : M \to \mathbb{R}^3$ be an algebraic minimal surface and $\widetilde{M} = \mathbb{D}$. Let $g : \mathbb{D} \to \mathbb{P}^1$ be the **lifted Gauss map**. Define

$$\kappa := \inf\left\{\overline{\kappa} \,|\, \int_0^1 \exp(\overline{\kappa} T_g(t))dt = \infty\right\} .$$

Then we have

Theorem 4 (Nevanlinna's LLD). [Analytic Version]
$m_{\frac{g'}{g},\infty}(r) \leq (\kappa + \delta)T_g(r)||_\delta$.

A strategy to get geometric conclusion from analytic LLD : We decompose LHS into geometric terms. To do so, we consider appropriate logarithmic differential on $\mathbb{P}^1$ which we regard as a meromorphic function on $\overline{T\mathbb{P}^1}$ and extract geometric information from the resolution of its indeterminancy.

Lemma on Logarithmic Derivative (Geometric Version)

(1) the case $D = \{0, \infty\}$. Consider $\frac{dt}{t}$ a meromorphic function on $\overline{T\mathbb{P}^1}$. The indeterminancy disappears after blowing up the jet space of $\{0, \infty\}$ in $\overline{T\mathbb{P}^1}$. We get
$m_{\frac{g'}{g},\infty}(r) = (m_{g,D}(r) - m_{g^{(1)},D^{(1)}}(r)) + m_{g^{(1)},S_\infty}(r)$.
(2) the case $D = \{a_1, \ldots, a_n\}$. Consider a rational function $\phi_b(t) := \frac{\prod_{i=1}^n (t-a_i)}{(t-b)^n}$ and consider $g_b := g \circ \phi_b : \mathbb{C} \to \mathbb{P}^1$. Then we have a decomposition $m_{\frac{g_b'}{g_b}}(r) = \sum_{i=1}^n (m_{g,\{a_i\}}(r) - m_{g^{(1)},\{a_i\}^{(1)}}(r)) + (m_{g,\{b\}}(r) - m_{g^{(1)},\{b\}^{(1)}}(r)) + m_{g^{(1)},S_\infty}(r)$ and apply LLD analytic version to LHS. Moving b in $\mathbb{P}^1$ and taking average we get

Theorem 5 (LLD). [Geometric Version] $0 \leq \exists \alpha(r) \leq 1$ *s.t.*
$m_{g,D}(r) \leq m_{g^{(1)},D^{(1)}}(r) + \alpha(r)(\kappa + \delta)T_g(r)||_\delta$,
$m_{g^{(1)},S_\infty}(r) \leq (1 - \alpha(r))(\kappa + \delta)T_g(r)||_\delta$.

Approximation and Geometric LLD

Let $g : \mathbb{D} \to \mathbb{P}^1$ be the lifted Gauss map of a given algebraic minimal surface and D a divisor (which we will take as the exceptional set of the Gauss map). The approximation to D is measured by $m_{g,D}(r) + N_{g,\mathrm{Ram}}(r)$. Here $N_{g,\mathrm{Ram}}(r) = \int_0^r \frac{dt}{t}\{n_{g,\mathrm{Ram}}(t) - n_{g,\mathrm{Ram}}(0)\} + n_{g,\mathrm{Ram}}(0)\log r$ is the ramification counting function. This quantity measures the approximation of g to D where $m_{g,D}(r)$ does in the metric sense and $N_{g,\mathrm{Ram}}(r)$ in the intersection sense. We start the estimate by noting that $m_{g,D}(r) \leq m_{g^{(1)},D^{(1)}}(r) + \alpha(r)(\kappa+\delta)T_g(r)$ and $N_{g,\mathrm{Ram}}(r) = N_{g^{(1)},S_0}(r)$. Next we set $g^{(1)} = \frac{g^{(1)}}{h}h$ where (g, hdz) is the Weierstrass data lifted on $\mathbb{D}$. $g^{(1)} = (g, g')$ and $h = (g, h)$ are maps to $\overline{T\mathbb{P}^1}$. Therefore we consider two compactified vector bundles $\overline{T\mathbb{P}^1}$ where $g^{(1)}$ and h lives and $\overline{\mathbb{P}^1 \times \mathbb{C}}$ where $\frac{g^{(1)}}{h}$ lives.

