Discrete surfaces and architecture (figures 1, 2, 3)

Useful to have offsets of discrete surfaces in architecture:
- structurally stronger
- heat insulation (figure 4)

Triangular meshes

**Pluses**
- variations easy
- offsets always exist

**Minuses**
- less esthetically pleasing: can get long thin triangles when reducing area (Evolver 5, figure 6)
- steel to glass ratio is high $\Rightarrow$ heavier structures
- making glass triangles is wasteful:
Quad meshes

**PLUSES**
- impose restrictions like: "4 vertices of each quad lie in a circle"
  \[ \Rightarrow \text{more pleasing esthetics} \]
- lighter structures
- glass quadrilaterals are less wasteful:

**MINUSES**
- variations and offsets are trickier:
  (figure 7)

**Torsion**

Take a discrete surface; thicken the edges into 2D planar quadrilaterals.

**Defn** No **node torsion**: these thickened edges meet along a line at each vertex. Otherwise, there is node torsion.

**Fact** Generically, for triangulated surfaces, torsion-free offsets are only translated and scaled copies of the original surface.

**Fact** For quad surfaces, there is more freedom.
Higher order meshes (figures 8, 9, 10, 11)

Let \( f : D \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}^3 \), \( N = \text{unit normal vector} \).

Fundamental forms:
\[
I = \begin{pmatrix} f_x \cdot f_x & f_x \cdot f_y \\ f_y \cdot f_x & f_y \cdot f_y \end{pmatrix} = (g_{ij}), \quad II = \begin{pmatrix} f_{xx} \cdot N & f_{xy} \cdot N \\ f_{yx} \cdot N & f_{yy} \cdot N \end{pmatrix} = (b_{ij})
\]

\( k_1, k_2 \) are eigenvalues of \( I^T II \), \( k = k_1 \cdot k_2 \),

eigenlines are principle curvature directions, \( H = \frac{k_1 + k_2}{2} \).
Fact: There exist conformal (isothermal) coordinates $(x, y)$, i.e., $I = \begin{pmatrix} g^1 & 0 \\ 0 & g^2 \end{pmatrix}$
- The map $f$ preserves angles and stretches uniformly.
- 2-dim'l Riemannian manifolds are Riemann surfaces, and can consider holomorphic functions, ....

Fact: Away from umbilic points (where $k_1 = k_2$), there exist curvature line coordinates $(x, y)$, i.e., $I = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$, $\mathbf{I} = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}$.

For the rest of this talk, we assume $f$ has no umbilics.

Question: Do there exist isothermal curvature line coordinates $(x, y)$, i.e., isothermic coordinates?

Remark: Isothermic $\neq$ isothermal

Remark: If yes, $f$ is an isothermic surface, even if we haven't found those isothermic coordinates yet.

Remark: "Isothermic coordinates" is a Möbius invariant notion.

(This is nice.)
Question: Do there exist isothermic coordinates \((x,y)\)?

Answer:

Start with curvature line coordinates \(f = f(x,y)\).
Stretch \(x\) and \(y\) to \(\tilde{x}(x)\) and \(\tilde{y}(y)\), with \(\tilde{x}_x > 0, \tilde{y}_y > 0\).

\[
f_x \circ f_y = 0 \Rightarrow f_x \circ f_y = \left( f_x \cdot x_x + f_y \cdot y_x \right) \circ \cdot (\cdots) = \cdots = 0,
\]

\[
f_{xy} \circ N = 0 \Rightarrow f_{xy} \circ N = \left( f_x \cdot x_x + f_y \cdot y_x \right)_y \circ N = \cdots = 0,
\]

So \((\tilde{x},\tilde{y})\) are curvature line coordinates too: \(I = \begin{pmatrix} \tilde{a} & 0 \\ 0 & \tilde{b} \end{pmatrix}, II = \begin{pmatrix} \tilde{a} & 0 \\ 0 & \tilde{b} \end{pmatrix}\).

Goal: Stretch by \(\tilde{x},\tilde{y}\) until \(\tilde{q}_{11} = \tilde{q}_{22}\), i.e. \(q_{11} \cdot (x_x)^2 = q_{22} \cdot (y_y)^2\).

So \(f\) is isothermic \(\iff \frac{q_{11}}{q_{22}} = \frac{a(x)}{b(y)}\).

Remark: We see that some surfaces are not isothermic.

