

QUANTIZATION

OF

UNIVERSAL

TEICHMÜLLER

SPACE

$$\mathcal{T} = \frac{QS(S^1)}{\text{M\"ob}(S^1)}$$

# I. UNIVERSAL TEICHMÜLLER SPACE

## 1. Definition of universal Teichmüller space

Def: Homeo  $f: S^1 \rightarrow S^1$ , preserving orientation,  
is quasisymmetric, if it extends to a  
quasiconformal homeo of  $\Delta$ .

Recall: homeo  $w: \Delta \rightarrow w(\Delta)$  with  $L_{loc}^1$ -derivatives  
is quasiconformal if there exists  
 $\mu \in L^\infty(\Delta)$  with  $\|\mu\|_\infty < 1$  s.t. the  
following Beltrami equation

$$w_{\bar{z}} = \mu w_z \quad \text{holds for a.a. } z \in \Delta.$$

$\Rightarrow \mu$  is called Beltrami differential of  $w$ .

Def:  $QS(S^1)$  is the group of quasisymmetric  
homeo of  $S^1$ .

The universal Teichmüller space

$$\mathcal{T} = QS(S^1) / \text{M\"ob}(S^1), \text{ where}$$

$\text{M\"ob}(S^1) = \{ \text{fract.-linear auto of } \Delta, \text{ restricted to } S^1 \}$

In other words,

$\mathcal{T} = \{ \text{normalized quasiregular homeo of } S^1, \text{ fixing the points } \pm 1, i \}$

Since

$QS(S^1) \supset \text{Diff}_+(S^1) = \{ \text{orient.-preserving diffeo of } S^1 \}$

$\Rightarrow \mathcal{T} \supset \mathcal{S} := \text{Diff}_+(S^1) / \text{M\"ob}(S^1)$

where  $\mathcal{S}$  may be considered as a "smooth" part of  $\mathcal{T}$ .

Def. of  $\mathcal{T}$  in terms of Beltrami differentials //  $B(\Delta) = \{ \text{space of Beltrami diff.} \}$   
 $\approx \text{unit ball in } L^\infty(\Delta)$

Given  $\mu \in B(\Delta)$ , we extend it by zero outside  $\Delta$  to a potential  $\check{\mu} \in L^\infty(\overline{\mathbb{C}})$

Solving Beltrami equation with potential  $\check{\mu}$ , we get a quasiconf. homeo  $w^\mu$  on  $\overline{\mathbb{C}}$ , which is conformal on  $\Delta_- := \overline{\mathbb{C}} \setminus \Delta$ .

The image  $\Delta^\mu := w^\mu(\Delta)$  is called the quasidisc.

(3)

Introduce an equivalence relation between Beltrami differentials:

$$\mu \sim \nu \iff w^\mu|_{\Delta_-} = w^\nu|_{\Delta_-}$$

Then:

$$\mathcal{T} = B(\Delta) / \sim = \{ \text{normalized quasidiscs in } \overline{\mathbb{C}} \}.$$

## 2. Kähler structure of $\mathcal{T}$

Bers embedding || Consider  $\mathcal{T}$  as the space of norm. quasidiscs

$\mathcal{T} \ni [\mu] \longleftrightarrow \text{norm. quasidisc } w^\mu(\Delta)$

Consider the map:

$$\mu \mapsto S(w^\mu|_{\Delta_-}), \text{ where } S \text{ denotes}$$

the Schwarzian

By Schwarzian properties, the image of this map depends only on the class  $[\mu]$  in  $\mathcal{T}$  and is a holom. quadratic differential on  $\Delta_-$ .

Composing this map with a fract.-linear biholom.  $\Delta_- \rightarrow \Delta$ , we obtain an embedding:

$$\Psi: \mathcal{T} = B(\Delta) / \sim \longrightarrow B_2(\Delta)$$

(4)

where

$B_2(\Delta) = \{ \text{holom. quadratic diff. on } \Delta \text{ with finite norm}$

$$\|\psi\|_2 = \sup_{z \in \Delta} (1 - |z|^2)^2 |\psi(z)|$$

The image of  $\Psi$  is an open bounded connected subset in  $B_2(\Delta)$ , so we can provide  $\mathcal{T}$  with complex structure, induced from  $B_2(\Delta)$ .

The smooth part of  $\mathcal{T}$  can be provided with a Kähler metric, which can be given explicitly at the origin of  $S$  in terms of Fourier decompositions.