Local Parameter vs Linear Coordinate

Let ξ be a local parameter of $\overline{M}$ around the puncture and z the linear coordinate of $\mathbb{C}$. The coordinate change between ζ and z is of the form $\zeta = \exp(\frac{1}{m}\frac{z+1}{z-1})$ where $m \in \mathbb{Z}|_{>0}$ is a parameter. Then

$$h(z) = \omega(\frac{\partial}{\partial z}) = \omega(\frac{\partial}{\partial \zeta})[-2\frac{\zeta}{m}\frac{1}{(z-1)^2}] = h(\zeta)[-2\frac{\zeta}{m}\frac{1}{(z-1)^2}] \ .$$

If we replace $m^{\zeta}_{h,S_\infty}(r)$ by $m_{h,S_\infty}(r)$, we have an extra approximation of magnitude

$$[J] := \overline{m}^{\zeta}_{h,S_\infty}(r) - \log\frac{1}{1-r} \ .$$

Here $\overline{m}$ means to ignore the multiplicity of poles of $h(\zeta)$ and m^{ζ} means that the coordinate used is the local parameter ζ.

Quantity $m_{h,S_0}(r) + N_{h,S_0}(r) - m_{h,S_\infty}(r)$

Apply (1) geometric LLD, (2) T is linear w.r.to linear equivalence class of divisors, (3) the linear equivalence between S_0, S_∞ and $K_{\mathbb{P}^1}$ on $\overline{T\mathbb{P}^1}$ and $\overline{\mathbb{P}^1 \times \mathbb{C}}$, (4) the difference of the effect of coordinate change on $g^{(1)}/h$ and h
in the estimate of $m_{g,D}(r) + N_{g,\mathrm{Ram}}(r)$. The result is

$$m_{g,D}(r) + N_{g,\mathrm{Ram}}(r) \leq (\kappa + \delta) T_g(r)$$
$$+ (m_{h,S_0}(r) + N_{h,S_0}(r) - m_{h,S_\infty}(r)) - [J]||_\delta \ .$$

FMT and the linear equivalence $[S_0] - [S_\infty] = -[K_{\mathbb{P}^1}]$ on $\overline{T\mathbb{P}^1} \Rightarrow$ $2T_g(r) = m_{h,S_0}(r) + N_{h,S_0}(r) - m_{h,S_\infty}(r)$.

Theorem 6.

$$T_g(r) \leq \frac{1}{2}[J] \ .$$

Proof of Theorem 6 and Geometric LLD

Set $H := \int((1-g^2), i(1+g^2), 2g)\omega$. We consider $(1-g^2)h$ and $i(1+g^2)h$ at the same time. The period condition is expressed as $H(\gamma z) - H(z) \in \sqrt{-1}\mathbb{R}$, $\forall \gamma \in \pi_1(M)$.
Let $A_r := \{a_1, \ldots, a_k\}$ be the values of e^H at the poles of g "near" $|z| = r$. The period condition implies that the set A_r has only finitely many variations for $|a_i|$ even if $k \to \infty$. Introduce a logarithmic 1-form $\sum_{i=1}^k \frac{dt}{t-a_i}$ and regard this as a meromorphic function on $\overline{T\mathbb{P}^1}$. Arguing as in the proof of geometric LLD, we have

$$\sum_{i=1}^k m^{\zeta}_{\frac{(e^H)'}{e^H - a_i},\infty}(r) = \log k + \sum_{i=1}^k (m^{\zeta}_{e^H,\{a_i\}}(r) - m^{\zeta}_{(e^H)^{(1)},\{a_i\}^{(1)}}(r))$$
$$+ \sum_{i=1}^k (m^{\zeta}_{e^H,\infty}(r) - m^{\zeta}_{(e^H)^{(1)},\{\infty\}^{(1)}}(r)) + m^{\zeta}_{(e^H)',S_\infty}(r)$$

Computing in Two Ways, RHS

$$(\mathrm{RHS}) = \log k + m^{\zeta}_{h,S_\infty}(r) + \frac{3}{2} m^{\zeta}_{H',S_0}(r) + [\mathrm{OTH}]_{\mathrm{RHS}}$$

holds, where $m^{\zeta}_{h,S_\infty}(r)$ is the contribution from the cusps, $\frac{3}{2} m^{\zeta}_{H',S_0}(r)$ is the contribution from the poles of g and $[\mathrm{OTH}]_{\mathrm{RHS}}$ is the contribution from the solutions of $e^H = a_i$ other than poles of g.

