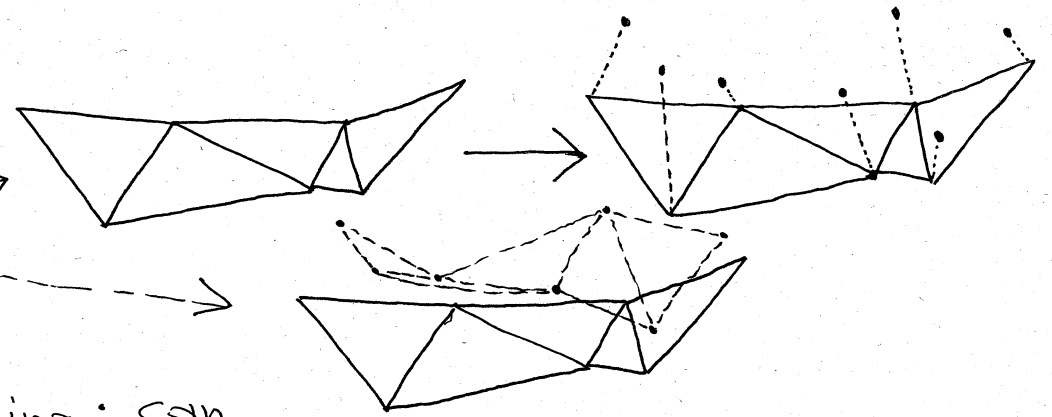
Discrete surfaces and architecture

(figures 1, 2, 3)

Useful to have offsets of discrete surfaces in architecture:

- structurally stronger
- heat insulation (figure 4)

Triangular meshes

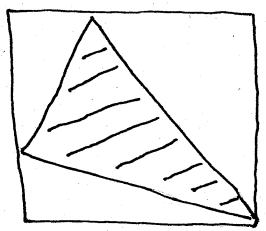


PLUSES

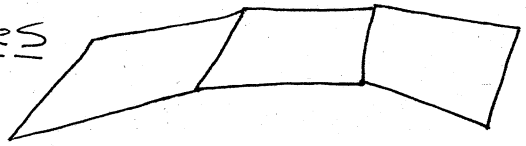
- Variations easy
- offsets always exist

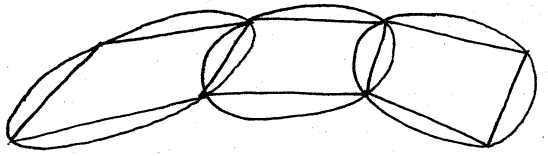
MINUSES

- less esthetically pleasing: can get long thin triangles when reducing area (Evolver 5, figure 6)
- steel to glass ratio is high $\Rightarrow$ heavier structures
- making glass triangles is wasteful:



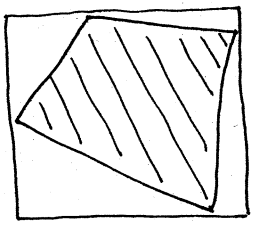
Quad meshes





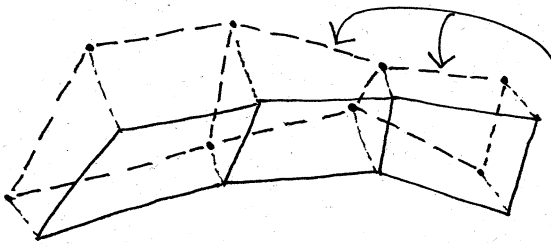
PLUSES

- impose restrictions like: "4 vertices of each quad lie in a circle"
  $\Rightarrow$ more pleasing esthetics
- lighter structures
- glass quadrilaterals are less wasteful:



MINUSES

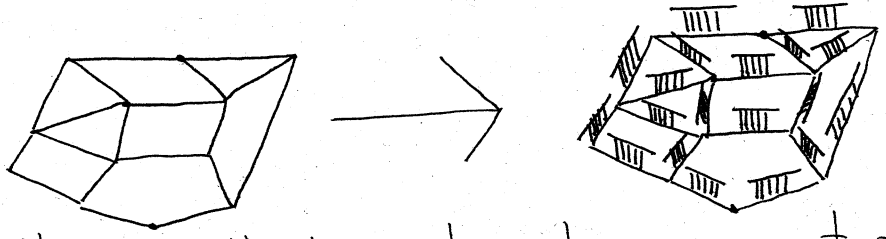
- variations and offsets are trickier: (figure 7)



not necessarily planar (and vertices might not be concircular)

Torsion

Take a discrete surface; thicken the edges into 2D planar quadrilaterals.