Tangent vectors  $u \in T_0 S$  have Fourier decompositions

$$u = \sum_{n=-1,0,1} u_n e_n$$

where  $e_n := ie^{int} d/d\theta$ ,  $n=0,\pm 1,\pm 2\dots$ , is a natural basis of the Lie algebra  $\text{Vect}(S^1)$  of  $\text{Diff}_+(S^1)$ .

$$\Rightarrow g(u,v) = 2 \operatorname{Re} \sum_{n=2}^{\infty} u_n \overline{v_n} (n^3 - n), \quad u, v \in T_0 S.$$

#### 4. QS-action on Sobolev space.

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Sobolev space of half-differentiable functions  $\parallel V := H_0^{1/2}(S^1, \mathbb{R}) = \{ f \in L^2(S^1, \mathbb{R}) \text{ with derivatives of order } 1/2 \text{ in } L^2(S^1, \mathbb{R}) \text{ and zero average } f \}$

In terms of Fourier series:

$$f(z) = \sum_{k \neq 0} f_k z^k \in V \iff \sum_{k \neq 0} |k| |f_k|^2 < \infty$$

$\parallel$   
 with  $f_k = \overline{f_{-k}}$        $\parallel$   
 $\parallel f \parallel_{1/2}^2$

This is a Kähler Hilbert space with symplectic form

$$\omega(f, g) = 2 \operatorname{Im} \sum_{k=1}^{\infty} k f_k \bar{g}_k$$

and complex structure

$$J^0 f(z) = -i \sum_{k>0} f_k z^k + i \sum_{k<0} f_k z_k^{-k}$$

This complex structure generates a decomposition of  $V^{\mathbb{C}}$  into the direct sum

$$V^{\mathbb{C}} = \overline{W}_+ \oplus \overline{W}_- =: \overline{W}_0 \oplus \overline{W}_0$$

of  $(\mp i)$ -eigenspaces of  $J^0$ .

$QS$ -action on  $V$  // For an orientation-preserving homeo of  $S^1$  (9)  
 we define an operator

$$T_f(h) = \text{hof-average}(hof) \text{ for } f \in L^1(S^1)$$

It has the following properties:

$$(1) T_f : V \rightarrow V \Leftrightarrow f \in QS(S^1)$$

(2) for  $f \in QS(S^1)$  the operator  $T_f$  preserves  $\omega$  and its complex-linear extension to  $V^\mathbb{C}$  preserves  $\omega_+$   $\Leftrightarrow f \in \text{M\"ob}(S^1)$

$\Rightarrow QS(S^1)$  acts on  $V$  by bounded symplectic operators and

$$\mathcal{T} = QS(S^1) / \text{M\"ob}(S^1) \hookrightarrow \frac{Sp(V)}{U(\omega_+)} = \mathfrak{D}$$

The space on the r.h.s. may be identified with the space  $\mathcal{Y}(V)$  of complex structures  $J$  on  $V$ , compatible with  $\omega$ .

For the smooth part  $\mathcal{S} \subset \mathcal{T}$  we have an analogous embedding

$$\mathcal{S} = \text{Diff}_+(S^1) / \text{M\"ob}(S^1) \hookrightarrow \frac{Sp_{HS}(V)}{U(\omega_+)} = \mathfrak{D}_{HS}$$

into a "regular" part  $\mathcal{Y}_{HS}(V) \subset \mathcal{Y}(V)$ , where

$$\text{Sp}_{HS}(V) = \{A \in \text{Sp}(V) : \pi_+ \circ A \text{ is Hilbert-Schmidt}\}$$

(10)

## II. QUANTIZATION OF STANDS.

### 5. Dirac quantization.

Def: A classical system is a pair  $(M, \mathcal{A})$ ,

where

$M$  is the phase space = symplectic manifold with symplectic form  $\omega$

Locally, it is equiv. to the standard model

$$(\mathbb{R}^{2n}, \omega_0) \text{ with } \omega_0 = \sum_{i=1}^n dp_i \wedge dq_i$$

$\mathcal{A}$  is the algebra of observables = a subalgebra of the Poisson algebra  $C^\infty(M, \mathbb{R})$  with Poisson bracket, determined by  $\omega$

