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THE MOUTARD TRANSFORMATION AND TWO-DIMENSIONAL RATIONAL SOLITONS

I.A.T., S.P. Tsarev (Krasnoyarsk):

- Two-dimensional Schrödinger operators with fast decaying rational potential and multi-dimensional L_2 -kernel. Russian Mathematical Surveys 62:3 (2007), 631-633.
- Blowing up solutions of the Novikov-Veselov equation. Doklady Math. 77 (2008), 467-468.
- Two-dimensional rational solitons and their blow-up via the Moutard transformation. Theoretical and Math. Physics (2008)

The Darboux transformation of H is the swapping of A^\top and A :

$$H = A^\top A \rightarrow \tilde{H} = AA^\top,$$

or in terms of u :

$$u = v^2 + v_x \rightarrow \tilde{u} = v^2 - v_x.$$

if φ satisfies the equation $H\varphi = E\varphi$ with $E = \text{const}$ then $\tilde{\varphi} = A\varphi$ meets $\tilde{H}\tilde{\varphi} = E\tilde{\varphi}$

EXAMPLE: $u = 0, \omega = \omega_1 = x$.

$$v = \frac{1}{x}, \quad v_x = -\frac{1}{x^2}, \quad u_1 = \tilde{u} = \frac{2}{x^2}.$$

By iterations defined by $\omega_n = x^n$ we obtain all rational solitons of the form

$$u_n = \frac{n(n+1)}{x^2}$$

- THE DARBOUX TRANSFORMATION

$$H = -\frac{d^2}{dx^2} + u(x)$$

$$H\omega = 0$$

ω defines a factorization of H :

$$H = A^\top A$$

$$A = -\frac{d}{dx} + v, \quad A^\top = \frac{d}{dx} + v, \quad v = \frac{\omega_x}{\omega}$$

$$\begin{aligned} A^\top A &= \left(\frac{d}{dx} + v \right) \left(-\frac{d}{dx} + v \right) = \\ &= -\frac{d^2}{dx^2} + v^2 + v_x \end{aligned}$$

and

$$v_x + v^2 = u \Leftrightarrow H\omega = 0$$

• THE LAPLACE TRANSFORMATION

$$H = - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + u = -4\partial\bar{\partial} + u$$

$$\begin{aligned} H &= -4(\bar{\partial} + \beta)(-\partial + \alpha) + v \rightarrow \\ \rightarrow \tilde{H} &= 4v(-\partial + \alpha)v^{-1}(\bar{\partial} + \beta) + v \end{aligned}$$

If φ satisfies the equation $H\varphi = 0$ then
 $\tilde{\varphi} = (-\partial + \alpha)\varphi$ meets $\tilde{H}\tilde{\varphi} = 0$

$$\mathcal{H} = -\frac{H}{4} = (\bar{\partial} + B)(\partial + A) + 2V,$$

$$A = -\alpha, B = \beta, V = -8v,$$

where $F = \frac{1}{2}(B_z - A_{\bar{z}})$ is the magnetic field
and V is the electric potential. Then

$$F \rightarrow \tilde{F} = F + \frac{1}{2}\partial\bar{\partial}(\log V),$$

$$V \rightarrow \tilde{V} = V + \tilde{F}$$

•THE MOUTARD TRANSFORMATION

$$H\omega = (-\Delta + u)\omega = 0$$

The Moutard transformation takes the form

$$\begin{aligned}\tilde{H} &= -\Delta + u - 2\Delta \log \omega = \\ &-\Delta - u + 2\frac{\omega_x^2 + \omega_y^2}{\omega^2}.\end{aligned}$$

If φ satisfies the equation $H\varphi = 0$, then the function θ defined from the system

$$\begin{aligned}(\omega\theta)_x &= -\omega^2 \left(\frac{\varphi}{\omega}\right)_y, \\ (\omega\theta)_y &= \omega^2 \left(\frac{\varphi}{\omega}\right)_x\end{aligned}$$

satisfies $\tilde{H}\theta = 0$.