<u>Defn</u> No <u>node torsion</u>: these thickened edges meet along a line at each vertex. Otherwise, there is node torsion.

<u>Fact</u> Generically, for triangulated surfaces, torsion free offsets are only translated and scaled copies of the original surface.

<u>Fact</u> For quad surfaces, there is more freedom.

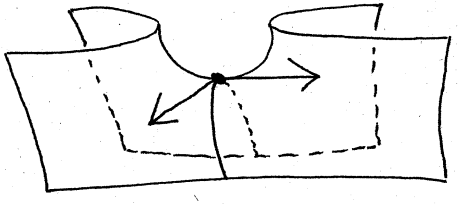
Higher order meshes (figures 8, 9, 10, 11)

Let $f : D \subseteq \mathbb{R}^2 \xrightarrow{\text{smooth immersion}} \mathbb{R}^3$, N = unit normal vector.

Fundamental forms: $\mathrm{I} = \begin{pmatrix} f_x \circ f_x & f_x \circ f_y \\ f_y \circ f_x & f_y \circ f_y \end{pmatrix} = (g_{ij})$, $\mathrm{II} = \begin{pmatrix} f_{xx} \circ N & f_{xy} \circ N \\ f_{yx} \circ N & f_{yy} \circ N \end{pmatrix} = (b_{ij})$

k_1, k_2 are eigenvalues of $\mathrm{I}^{-1}\mathrm{II}$, $K = k_1 k_2$,

eigenlines are principle curvature directions, $H = \dfrac{k_1 + k_2}{2}$,



<u>Fact</u> $\exists$ conformal (isothermal) coordinates (x,y), i.e. $I = \begin{pmatrix} g_{11} & 0 \\ 0 & g_{11} \end{pmatrix}$

- the map f preserves angles and stretches uniformly
- 2-dim'l Riemannian manifolds are Riemann surfaces, and can consider holomorphic functions, ...

<u>Fact</u> Away from umbilic points (where $k_1 = k_2$), $\exists$ curvature line coordinates (x,y), i.e. $I = \begin{pmatrix} g_{11} & 0 \\ 0 & g_{22} \end{pmatrix}$, $II = \begin{pmatrix} b_{11} & 0 \\ 0 & b_{22} \end{pmatrix}$.

$\boxed{\text{For the rest of this talk, we assume } f \text{ has no umbilics.}}$

<u>Question</u> Do there exist isothermal curvature line coordinates (x,y), i.e. <u>isothermic</u> coordinates?

<u>Remark</u> isothermic $\neq$ isothermal

<u>Remark</u> If yes, f is an <u>isothermic surface</u>, even if we haven't found those isothermic coordinates yet.

<u>Remark</u> "isothermic coordinates" is a Möbius invariant notion.
(This is nice.)

<u>Question</u> Do there exist isothermic coordinates (x,y)?

<u>Answer</u>

Start with curvature line coordinates $f = f(x,y)$.