For the standard model one can take for  $\mathcal{A}$  the Heisenberg algebra  $hei(\mathbb{R}^{2n})$ , generated by  $p_i, q_i$  and 1 with Poisson bracket, determined by:

$$\{p_i, q_j\} = \delta_{ij}, \quad \{p_i, p_j\} = \{q_i, q_j\} = 0.$$

Def: The Dirac quantization of  $(M, \mathcal{A})$  is an irreducib.<sup>(11)</sup> Lie-algebra representation

$$\tau: \mathcal{A} \longrightarrow \text{End}^* H$$

of  $\mathcal{A}$  in the Lie algebra of self-adjoint operators in a complex Hilbert space  $H$ , called the quantization space, with Lie bracket given by commutator  $\frac{1}{i}[A, B]$ , satisfying the normalization condition:  $\tau(1) = \text{id}$ .

NB: In the case of inf-dim. algebras of observables  $\mathcal{A}$  it's more natural to look for projective representations, which is equiv. to the construction of representations of suitable central extensions  $\tilde{\mathcal{A}}$  of  $\mathcal{A}$

## 6. Quantization of $V$ and $S$ :

Consider an inf-dim. classical system

$$(V, \mathcal{X})$$

where the phase space  $V = H_0^{1/2}(S^1, \mathbb{R})$  and algebra of observables

$$\mathcal{A} = \text{heis}(V) \rtimes \text{sp}_{HS}(V).$$

**Quantization space** || Fix a complex structure  $J \in \mathcal{Y}_{HS}(V)$ , (12) generating the decomposition

$$V^{\mathbb{C}} = W \oplus \overline{W}$$

into the direct sum of  $(\pm i)$ -eigenspaces of  $J$ .

The role of quantization space  $H$  will be played by the Fock space

$F_J \equiv F(V^{\mathbb{C}}, J) =$  completion of  $S(W)$  w.r. to the norm, generated by  $\langle \cdot, \cdot \rangle_J$

Here,  $S(W)$  is the algebra of symmetric polynomials on  $W$ , which elements are considered as holom. functions on  $\overline{W}$  by identifying  $z \in W$  with holom. function  $\overline{W} \mapsto \langle z, w \rangle_J$  on  $\overline{W}$ .

A Lie-algebra representation of the first component  $heis(V)$  of the algebra  $\mathfrak{A}$  is given by Heisenberg representation

$$\gamma_J : heis(V) \longrightarrow \text{End}^* F_J,$$

$$\gamma_J(v) f(\bar{w}) = -\partial_v f(\bar{w}) + \langle v, w \rangle_J f(\bar{w}), \quad v \in V,$$

where  $\partial_v$  is the derivative in direction of  $v$  and we set for the central element  $c$  of  $heis(V)$ :  $\gamma_J(c) = \lambda \cdot \text{id}$ ,  $\lambda \in \mathbb{R} \setminus 0$ .

Symplectic group action on Fock spaces || In order to construct a Lie-algebra repr. of the second component  $sp_{HS}(V)$  of  $\mathcal{A}$  in  $F_J$ , we consider an  $Sp_{HS}(V)$ -action on Fock spaces,

provided by the following theorem of Shale:

representations  $r_0$  in  $F_0$  and  $r_J$  in  $F_J$  are unitary equivalent  $\Leftrightarrow J \in Sp_{HS}(V)$ . Under this

condition, there exists a unitary intertwining operator  $U_J : F_0 \rightarrow F_J$  such that

$$r_J = U_J \circ r_0 \circ U_J^{-1}.$$

Shale theorem  $\Rightarrow$  there is a projective unitary action of  $Sp_{HS}(V)$  on the Fock bundle

$$\mathcal{F} = \bigcup_{J \in \gamma_{HS}(V)} F_J \longrightarrow \gamma_{HS}(V) = \frac{Sp_{HS}(V)}{U(W_+)}$$

"

$\mathcal{D}_{HS}$

An infinitesimal version of this action yields a projective unitary representation of the Lie algebra  $sp_{HS}(V)$  in the Fock space  $F_0 \Rightarrow$  Dirac quantization of

$$(V, \mathcal{B})$$

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Dirac quantization of  $\mathcal{S}$

Restricting this construction to  $\mathcal{S} \subset \mathcal{Y}_{HS}(V)$ , we obtain the Fock bundle

$$\mathcal{F} = \bigcup_{J \in \mathcal{S}} F_J \longrightarrow \mathcal{S} = \text{Diff}_+(S^1) / \text{M\"ob}(S^1)$$

with a projective unitary action of  $\text{Diff}_+(S^1)$ .

