Riemann Surfaces, Harmonic Maps and Visualization

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Fourier-Mukai transforms and spectral data of harmonic tori into compact symmetric spaces

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Contents.

1. An analogue of Fourier-Mukai transforms

Let M be a smooth manifold and D a linear differential operator acting on a vector bundle E over M. Let $\hat{M}$ be the moduli space of flat $\mathbb{C}^*$-bundles over M. For $F \in \hat{M}$, we denote by D_F and E_F the natural differential operator actiong on the vector bundle $E \otimes F$ and the sheaf of sections of the kernel of D_F, respectively.

Ex. $M = T^1 = \mathbb{R}/\mathbb{Z}\Lambda$. $\hat{M} = H^1(M, \mathbb{C}^*) = Hom(\pi_1(M), \mathbb{C}^*) \cong \mathbb{C}^*$.
$D = \frac{\partial}{\partial x}$.
$D_F(e \otimes f) = D(e) \otimes f$, where e and f are local sections of E and F, respectively.

$\mathcal{P}$: the modified Poincare bundle over $\hat{M} \times M$.

In particular, if $F \in \hat{M}$ corresponds to a flat $\mathbb{C}^*$-bundle, then $\mathcal{P}|_{\{F\}\times M}$ is isomorphic to F.

Similarly, if $m \in M$ corresponds to a point m of M, then $\mathcal{P}|_{\hat{M}\times\{m\}}$ is isomorphic to $Alb(m)$, $(Alb\colon M \to \widehat{\hat{M}})$.

For the pair of (E, D), we define a spectral pair $(S, \mathcal{L})$(which we call the Fourier-Mukai transorm of (E, D)). Here, S is a set in $\widehat{M}$ defined by $S = \{F \in \widehat{M} \mid H^0(M, E_F) \neq 0\}$. And $\mathcal{L}$ is a sheaf on S which is the pull-back $\iota^*\mathcal{H}$ of $\mathcal{H}$ by the inclusion map $\iota\colon S \hookrightarrow \widehat{M}$. Moreover we also get a map L_S from M to the category of sheaves on S, which is defined by $m \in M \mapsto L_S(m) = \iota^* Alb(m)$.

Ex. $M = T^1$. $E = \mathbb{C} \times M$. $D = \partial/\partial x$.
$S = \{F \in \widehat{M} \cong \mathbb{C}^* \mid H^0(M, E \otimes F) \neq 0\}$
$= \{F = 1(\textit{ the trivial bundle over } T^1)\}$.
$\mathcal{L}_1 = \mathbb{C}$.