Stretch x and y to $\tilde{x}(x)$ and $\tilde{y}(y)$, with $\tilde{x}_x > 0$, $\tilde{y}_y > 0$.

$$f_x \circ f_y = 0 \Rightarrow f_{\tilde{x}} \circ f_{\tilde{y}} = (f_x \cdot x_{\tilde{x}} + f_y \cdot y_{\tilde{x}}) \circ (\cdots) = \cdots = 0,$$

$$f_{xy} \circ N = 0 \Longrightarrow f_{\tilde{x}\tilde{y}} \circ N = (f_x \cdot x_{\tilde{x}} + f_y \cdot y_{\tilde{x}})_{\tilde{y}} \circ N = \cdots = 0,$$

$\{$ using $f_x \circ N = f_y \circ N = 0 \}$

so $(\tilde{x},\tilde{y})$ are curvature line coordinates too: $\tilde{\mathrm{I}} = \begin{pmatrix} \tilde{g}_{11} & 0 \\ 0 & \tilde{g}_{22} \end{pmatrix}$, $\tilde{\mathrm{II}} = \begin{pmatrix} \tilde{b}_{11} & 0 \\ 0 & \tilde{b}_{22} \end{pmatrix}$

<u>Goal</u>: Stretch by $\tilde{x}, \tilde{y}$ until $\tilde{g}_{11} = \tilde{g}_{22}$, i.e. $g_{11} \cdot (x_{\tilde{x}})^2 = g_{22} \cdot (y_{\tilde{y}})^2$.

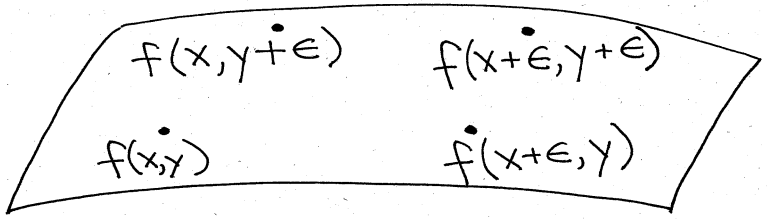
So $$\boxed{f \text{ is isothermic} \Longleftrightarrow \frac{g_{11}}{g_{22}} = \frac{a(x)}{b(y)}}$$

<u>Remark</u> We see that some surfaces are not isothermic.
Example: Dini's helices. $K = -1$, $g_{12} = b_{12} = 0$, $\frac{g_{11}}{g_{22}} = \sinh^2\left(\frac{x+by}{\sqrt{1-b^2}}\right)$.

(figures 12, 13)

<u>Remark</u> CMC surfaces and surfaces of revolution are always isothermic.

For the benefit of the discrete case, take coordinates s.t. $f_x \perp f_y$, and set



$$q_\epsilon := cr(\, f(x,y),\, f(x+\epsilon,y),\, f(x+\epsilon,y+\epsilon),\, f(x,y+\epsilon)\,)$$
$$= \big(f(x+\epsilon,y)-f(x,y)\big)\cdot\big(f(x+\epsilon,y+\epsilon)-f(x+\epsilon,y)\big)^{-1}\cdot\big(f(x,y+\epsilon)-f(x+\epsilon,y+\epsilon)\big)\cdot\big(f(x,y)-f(x,y+\epsilon)\big)^{-1}$$

Meaning of $\cdot$ and $(\,''\,)^{-1}$:

$$\mathbb{R}^3 \approx \mathrm{Im}(H) \longrightarrow f(x,y) = f_1(x,y)\cdot \mathbb{i} + f_2(x,y)\cdot \mathbb{j} + f_3(x,y)\cdot \mathbb{k}$$

$$\text{or} \quad f = f_1\cdot\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} + f_2\cdot\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + f_3\cdot\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

<u>Lemma</u> $\lim_{\epsilon\to 0} q_\epsilon = -\frac{g_{11}}{g_{22}}$. Pf/ $f_x \circ f_y = 0 \iff f_x f_y = -f_y f_x \iff f_x\cdot\frac{f_y}{f_y^2} = \frac{-f_y}{f_y^2}\cdot f_x$
$\iff f_x f_y^{-1} = -f_y^{-1} f_x$, so $f_x f_y^{-1} f_x f_y^{-1} = -\frac{f_x^2}{f_y^2} = -\frac{g_{11}}{g_{22}}$. //