Its infinitesimal version yields a projective unitary representation of the Lie algebra  $\text{Vect}(S^1)$  in  $F_0$ .

This may be considered as the Dirac quantization of the classical system

$$(\mathcal{S}, \text{vir}),$$

where  $\text{vir} = \text{Virasoro algebra}$  is a central extension of  $\text{Vect}(S^1)$ .

## 7. Quantization of $\mathcal{T}$ .

Unfortunately, this quantization method does not apply to the whole of  $\mathcal{T}$ . Though we still have an embedding

$$\mathcal{T} \hookrightarrow \mathcal{Y}(V) = \text{Sp}(V) / \text{U}(W_+),$$

We cannot construct a Fock Bundle over  $\mathcal{V}(V)$  together with a projective action of  $Sp(V)$ . By Shale theorem it is possible only for the subgroup  $SP_{HS}(V)$  of  $Sp(V)$ . We used instead another method, based on Connes quant.

### Dirac quant.

Classical system:  
 $(M, \mathfrak{A})$  where  
 $\mathfrak{A}$  is a Lie algebra  
of observables

Quantization:  
 $r: \mathfrak{A} \rightarrow \text{End}^* H$   
sending  
 $\{f, g\} \mapsto$   
 $\frac{1}{i} [r(f), r(g)]$

### Connes quantization

Classical system:  
 $(M, \mathcal{O})$  where  $\mathcal{O}$  is an associative  
involutive algebra of observables with  
an exterior differential  
 $d: \mathcal{O} \rightarrow \Omega^1(\mathcal{O})$ ,  
given by a linear map, sat. Leibnitz rule

Quantization:  
 $\pi: \mathcal{O} \rightarrow \text{End } H$   
sending  
 $df \mapsto [S, \pi(f)]$ ,  
where  $S$  is the symmetry operator,  
s.t.  $S = S^*$ ,  $S^2 = I$   
In other words, it is a representation  
 $\pi: \text{Der}(\mathcal{O}) \rightarrow \text{End } H$ ,  
where  $\text{End } H$  is considered as a Lie algebra

If the associative algebra  $\mathcal{A}$  consists of smooth functions on the phase space  $M$ , then  $df$  is symplectically dual to the Hamiltonian vector field  $X_f$ . Hamiltonian vector fields form a Lie algebra on  $M$ , and that gives a relation between two approaches in this case. (16)

But Connes' approach provides also a natural method of quantization of the algebras, containing non-smooth functions, in which case Dirac's approach does not work. Consider the following example:

$$\mathcal{A} = L^\infty(S^1)$$

$H = L^2(S^1)$  with symmetry operator, given by Hilbert transform  $S : L^2(S^1) \rightarrow L^2(S^1)$

$\pi : \mathcal{A} \ni f \longmapsto$  multiplication operator

$$M_f : h \in H \longmapsto fh \in H,$$

which is a linear bounded operator in  $H$

The differential of a general  $f \in \mathcal{A}$  is not defined in the classical sense, but its quantum analogue

$$d^q f := [S, M_f]$$

is well-defined for any  $f \in \mathcal{A}$ .

Let us apply Connes approach to  $\mathcal{T}$ . To simplify the formulas, we switch from  $S^1$  to  $\mathbb{R}$ , replacing (17)

$$V = H_0^{1/2}(S^1, \mathbb{R}) \text{ by } V_{|\mathbb{R}} := H^{1/2}(\mathbb{R}, \mathbb{R}) \equiv H^{1/2}(\mathbb{R}).$$

$$Sf(s) = \frac{1}{\pi i} V.P. \int \frac{f(t)dt}{s-t} , \quad f \in H^{1/2}(\mathbb{R}).$$

The quantum differential

$$d^q f(h) = [S, M_f] : H^{1/2}(\mathbb{R}) \rightarrow H^{1/2}(\mathbb{R})$$

is an integral operator

$$d^q f(h) = \frac{1}{\pi i} \int k(s,t) h(t) dt$$

with kernel, given by

$$k(s,t) = \frac{f(s) - f(t)}{s - t} , \quad s, t \in \mathbb{R}.$$

The quasiclassical limit of this operator, defined by taking the value of the kernel on diagonal (i.e. by taking the limit for  $s \rightarrow t$ ), coincides with the multiplication operator:  $h \mapsto f \cdot h$ , so the quantization reduces in this case to the replacement of derivative by its finite-difference analogue.