2. Construction of spectral data of harmonic tori into compact symmetric spaces

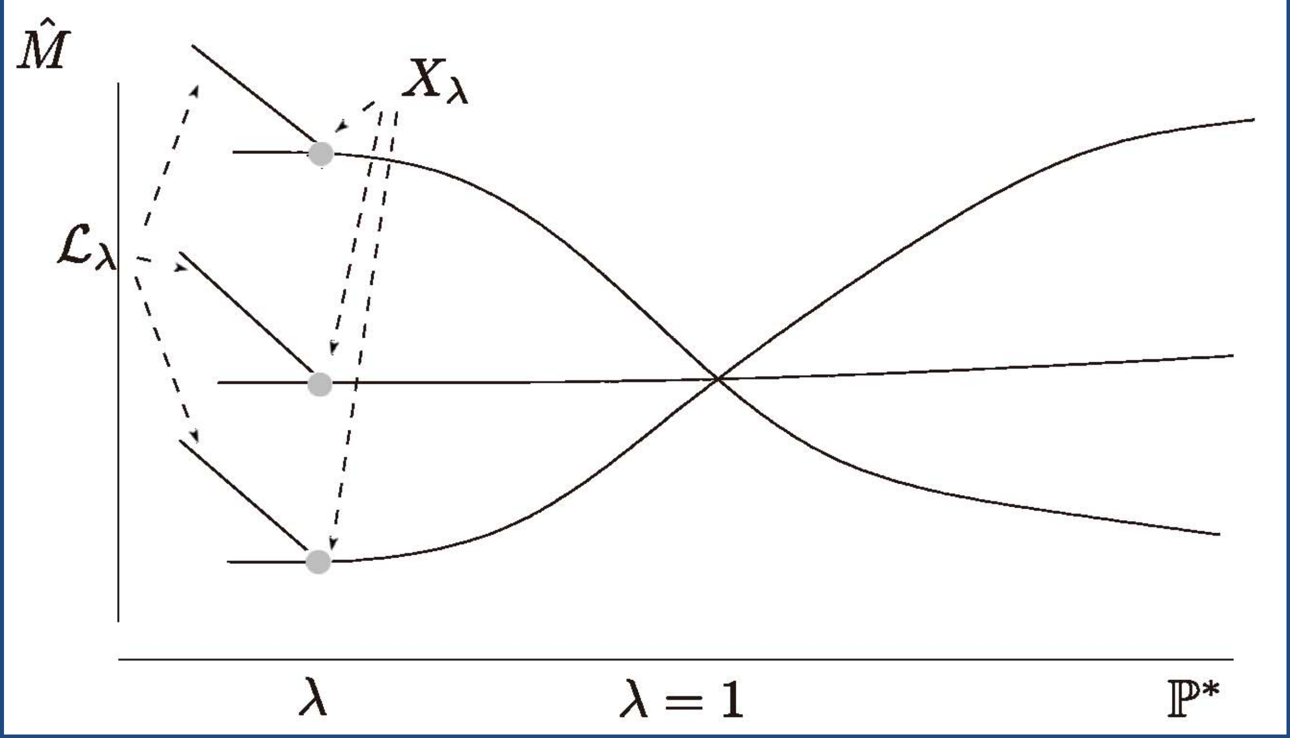
Let $T = \mathbb{C}/(\mathbb{Z}\oplus\mathbb{Z}\tau)$ be a 1-dimensional complex torus. We regard T as the 2-dimensional real torus with the conformal structure induced by τ. Let ϕ be a harmonic map from T to a compact symmetric space G/K. Let $\Phi_\lambda : \widetilde{T} \to G^{\mathbb{C}}$ be an extended frame of ϕ ($\lambda \in \mathbb{P}^* = \mathbb{C}\backslash\{0\}$) where $\widetilde{T}$ is the universal cover $\mathbb{C} \to T$. We denote the Maurer-Cartan form of Φ_λ by $\alpha_\lambda = \Phi_\lambda^{-1} d\Phi_\lambda$.

$$\phi^{-1}d\phi = \alpha_{\mathfrak{p}}' + \alpha_{\mathfrak{k}} + \alpha_{\mathfrak{p}}''.$$

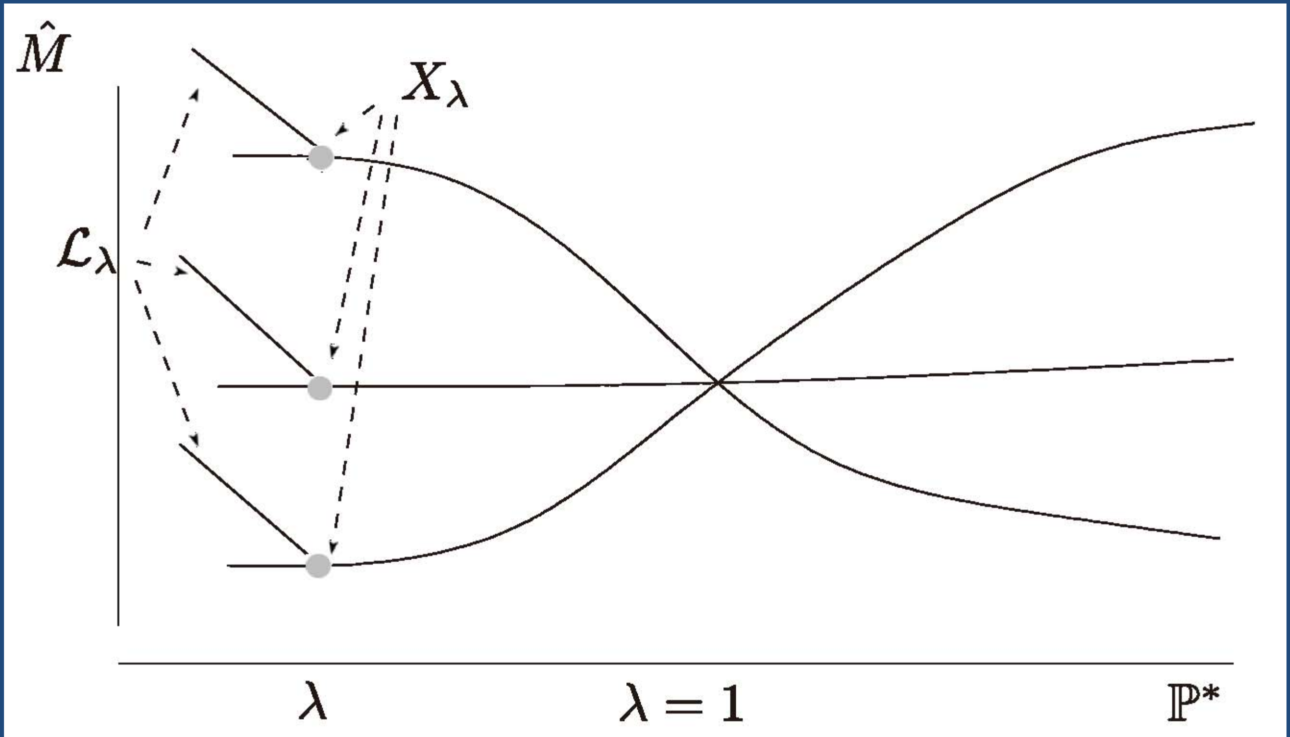
$$\alpha_\lambda = \Phi_\lambda^{-1}d\Phi_\lambda = \lambda^{-1}\alpha_{\mathfrak{p}}' + \alpha_{\mathfrak{k}} + \lambda\alpha_{\mathfrak{p}}''.$$