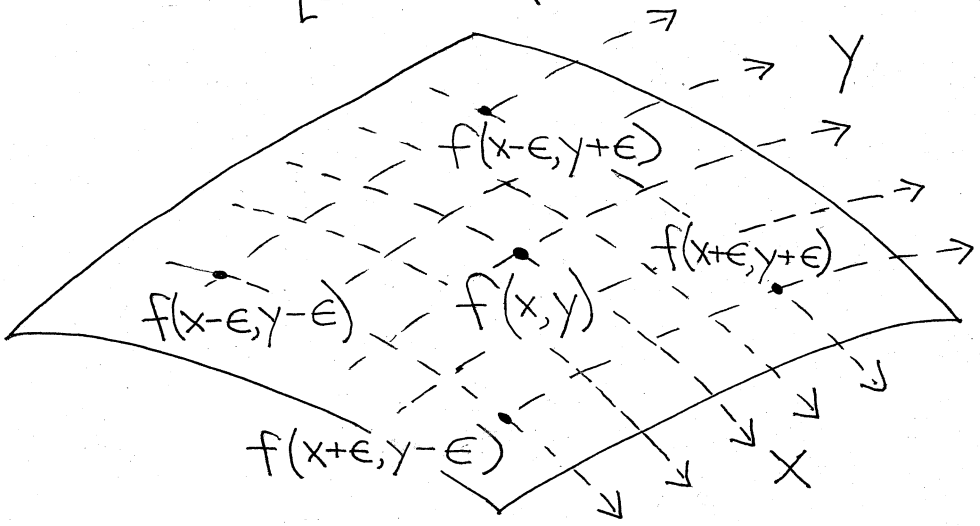
<u>Cor</u> $\boxed{f \text{ is isothermic} \iff \lim_{\epsilon\to 0} q_\epsilon = \frac{a(x)}{b(y)}.}$

<u>Remark</u> This condition is stated <u>without</u> stretching x and y, and this is useful for the discrete case, where we can't stretch.

Bobenko-Pinkall lemma

(figure 14)

Define $q^d_\epsilon := cr\big(f(x-\epsilon,y-\epsilon), f(x+\epsilon,y-\epsilon), f(x+\epsilon,y+\epsilon), f(x-\epsilon,y+\epsilon)\big)$.



Lemma

$q^d_\epsilon = -1 + \mathcal{O}(\epsilon) \iff (x,y)$ is conformal,

$q^d_\epsilon = -1 + \mathcal{O}(\epsilon^2) \iff (x,y)$ is isothermic.

Pf/ WLOG $f(x,y) = \vec{0}$, then for $\rho_x, \rho_y \in \{\pm 1\}$,

$$f(x+\rho_x\cdot\epsilon, y+\rho_y\cdot\epsilon) = \epsilon\rho_x f_x + \epsilon\rho_y f_y + \frac{\epsilon^2}{2}\left(f_{xx}+f_{yy}+2\rho_x\rho_y f_{xy}\right) + \mathcal{O}(\epsilon^3),$$

so

$$q^d_\epsilon = \underbrace{f_x f_y^{-1} f_x f_y^{-1}}_{=-1 \text{ for conformal coordinates}} + \epsilon\cdot\left(f_x f_y^{-1} f_{xy} f_y^{-1} + f_x f_y^{-1} f_x f_y^{-1} f_{xy} f_y^{-1} - f_{xy} f_y^{-1} f_x f_y^{-1} - f_x f_y^{-1} f_{xy} f_y^{-1} f_x f_y^{-1}\right) + \mathcal{O}(\epsilon^2)$$

$$= -1 + \epsilon\cdot f_x^{-4}\cdot\left(f_x f_y f_{xy}(f_x+f_y) + f_x^2 f_{xy}(f_x - f_y)\right) + \mathcal{O}(\epsilon^2)$$

$= -1 + \mathcal{O}(\epsilon^2)$ for isothermic coordinates, since

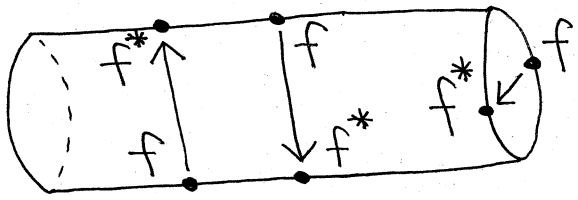
$b_{12} = 0 \Rightarrow f_{xy} = \alpha_1 f_x + \alpha_2 \cdot f_y$. //