Example: Dini's helices. \(K = -1\), \(q_{12} = b_{12} = 0\), \(\frac{q_{11}}{q_{22}} = \sinh^2\left(\frac{x+by}{\sqrt{1-b}}\right)\). (figures 12, 13)

Remark: CMC surfaces and surfaces of revolution are always isothermic.
For the benefit of the discrete case, take coordinates s.t. $f_x \perp f_y$, and set

$$q_e := \text{cr}(f(x, y), f(x + \varepsilon, y), f(x + \varepsilon, y + \varepsilon), f(x, y + \varepsilon))^{-1}.$$  

$$= (f(x + \varepsilon, y) - f(x, y) \cdot (f(x + \varepsilon, y + \varepsilon) - f(x + \varepsilon, y)) \cdot (f(x, y + \varepsilon) - f(x + \varepsilon, y + \varepsilon)) \cdot (f(x, y) - f(x, y + \varepsilon))^{-1}.$$  

Meaning of $\cdot$ and $^{\prime\prime}$:

$$\mathbb{R}^3 \approx \text{Im}(H) \rightarrow f(x, y) = f_1(x, y) \cdot i + f_2(x, y) \cdot j + f_3(x, y) \cdot k.$$  

or $f = f_1 \cdot (0, -i) + f_2 \cdot (0, 1) + f_3 \cdot (i, 0)$

Lemma $\lim_{\varepsilon \to 0} q_e = -\frac{g_{0\varepsilon}}{g_{0\varepsilon}}$. \quad Pf $f_x = f_y = 0$ \iff $f_x f_y = f_y f_x \iff f_x f_y = -f_x f_y \iff f_x f_y = -f_x f_y$. \iff $f_x f_y = -f_y f_x$, so $f_x f_y f_x f_y = -\frac{f_x f_y f_x f_y}{g_{0\varepsilon}} = -\frac{g_{0\varepsilon}}{g_{0\varepsilon}}$.

Cor $f$ is isothermic $\iff \lim_{\varepsilon \to 0} q_e = \frac{a(x)}{b(y)}$.

Remark This condition is stated without stretching $x$ and $y$, and this is useful for the discrete case, where we can't stretch.
Bobenko-Pinkall lemma

Define \( q^d_\varepsilon := cr(f(x-\varepsilon, y-\varepsilon), f(x+\varepsilon, y-\varepsilon), f(x+\varepsilon, y+\varepsilon), f(x-\varepsilon, y+\varepsilon)) \).

Lemma

\[
q^d_\varepsilon = -1 + \Theta(\varepsilon^2) \iff (x, y) \text{ is conformal,}
q^d_\varepsilon = -1 + \Theta(\varepsilon^2) \iff (x, y) \text{ is isothermic.}
\]

Proof

WLOG \( f(x, y) = 0 \), then for \( p_x, p_y \in \mathbb{R} \),

\[
f(x + p_x \cdot \varepsilon, y + p_y \cdot \varepsilon) = \varepsilon p_x f_x + \varepsilon p_y f_y + \frac{\varepsilon^2}{2} (f_{xx} + 2 p_x p_y f_{xy}) + O(\varepsilon^3).
\]

So \( q^d_\varepsilon = \frac{f_x f^{-1}_y + f_y f^{-1}_x}{f_x f_y} + \varepsilon \cdot (f_{x} f^{-1}_y f_{xy} f_x f^{-1}_y f_{xy} - f_y f_x f_{xy} f_y f_{xy} f_x f_{xy}) + O(\varepsilon^3) \).

For conformal coordinates, \( q^d_\varepsilon = -1 \) for \( \varepsilon \neq 0 \).

For isothermic coordinates, since \( b_{12} = 0 \Rightarrow f_{xy} = \alpha_1 f_x + \alpha_2 f_y \).
**Dual surfaces**

**Defn** The dual surface (or Christoffel transform) $f^*$ of $f$ satisfies $df^* = f^{-1}_x dx - f^{-1}_y dy$.

**Properties**
1. $f$ and $f^*$ have the same conformal structure
2. they have opposite orientations
3. they have parallel tangent planes at corresponding points

**Examples**

**Lemma** $f^*$ exists $\iff f$ is isothermic.

**Pf** $\iff$

Let $(x,y)$ be isothermic coordinates. $f_{xy} = \varepsilon_1 x_1 f_x + \varepsilon_2 x_2 f_y$. Let $d(f^{-1}_x dx - f^{-1}_y dy) = 16^{-1} \delta:\varepsilon (f_x f_{xy} f_x + f_y f_{xy} f_y) dx \wedge dy = 0$. So there exists $f^*$ s.t. $df^* = f^{-1}_x dx - f^{-1}_y dy$.

