QUANTIZATION OF THE UNIVERSAL TEICHMÜLLER SPACE

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The universal Teichmüller space $\mathcal{T}$, introduced by Ahlfors and Bers, plays a key role in the theory of quasiconformal maps and Riemann surfaces. It can be defined as the space of quasisymmetric homeomorphisms of the unit circle S^1 (i.e. homeomorphisms of S^1, extending to quasiconformal maps of the unit disc) modulo Möbius transformations (i.e. fractional-linear automorphisms of Δ). The space $\mathcal{T}$ has a natural complex structure, induced by its realization as an open subset in the complex Banach space of holomorphic quadratic differentials in the unit disc. The universal Teichmüller space $\mathcal{T}$ contains all classical Teichmüller spaces $T(G)$, where G is a Fuchsian group, in particular, all finite-dimensional Teichmüller spaces, associated with compact Riemann surfaces of finite genus. The spaces $T(G)$ are embedded into $\mathcal{T}$ as complex submanifolds. We can also consider the homogeneous space $\mathcal{S} := \mathrm{Diff}_+(S^1)/\mathrm{Möb}(S^1)$, which is the quotient of the diffeomorphism group $\mathrm{Diff}_+(S^1)$ of the circle modulo Möbius transformations, as a "smooth" part of $\mathcal{T}$.

The smooth part $\mathcal{S}$ may be quantized by embedding it into the Hilbert-Schmidt Siegel disc $\mathcal{D}_{\mathrm{HS}}$. By this embedding, the diffeomorphism group $\mathrm{Diff}_+(S^1)$ is realized as a subgroup of the Hilbert–Schmidt symplectic group, acting on the Siegel disc by operator fractional-linear transformations. We define a holomorphic Fock bundle over the Siegel disc $\mathcal{D}_{\mathrm{HS}}$, provided with a projective action of the Hilbert–Schmidt symplectic group, covering its action on $\mathcal{D}_{\mathrm{HS}}$. The infinitesimal version of this action yields a projective representation of the Hilbert–Schmidt symplectic Lie algebra in a fibre F_0 of the Fock bundle at the origin. This construction can be considered as geometric quantization of the Siegel disc. Its restriction to $\mathcal{S}$ gives a projective representation of the Lie algebra $\mathrm{Vect}(S^1)$ of the group $\mathrm{Diff}_+(S^1)$ in the Fock space F_0, which defines the geometric quantization of $\mathcal{S}$.

However, the described quantization procedure does not apply to the whole universal Teichmüller space $\mathcal{T}$. Nevertheless, this space can be quantized, using the "quantized calculus" of Connes and Sullivan. The idea of this approach is to construct a representation π of the associative algebra of observables in a Hilbert space H, sending the differential df of an observable f into the commutator $[S, \pi(f)]$ of $\pi(f)$ with a self-adjoint symmetry operator S, determined by polarization of H.

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