Let us consider the family $\mathcal{E}$ of vector bundles $E(\lambda)$ on T associated to the representation of fundamental group of T induced by α_λ ($\lambda \in \mathbb{C}^*$).

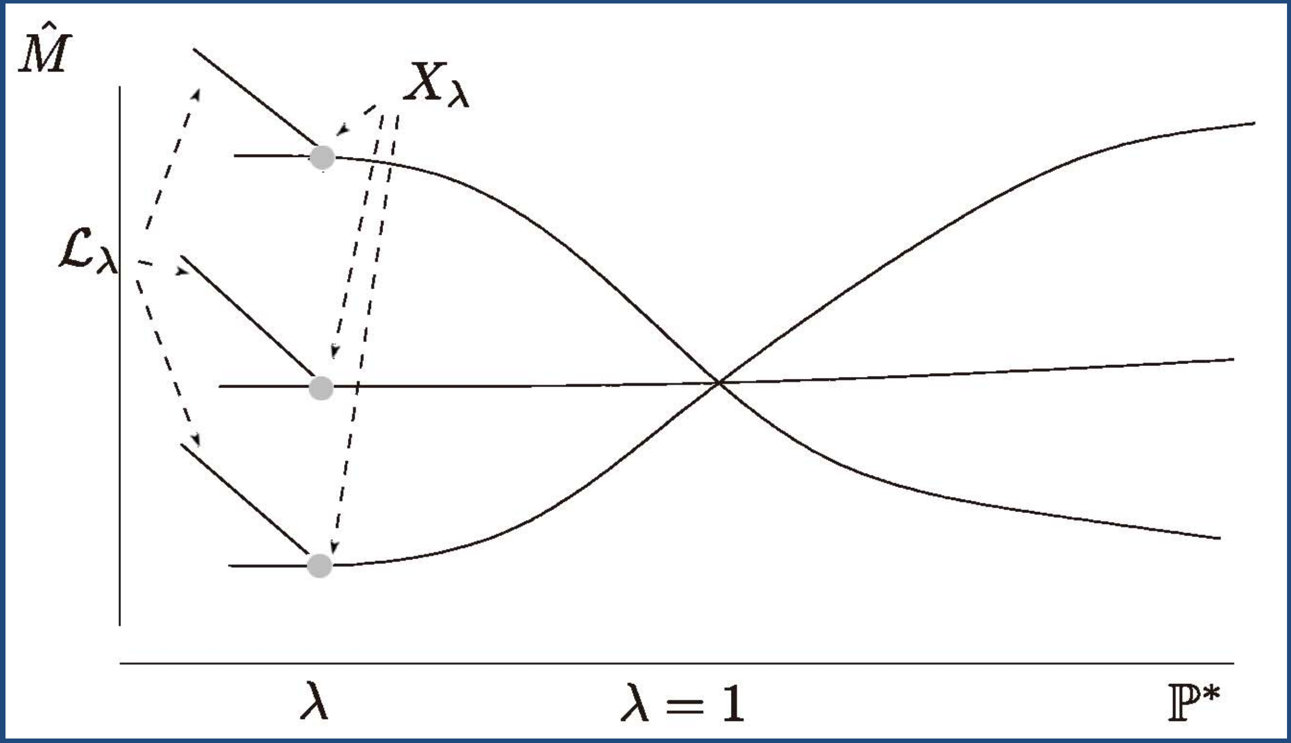
For each $\lambda \in \mathbb{P}^*$, by setting $M = T$ and applying the above analouge of Fourier-Mukai transform, we shall construct a spectral pair $(X_\lambda, \mathcal{L}_\lambda)$ which correponds to the pair of $(E(\lambda), D = d + \alpha_\lambda)$.