$\implies$ Assume $f^*$ exists. Take curvature line coordinates $(x,y)$ for $f$. 
The Codazzi equations are

\[ 2(k_1)_y = \frac{(g_{11})_y}{g_{11}} \cdot (k_2 - k_1), \quad 2(k_2)_x = \frac{(g_{22})_x}{g_{22}} \cdot (k_1 - k_2). \]

Existence of \( f^* \Rightarrow \left( \frac{(k_2 + k_1)_y}{k_1 - k_2} \right)_x = \left( \frac{(k_1 + k_2)_x}{k_2 - k_1} \right)_y \Rightarrow \]

\[ 2((k_1)_y + (k_2)_x)(k_1 - k_2)^{-1} + 2(k_2 - k_1)^{-2} \cdot [(k_1)_y \cdot (k_2 - k_1)_x + (k_2)_x \cdot (k_2 - k_1)_y] = 0. \]

Substituting the Codazzi equations into this, we have

\[ (\log \frac{g_{11}}{g_{22}})_x = 0, \quad \text{i.e.} \quad \frac{g_{11}}{g_{22}} = \frac{a(x)}{b(y)}. \]

Remark: Main point is that we get to the condition \( \frac{g_{11}}{g_{22}} = \frac{a(x)}{b(y)}. \)

Remark: Isothermic coordinates \((x,y)\) for \( f \) are isothermic for \( f^* \).

Remark: Other transformations:

- Calapso, Darboux, Bäcklund, Goursat, Lawson, Ribaucour, ...

  \Rightarrow \text{a lot of interesting mathematics.}

But the natural setting for most of this is

Möbius geometry.
Steiner's Formula

The area of \( f \) is \( A(f) = \iint_D \det(f_x, f_y, N) \, dx \, dy \). Parallel surfaces are \( f^+ = f + t \cdot N \).

**Lemma** \( A(f^+) = A(f) + 2t \cdot \iint_D H \, dx \, dy + t^2 \iint_D K \, dx \, dy \).

**Proof 1**

\[
A(f^+) = \iint_D \det(f_x + tN_x, f_y + tN_y, N) \, dx \, dy \quad (\text{note: } N^+ = N)
\]

\[
= \iint_D \det(f_x + t(af_x + cf_y), f_y + t(bf_x + df_y), N) = \ldots ,
\]

where \( I^{-1} II = (a \ b) \), \( H = \frac{1}{2}(a + d) \), \( K = ad - bc \).

**Proof 2**

Use that \( f \rightarrow f^+ \)

- Area form \( \omega = \det(f_x, f_y, N) dx \, dy \rightarrow \omega^+ = \det(f_x^+, f_y^+, N) dx \, dy \)
- \( K \cdot \omega \rightarrow K^+ \cdot \omega^+ = K \cdot \omega \)
- "Radius" \( \frac{1}{k_j} \rightarrow \frac{1}{k_j^+} + t \)

\[
A(f) = \iint_D \omega \rightarrow A(f^+) = \iint_D \omega^+\]

Taking \( f \) with small domains, we know \( H \) and \( K \) via Steiner's formul...
Remark H constant means we have a variational property:

The surface is critical for area w.r.t. volume preserving variation

(Mesh 15)

Linear conserved quantities

Theorem (special case of a Burstall-Calderbank result)
Suppose f is isothermic. Then H is constant \iff there exists a linear conserved quantity

\[
P = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \lambda \begin{pmatrix} \hat{z} & u \\ v & -z \end{pmatrix}, \quad z \in \text{Im}(H), \quad u, v \in \mathbb{R}
\]

\[\text{unexplained normalization here}\]

Solving

\[
dP = P \cdot \lambda \gamma - \lambda \gamma \cdot P \quad \forall \lambda \in \mathbb{R}, \text{ where } \gamma := \begin{pmatrix} f \cdot df^* - f \cdot df^* \\ df^* & -df^* \cdot f \end{pmatrix}
\]

Then the normal vector is

\[
N = H \left( \frac{z}{v} - f \right).
\]
Cross ratio of the face pqrs is \( q_{pqrs} = (f_q - f_p)(f_r - f_q)^{-1}(f_s - f_r)(f_p - f_s)^{-1} \).