We have an action of quasisymm. homeo on  $V_{IR}$ .

This action does not admit differentiation, so there is no Lie algebra, associated to  $QS(IR)$ .

The situation is similar to the one, considered in example above. As in this example, we shall define a quantum Lie algebra, associated to  $QS(IR)$ .

We extend first the  $QS(IR)$ -action on  $V_{IR}$  to symmetry operators by

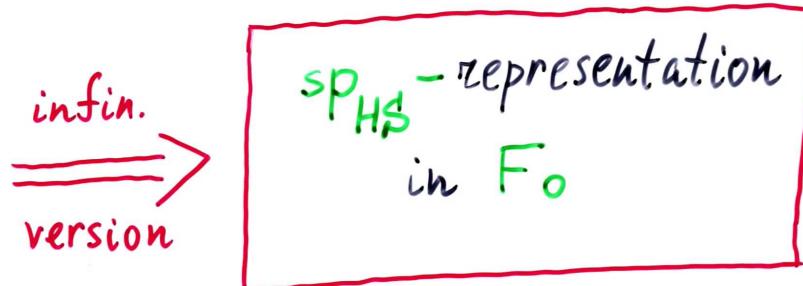
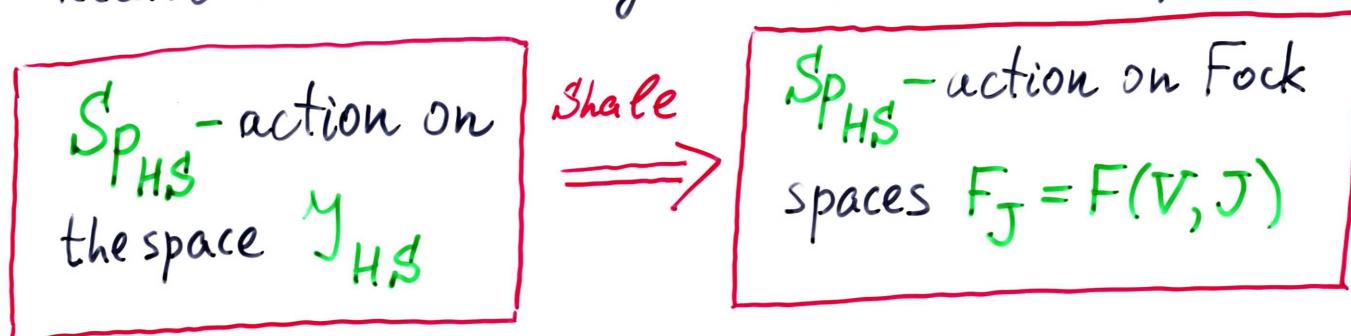
$$QS(IR) \ni h : S \longmapsto S^h := h \circ S \circ h^{-1}.$$

The quantized infinitesimal version of this action is given by the integral operator

$$d^q h : V_{IR} \rightarrow V_{IR},$$

introduced above.

Recall now how the system  $(V, \mathcal{A})$  was quantized.



IN THE CASE OF A WE PAGE:

NO POSITION ON

$F = \sum F_x$  FOR SPACES

NO POSITION ON

$\sum$  SUMMATION OF ALL

POSITIONS IN DIRECTION

NO POSITION ON

FOR SPACE  $F$

POSITIONS ↓ FOR

DIRECTION

NO POSITION ON

FOR POSITION, NO POSITION

$\sum V = V_1 + V_2 + \dots + V_n$

FORWARD  
BACKWARD

TO EXTEND A LINE OF ACTION OF THE OBJECTIVE  
WE NEED TO FIND THE POSITION AND DIRECTION  
OF THE OBJECTIVE ON THE POSITION OF THE  
COMBINED EFFECT OF THE ELEMENTS OF  
THE OBJECTIVE WITH A LINE OF ACTION  
DEFINING THE POSITION OF THE OBJECTIVE  
ON THE PAGE: (2) (2) (2)

POSITION

(2) (2) (2)



THIS OBJECTIVE CAN BE OBTAINED AS A COMBINATION OF THE OBJECTIVE OF POSITION AND DIRECTION.