If θ satisfies this system $\theta + \frac{C}{\omega}$ satisfies the system for any constant C .

We shall use the following notation:

$$M_\omega(u) = \tilde{u} = u - 2\Delta \log \omega,$$

$$M_\omega(\varphi) = \{\theta + \frac{C}{\omega}, \ C \in \mathbb{C}\}.$$

The one-dimensional limit:

$$u = u(x), \quad \omega = f(x)e^{\sqrt{c}y},$$

$$H_0 f = \left(-\frac{d^2}{dx^2} + u \right) f = c f$$

and the Moutard transformation reduces to the Darboux transformation of H_0 defined by f :

$$H = H_0 - \frac{\partial^2}{\partial y^2} \quad \rightarrow \quad \tilde{H} = \widetilde{H_0} - \frac{\partial^2}{\partial y^2}.$$

• MAIN CONSTRUCTION

$$H_0 = -\Delta = -\Delta + u_0$$

$$\overset{\omega_1}{H_0} \omega_1 = H_0 \omega_2 = 0.$$

$$\begin{aligned} \overset{\omega_2}{H_1} &= -\Delta + u_1, \quad u_1 = M_{\omega_1}(u_0), \\ \overset{\theta_2}{H_2} &= -\Delta + u_2, \quad u_2 = M_{\omega_2}(u_0). \end{aligned}$$

By the construction, we have

$$H_1 M_{\omega_1}(\omega_2) = 0, \quad H_2 M_{\omega_2}(\omega_1) = 0.$$

Let us choose some function

$$\theta_1 \in M_{\omega_1}(\omega_2)$$

and put

$$\theta_2 = -\frac{\omega_1}{\omega_2} \theta_1 \in M_{\omega_2}(\omega_1).$$

$$u_{12} = M_{\theta_1}(u_1), \quad u_{21} = M_{\theta_2}(u_2).$$

Lemma. 1) $u_{12} = u_{21} = u$, i.e. the diagram

$$\begin{array}{ccc} u_0 & \xrightarrow{\omega_1} & u_1 \\ \omega_2 \downarrow & & \downarrow \theta_1 \\ u_2 & \xrightarrow{\theta_2} & u_{12} = u_{21} \end{array}$$

$\theta_1 \in M_{\omega_1}(\omega_2)$, $\theta_2 = -\frac{\omega_1}{\omega_2}\theta_1 \in M_{\omega_2}(\omega_1)$, is commutative;

2) for $\psi_1 = \frac{1}{\theta_1}$ and $\psi_2 = \frac{1}{\theta_2}$ we have

$H\psi_1 = H\psi_2 = 0$ with $H = -\Delta + u_{12}$.

Theorem 1 Let $u_0 = 0$ and

$$\omega_1 = p_1(z) + \overline{p_1(z)}, \quad \omega_2 = p_2(z) + \overline{p_2(z)},$$

where p_1 and p_2 are holomorphic. Then

$$\begin{aligned} u = u_{12} = & -2\Delta \log i ((p_1 \bar{p}_2 - p_2 \bar{p}_1) + \\ & + \int ((p'_1 p_2 - p_1 p'_2) dz + (\bar{p}_1 \bar{p}'_2 - \bar{p}'_1 \bar{p}_2) d\bar{z}) \end{aligned}$$

EXAMPLE 1. Let $p_1(z) = \left(1 - \frac{i}{4}\right)z^2 + \frac{z}{2}$,
 $p_2(z) = \frac{1}{4}(3 - 5i)z^2 + \frac{1-i}{2}z$. For some appropriate constant C in θ_1 we obtain

$$u = -\frac{5120|1 + (4 - i)z|^2}{(160 + |z|^2|2 + (4 - i)z|^2)^2},$$

$$\psi_1 = (x + 2x^2 + xy - 2y^2)/Q(x, y),$$

$$\psi_2 = (2x + 2y + 3x^2 + 10xy - 3y^2)/Q(x, y),$$

$$\begin{aligned} Q(x, y) = & 160 + 4x^2 + 4y^2 + 16x^3 + 4x^2y + \\ & + 16xy^2 + 4y^3 + 17(x^2 + y^2)^2. \end{aligned}$$

- u, ψ_1 , and ψ_2 are smooth. We have

$$u = O(r^{-6}), \quad \psi_1 = O(r^{-2}), \quad \psi_2 = O(r^{-2})$$

as $r \rightarrow \infty$.