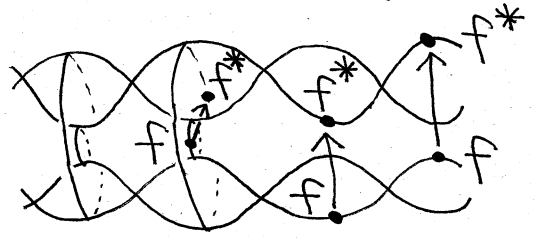
Dual surfaces

Defn The <u>dual surface</u> (or <u>Christoffel transform</u>) f^* of f satisfies $df^* = f_x^{-1}dx - f_y^{-1}dy$.

Properties
- f and f^* have the same conformal structure
- they have opposite orientations
- they have parallel tangent planes at corresponding points

Examples





Lemma f^* exists $\Longleftrightarrow$ f is isothermic.

Pf/ "$\Longleftarrow$" Let (x,y) be isothermic coordinates. $f_{xy} = {}^{\exists}\alpha_1 \cdot f_x + {}^{\exists}\alpha_2 \cdot f_y$. So
$d(f_x^{-1}dx - f_y^{-1}dy) = 16 \cdot g_{11}^{-2} \cdot (f_x f_{xy} f_x + f_y f_{xy} f_y)\, dx \wedge dy = 0$.
there exists f^* s.t. $df^* = f_x^{-1}dx - f_y^{-1}dy$.

"$\Longrightarrow$" Assume f^* exists. Take curvature line coordinates (x,y) for f.

The Codazzi equations are

$$2(k_1)_y = \frac{(g_{11})_y}{g_{11}} \cdot (k_2 - k_1) \quad , \quad 2(k_2)_x = \frac{(g_{22})_x}{g_{22}} \cdot (k_1 - k_2) \; .$$

Existence of $f^* \Rightarrow \left(\frac{(k_2+k_1)_y}{k_1-k_2} \right)_x = \left(\frac{(k_1+k_2)_x}{k_2-k_1} \right)_y \Longrightarrow$

$$2\left((k_1)_{yx} + (k_2)_{xy}\right)(k_1-k_2)^{-1} + 2(k_2-k_1)^{-2} \cdot \left[(k_1)_y \cdot (k_2-k_1)_x + (k_2)_x \cdot (k_2-k_1)_y\right] = 0 \, .$$

Substituting the Codazzi equations into this, we have

$$\left(\log \frac{g_{11}}{g_{22}}\right)_{xy} = 0 \, , \qquad \text{i.e.} \quad \frac{g_{11}}{g_{22}} = \frac{a(x)}{b(y)} \; . \qquad //$$

<u>Remark</u> Main point is that we get to the condition $\frac{g_{11}}{g_{22}} = \frac{a(x)}{b(y)}$.

<u>Remark</u> Isothermic coordinates (x,y) for f are isothermic for f^* too

<u>Remark</u> Other transformations:

Calapso, Darboux, Bäcklund, Gorsat, Lawson, Ribaucour, ...

$\Longrightarrow$ a lot of interesting mathematics.

But the natural setting for most of this is Möbius geometry.

Steiner's formula

The area of f is $A(f) = \int_D \det(f_x, f_y, N)\, dx\, dy$.

Parallel surfaces are $f^t = f + t \cdot N$.

Lemma $A(f^t) = A(f) + 2t \cdot \int_D H\, dx\, dy + t^2 \cdot \int_D K\, dx\, dy$.