**Defn.** \( f \) is *discrete isothermic* if \( \exists a : \text{edges} \mathbb{Z} \to \mathbb{R} \) s.t.

\[
q_{pqrs} = \frac{a_{pq}}{a_{ps}} \quad \text{and} \quad a_{pq} = a_{rs}, \quad a_{ps} = a_{qr} \quad \text{for all faces}.
\]

**Remark.** It’s analogous to smooth case, where \( f \) is isothermic iff

\[
\lim_{\varepsilon \to 0} q_\varepsilon = \frac{a(x)}{b(y)}.
\]

**Remark.** Bobenko-Pinkall condition \( q_\varepsilon^d = -1 + O(\varepsilon^2) \) for isothermicity in the smooth case suggests the definition \( q_{pqrs} = -1 \).

This was the first definition given, but it is not preserved under Calapso transformations.

**Remark.** Once \( q_{pqrs} \in \mathbb{R} \), \( q_{pqrs} \) is a Möbius invariant. In fact, \( f_p, f_q, f_r \) and \( f_s \) are concircular.
Discrete Christoffel transforms

In the smooth case, $f$ and $f^*$ satisfy
$$df = f_x dx + f_y dy \quad \text{and} \quad df^* = f_x^- dx - f_y^- dy.$$

So
$$df(\partial_x) \cdot df^*(\partial_x) = 1 \quad \text{and} \quad df(\partial_y) \cdot df^*(\partial_y) = -1.$$ 

Also, $\lim_{\varepsilon \to 0} q_{\varepsilon} = -1$.

In the discrete case, we loosened $\lim_{\varepsilon \to 0} q_{\varepsilon} = -1$ to $q_{pqrs} = \frac{a_{pq}}{a_{ps}}$.

So we want this:

**Defn** The discrete Christoffel transform $f^*$ of $f$ satisfies $f^*$ for all edges $pq$.

$$(f_q - f_p) \cdot (f_q^* - f_p^*) = a_{pq}$$

**Lemma** $f$ is discrete isothermic iff $f^*$ exists.

When $f^*$ exists, it has the same cross ratios as $f$. In particular, $f^*$ is isothermic too.
Discrete Steiner formula

**Defn** A **line congruence net** is a map \( f : D \subseteq \mathbb{Z}^2 \rightarrow \mathbb{R}^3 \) with planar quadrilaterals together with a map

\[ \lambda : D \rightarrow \exists \text{ set of lines in } \mathbb{R}^3 \text{ s.t.} \]

\[ f_p \in l_p \text{ and } l_p \cap l_q \neq \emptyset \text{ and } l_p \cap l_s \neq \emptyset \]

on all quadrilaterals.

**Remark** \( l_p \) and \( l_q \), or \( l_p \) and \( l_s \), are allowed to be parallel, i.e. they may intersect at \( \infty \).

**Lemma** A line congruence net \( f \) has a 1-parameter family of parallel surfaces \( f^+ \) with

\[ f^+_p \in l_p \]

and

\[ (f^+_q - f^+_p) \parallel (f^+_q - f_p), \quad (f^+_s - f^+_p) \parallel (f^+_s - f_p) \].
This offset can be used in architecture.

We can now compare \( A(f) \) and \( A(f^+) \).

- use areas of quadrilaterals
- use mixed areas (used in research on convex bodies)

Get quadratic equation in \( t \): discrete Steiner formula

\[ H \text{ and } K \text{ defined on faces of } f \] (figures 16, 17)

(initially defined by Wolfgang Schief)

Can get principal curvatures as well: \( H = \frac{k_1 + k_2}{2}, \quad K = k_1 \cdot k_2 \Rightarrow k_1, k_2 = H \pm \sqrt{H^2 - K} \)
Discrete linear conserved quantities

The theorem in the smooth case becomes a definition in the discrete case:

**Defn** A discrete isothermic surface \( f \) is CMC if there exists a conserved quantity

\[
P_p = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \lambda \begin{pmatrix} z_p & u_p \\ v_p & -z_p \end{pmatrix}, \quad z_p \in \text{Im}(H), \quad u_p, v_p \in \mathbb{R}
\]

solving

\[
(I + \lambda \tau_{pq}) P_q = P_p (I + \lambda \tau_{pq}) \quad \tau_{pq} := \begin{pmatrix} f_p d f_{pq}^* - f_p d f_{pq}^* f_q \\ d f_{pq}^* - d f_{pq}^* f_q \end{pmatrix}
\]

\[
d f_{pq}^* := f_q - f_p
\]

\[
\text{discretized version of eqn in smooth case}
\]

Then we define \( N = z - v \cdot f \) to be a normal vector at vertices.

**Lemma** \( |N| \) is constant, and this picture holds:

So we have applications to architecture.

(figures 18, 19, 20)