EXAMPLE 2.

$$p_1(z) = (i - 1)z^3 + \left(\frac{1}{10} + \frac{3}{20}i \right) z^2 + \frac{z}{2},$$

$$p_2(z) = 2iz^3 + \left(\frac{1}{4} + \frac{i}{20} \right) z^2 + \frac{1-i}{2}z.$$

Then u , ψ_1 , and ψ_2 are smooth and

$$u = O(r^{-8}), \quad \psi_1 = O(r^{-3}), \quad \psi_2 = O(r^{-3})$$

as $r \rightarrow \infty$.

GUESS. For every $N > 0$ by applying this construction to other harmonic polynomials one can construct smooth rational potentials u and the eigenfunctions ψ_1 and ψ_2 decaying faster than $\frac{1}{r^N}$.

•THE NOVIKOV–VESELOV EQUATION

$$H = \partial\bar{\partial} + U$$

$$U_t = \partial^3 U + \bar{\partial}^3 U + 3\partial(VU) + 3\bar{\partial}(\bar{V}U) = 0,$$
$$\bar{\partial}V = \partial U$$

$$H_t = HA + BH$$

$$H\psi = 0,$$

$$\partial_t\psi = (\partial^3 + \bar{\partial}^3 + 3V\partial + 3\bar{V}\bar{\partial})\psi$$

$$\text{where } \bar{\partial}V = \partial U$$

One-dimensional limit: $U = U(x)$ — the KdV equation

$$\begin{aligned} H\varphi &= 0, \\ \partial_t\varphi &= (\partial^3 + \bar{\partial}^3 + 3V\partial + 3V^*\bar{\partial})\varphi \end{aligned} \quad (1)$$

where $\bar{\partial}V = \partial U$, $\partial V^* = \bar{\partial}U$.

The system (1) is invariant under the extended Moutard transformation

$$\varphi \rightarrow \theta = \frac{i}{\omega} \int (\varphi\partial\omega - \omega\partial\varphi)dz - (\varphi\bar{\partial}\omega - \omega\bar{\partial}\varphi)d\bar{z} +$$

$$\begin{aligned} & [\varphi\partial^3\omega - \omega\partial^3\varphi + \omega\bar{\partial}^3\varphi - \varphi\bar{\partial}^3\omega + \\ & + 2(\partial^2\varphi\partial\omega - \partial\varphi\partial^2\omega) - 2(\bar{\partial}^2\varphi\bar{\partial}\omega - \bar{\partial}\varphi\bar{\partial}^2\omega) + \\ & + 3V(\varphi\partial\omega - \omega\partial\varphi) + 3V^*(\omega\bar{\partial}\varphi - \varphi\bar{\partial}\omega)]dt, \end{aligned}$$

$$U \rightarrow U + 2\partial\bar{\partial}\log\omega,$$

$$V \rightarrow V + 2\partial^2\log\omega, \quad V^* \rightarrow V^* + 2\bar{\partial}^2\log\omega.$$

If ω is a real-valued function, then the latter transformations preserve the property $V^ = \overline{V}$.*

$$\frac{\partial p}{\partial t} = \frac{\partial^3 p}{\partial z^3}$$

Let us apply the extended Moutard transformation and obtain a solution to the NV equation.