Proof 1

$$A(f^t) = \int_D \det(f_x + tN_x, f_y + tN_y, N)\, dx\, dy \qquad (\text{note: } N^t = N)$$

$$= \int_D \det(f_x + t(af_x + cf_y), f_y + t(bf_x + df_y), N) = \cdots,$$

where $I^{-1}II = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $H = \frac{1}{2}(a+d)$, $K = ad - bc$. //

Proof 2

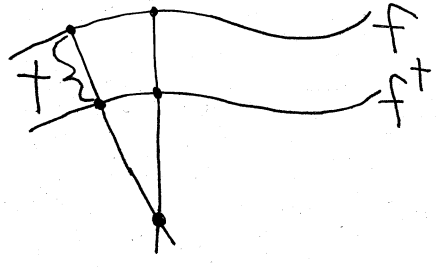
Use that $f \to f^t \implies$

area form $\omega = \det(f_x, f_y, N)\, dx\, dy \longrightarrow \omega^t = \det(f_x^t, f_y^t, N)\, dx\, dy$

$K \cdot \omega \longrightarrow K^t \cdot \omega^t = K \cdot \omega$

"radius" $\frac{1}{k_j} \longrightarrow \frac{1}{k_j^t} + t$

$A(f) = \int_D \omega \longrightarrow A(f^t) = \int_D \omega_t$



//

Taking f with small domains, we know H and K via Steiner's formula

<u>Remark</u> H constant means we have a variational property:

The surface is critical for area w.r.t. volume preserving variation (MESH 15)

Linear conserved quantities

<u>Theorem</u> (special case of a Burstall-Calderbank result)

Suppose f is isothermic. Then H is constant $\Longleftrightarrow$ there exists a linear conserved quantity

$$P = \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{\text{unexplained normalization here}} + \lambda \begin{pmatrix} z & u \\ v & -z \end{pmatrix}, \quad z \in \mathrm{Im}(H), \; u, v \in \mathbb{R}$$

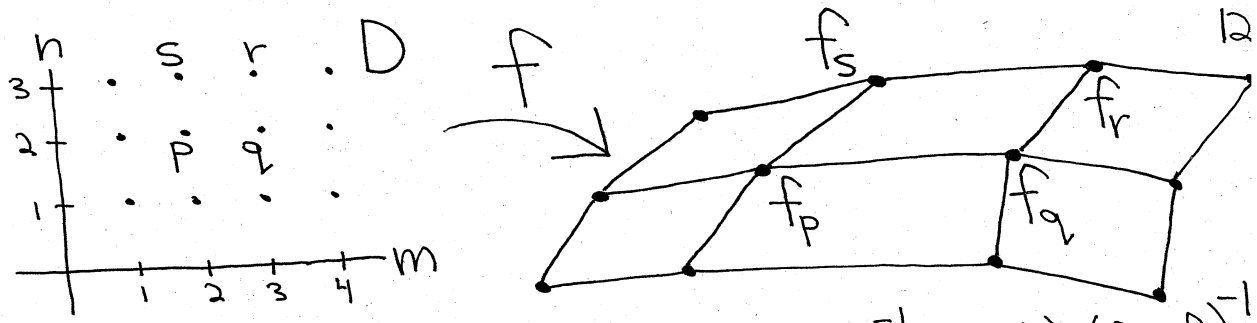
solving

$$dP = P \cdot \lambda \tau - \lambda \tau \cdot P \quad \forall \lambda \in \mathbb{R}, \text{ where } \tau := \begin{pmatrix} f \cdot df^* & -f \cdot df^* \cdot f \\ df^* & -df^* \cdot f \end{pmatrix}$$

Then the normal vector is $N = H\left(\frac{z}{v} - f\right)$.

Discrete surfaces

$f : D \subseteq \mathbb{Z}^2 \to \mathbb{R}^3$



Cross ratio of the face $pqrs$ is $q_{pqrs} = (f_q - f_p)(f_r - f_q)^{-1}(f_s - f_r)(f_p - f_s)^{-1}$.