In the computation we use the Fubini-Study distance on $\overline{T\mathbb{P}^1}$. We have $m^{\zeta}_{e^H,\infty}(r) - m^{\zeta}_{(e^H)^{(1)},\{\infty\}^{(1)}}(r) = \frac{1}{2} m^{\zeta}_{h,S_\infty}(r)$,
$m^{\zeta}_{(e^H)^{(1)},S_\infty}(r) = \frac{1}{2} m^{\zeta}_{h,S_\infty}(r)$,
$\sum_{i=1}^{k} m^{\zeta}_{e^H,\{a_i\}}(r) = \frac{3}{2} m^{\zeta}_{H',S_0}(r) + [\mathrm{OTH}]_{\mathrm{RHS}}$ and
$\sum_{i=1}^{k} m^{\zeta}_{(e^H)',\{a_i\}^{(1)}}(r) = O(1)$.

Computing in Two Ways, LHS

$$(\mathrm{LHS}) = \log k + m^{\zeta}_{h,S_\infty}(r) + \frac{1}{2} m^{\zeta}_{H',S_0}(r) + [\mathrm{OTH}]_{\mathrm{LHS}}$$

holds, where $m^{\zeta}_{h,S_\infty}(r)$ is the contribution from the cusps, $\frac{1}{2} m^{\zeta}_{H',S_0}(r)$ is the contribution from the poles of g and $[\mathrm{OTH}]_{\mathrm{RHS}}$ is the contribution from the solutions of the equation $e^H = a_i$ other than poles of g.
In the computation we use just the absolute value of functions.

Comparison of OTH's and Estimate of $[J]_0$

We have

$$(\mathrm{OTH})_{\mathrm{LHS}} - (\mathrm{OTH})_{\mathrm{RHS}} \leq \frac{1}{2} m^{\zeta}_{h,S_\infty}(r) = \frac{1}{2} m_{h,S_\infty}(r) + \frac{1}{2}[J] \ .$$

and

$$[J]_0 = N_{h,S_0}(r) \ .$$

We have also

$$m_{H',S_0}(r) = \frac{1}{2} m_{h,S_0}(r) \ .$$

Replacing m^ζ with m

Replacing m^ζ with m we have

$$(\mathrm{RHS}) = \log k + m_{h,S_\infty}(r) + \frac{3}{2} m_{H',S_0}(r) + [\mathrm{OTH}]_{\mathrm{RHS}} + [J]$$

and

$$(\mathrm{LHS}) = \log k + m_{h,S_\infty}(r) + \tfrac{1}{2} m_{H',S_0}(r) + [\mathrm{OTH}]_{\mathrm{LHS}} \\ + [J] - \tfrac{1}{2}[J]_0$$

holds, where $[J]_0$ is the part of the contribution from the Jacobian of the coordinate change stemming from zeros of ω.

Comparison of LHS and RHS

$$\log k + m_{h,S_\infty}(r) + \tfrac{1}{2} m_{H',S_0}(r) + [\mathrm{OTH}]_{\mathrm{LHS}} + [J] - \tfrac{1}{2}[J]_0$$
$$= \log k + m_{h,S_\infty}(r) + \tfrac{3}{2} m_{H',S_0}(r) + [\mathrm{OTH}]_{\mathrm{RHS}} + [J]$$

Therefore we have

$$[J] = \tfrac{1}{2} m_{h,S_0}(r) + \tfrac{1}{2} N_{h,S_0}(r) - \tfrac{1}{2} m_{h,S_\infty}(r)$$
$$+[\mathrm{OTH}]_{\mathrm{RHS}} - [\mathrm{OTH}]_{\mathrm{LHS}} + \tfrac{1}{2} m_{h,S_\infty}(r) + \tfrac{1}{2}[J]$$
$$+\tfrac{1}{2}[J]$$
$$\geq \tfrac{1}{2} m_{h,S_0}(r) + \tfrac{1}{2} N_{h,S_0}(r) - \tfrac{1}{2} m_{h,S_\infty}(r) + \tfrac{1}{2}[J]$$

and finally we have

$$T_g(r) \leq \frac{1}{2}[J] \ .$$

Conclusion and Remark

(1) We have

$$\frac{1}{\kappa}\log\frac{1}{1-r} \leq T_g(r) \leq \frac{1}{2}[J] \ .$$

We note that the upper bound depends only on the conformal structure of M and not on the individual Weierstrass data.
(2) We have

$$m_{g,D}(r) + N_{g,\mathrm{Ram}}(r) \leq (\kappa+\delta)T_g(r)||_\delta$$

holds. This is the Second Main Theorem for the Gauss map of algebraic minimal surfaces. In Particular the Gauss map of any algebraic minimal surface omits at most $[\kappa]$ values.
(3) $\kappa \leq e + \text{small number}$.
(4) $\sharp\{\text{exceptional values}\} \leq \text{TRVN} \leq e + \text{small number}$. Here TRVN means totally ramified value number of the Gauss map.