By applying the EMT defined by

$$\omega_1 = p_1(z, t) + \overline{p_1(z, t)}, \quad \omega_2 = p_2(z, t) + \overline{p_2(z, t)}$$

to $U = V = 0$ we obtain

$$\begin{aligned} U(z, \bar{z}, t) = & 2\partial\bar{\partial}\log i((p_1\bar{p}_2 - p_2\bar{p}_1) + \\ & + \int ((p'_1 p_2 - p_1 p'_2) dz + (\bar{p}_1 \bar{p}_2' - \bar{p}_1' \bar{p}_2) d\bar{z}) + \\ & + \int (p'''_1 p_2 - p_1 p'''_2 + 2(p'_1 p''_2 - p''_1 p'_2) + \\ & + \bar{p}_1 \bar{p}_2''' - \bar{p}_1''' \bar{p}_2 + 2(\bar{p}_1'' \bar{p}_2' - \bar{p}_1' \bar{p}_2'')) dt), \end{aligned}$$

a solution to the Novikov–Veselov equation.

•REMARK

$$p(z) = \sigma_0 z^N + \sigma_1 z^{N-1} + \cdots + \sigma_{N-1} z + \sigma_N$$

$\dot{\sigma}_k = (N - k + 3)(N - k + 2)(N - k + 1)\sigma_{k-3}$,
where $k = 0, \dots, N$.

The integrable (even linear) evolution of σ induces a dynamical system on $S^n\mathbb{C}$, a motion of roots.

$U(x, t) =$
 $-2 \frac{d^2}{dx^2} [(x - x_1(t))(x - x_2(t))(x - x_3(t))]$
, a solution of the KdV equation:

$$x_k(t) = \varepsilon^k \sqrt[3]{\varepsilon t}, \quad \varepsilon = e^{\frac{2\pi i}{3}}, \quad k = 1, 2, 3.$$

A solution to the σ -system corresponding to $p(z, t) = z^3 + 6t$:

$$z_k(t) = \varepsilon^k \sqrt[3]{6t}, \quad \varepsilon = e^{\frac{2\pi i}{3}}, \quad k = 1, 2, 3.$$

•BLOWING UP SOLUTIONS OF THE NOVIKOV–VESELOV EQUATION

The solution $U(z, \bar{z}, t)$ of the Novikov–Veselov equation obtained from the polynomials

$$p_1 = i z^2, \quad p_2 = z^2 + (1 + i)z$$

has the form $U = \frac{H_1}{H_2}$, with $H_1 = -12 \left(24tx^2 + 12tx + 24ty^2 + 12ty + x^5 - 3x^4y + 2x^4 - 2x^3y^2 - 4x^3y - 2x^2y^3 - 60x^2 - 3xy^4 - 4xy^3 - 30x + y^5 + 2y^4 - 60y^2 - 30y \right)$ and

$$H_2 = (3x^4 + 4x^3 + 6x^2y^2 + 3y^4 + 4y^3 + 30 - 12t)^2.$$

This solution decays at infinity as r^{-3} and obviously blows up for some $t > 0$.

•A REMARK ON PERIODIC SOLITONS

$$(-\Delta - k^2)\omega_1 = (-\Delta - k^2)\omega_2 = 0$$

$$\omega_1 = \sin kx, \quad \omega_2 = \sin(ax+by), \quad a^2+b^2 = k^2.$$

$$\theta_1 = \frac{b}{2 \sin kx} \left(\frac{\cos(ax + by + kx)}{a + k} - \frac{\cos(ax + by - kx)}{a - k} + C \right)$$

$$\tilde{u} = k^2 + 2 \frac{\cos^2 kx}{\sin^2 kx}.$$

Let us iterate the Moutard transformation using θ_1 :

$$\tilde{\tilde{u}} = -\tilde{u} + 2 \frac{(\theta_1)_x^2 + (\theta_1)_y^2}{\theta_1^2} = k^2 - 2\Delta \log(\omega_1 \theta_1)$$

which is nonsingular for C large enough.