Defn f is <u>discrete isothermic</u> if $\exists\, a : \{\text{edges}\} \to \mathbb{R}$ s.t.

$$q_{pqrs} = \frac{a_{pq}}{a_{ps}} \quad \text{and} \quad a_{pq} = a_{rs},\ a_{ps} = a_{qr} \quad \text{for all faces.}$$

Remark It's analogous to smooth case, where f is isothermic iff $\lim_{\epsilon \to 0} q_\epsilon = \frac{a(x)}{b(y)}$.

Remark Bobenko-Pinkall condition $q_\epsilon^d = -1 + \mathcal{O}(\epsilon^2)$ for isothermicity in the smooth case suggests the definition $q_{pqrs} = -1$. This was the first definition given, but it is not preserved under Calapso transformations.

Remark Once $q_{pqrs} \in \mathbb{R}$, q_{pqrs} is a Möbius invariant. In fact, f_p, f_q, f_r and f_s are concircular.

Discrete Christoffel transforms

In the smooth case, f and f^* satisfy

$$df = f_x\,dx + f_y\,dy \quad \text{and} \quad df^* = f_x^{-1}\,dx - f_y^{-1}\,dy .$$

So

$$df(\partial_x)\cdot df^*(\partial_x) = 1 \quad \text{and} \quad df(\partial_y)\cdot df^*(\partial_y) = -1 .$$

Also, $\lim_{\epsilon\to 0} q_\epsilon = -1$.

In the discrete case, we loosened $\lim_{\epsilon\to 0} q_\epsilon = -1$ to $q_{pqrs} = \frac{a_{pq}}{a_{ps}}$.

So we want this:

<u>Defn</u> The <u>discrete Christoffel transform</u> f^* of f satisfies

$$(f_q - f_p)\cdot(f_q^* - f_p^*) = a_{pq} \qquad \text{for all edges } \overline{pq}.$$

<u>Lemma</u> f is discrete isothermic iff f^* exists.
When f^* exists, it has the same cross ratios as f. In particular, f^* is isothermic too.

Discrete Steiner formula

Defn A <u>line congruence net</u> is a map $f : D \subseteq \mathbb{Z}^2 \to \mathbb{R}^3$ with planar quadrilaterals together with a map

$$l : D \to \{\text{set of lines in } \mathbb{R}^3\} \quad \text{s.t.}$$

$$f_p \in l_p \quad \text{and} \quad l_p \cap l_q \neq \phi \quad \text{and} \quad l_p \cap l_s \neq \phi$$

on all quadrilaterals.

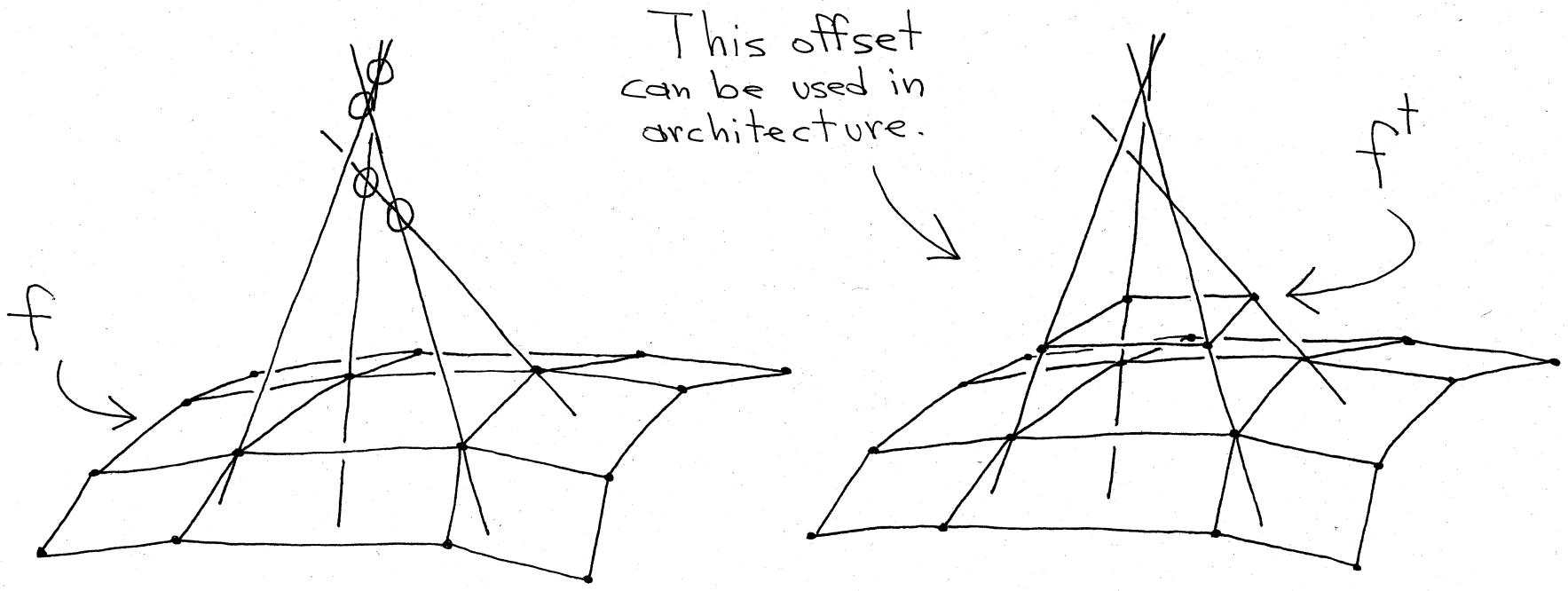
Remark l_p and l_q, or l_p and l_s, are allowed to be parallel, i.e. they may intersect at ∞.

Lemma A line congruence net f has a 1-parameter family of parallel surfaces f^t with

$$f_p^t \in l_p$$

and

$$(f_q^t - f_p^t) \parallel (f_q - f_p), \quad (f_s^t - f_p^t) \parallel (f_s - f_p).$$



We can now compare $A(f)$ and $A(f^t)$.

- use areas of quadrilaterals
- use mixed areas (used in research on convex bodies)

Get quadratic equation in t: discrete Steiner formula

$\longrightarrow$ H and K defined on faces of f (figures 16, 17)

(initially defined by Wolfgang Schief)

Can get principal curvatures as well: $H = \frac{k_1 + k_2}{2}$, $K = k_1 \cdot k_2 \Rightarrow k_1, k_2 = H \pm \sqrt{H^2 - K}$

Discrete linear conserved quantities

The theorem in the smooth case becomes a definition in the discrete case:

Defn A discrete isothermic surface f is CMC if there exists a conserved quantity

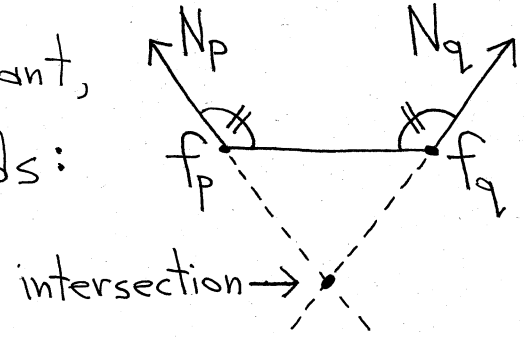
$$P_p = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \lambda \cdot \begin{pmatrix} z_p & u_p \\ v_p & -z_p \end{pmatrix}, \quad z_p \in \mathrm{Im}(\mathbb{H}), \; u_p, v_p \in \mathbb{R}$$

solving

$$\underbrace{(I + \lambda \tau_{pq}) P_q = P_p (I + \lambda \tau_{pq})}_{\text{discretized version of eqn in smooth case}}, \quad \tau_{pq} := \underbrace{\begin{pmatrix} f_p df_{pq}^* & -f_p df_{pq}^* f_q \\ df_{pq}^* & -df_{pq}^* f_q \end{pmatrix}}_{df_{pq}^* := f_q^* - f_p^*}$$

Then we define $N = z - v \cdot f$ to be a normal vector at vertices.

Lemma $|N|$ is constant, and this picture holds:



So we have applications to architecture.

(figures 18, 19, 20)