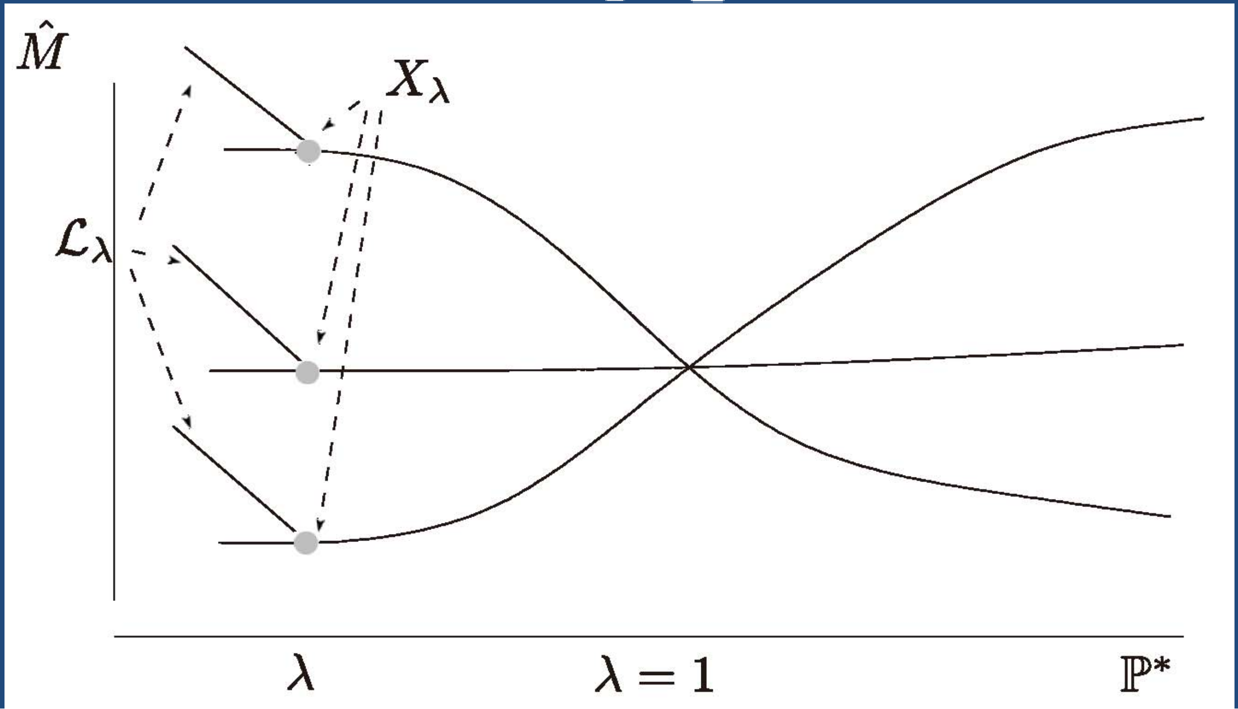
Let us define the spectral data $(X, \pi, \mathcal{L})$ as fol-lowos. We define X and $\mathcal{L}$ as the collections of $\{X_\lambda\}$ and $\{\mathcal{L}_\lambda\}$, respectively. More precisely, X is given by $X = \{(\lambda, F) \mid \lambda \in \mathbb{P}^*, F \in X_\lambda\}$ which is a complex curve in the product $\mathbb{P}^* \times \hat{T}$.



And π is a map from X to $\mathbb{P}^*$ induced by the first projection $pr_1\colon \mathbb{P}^* \times \hat{T} \to \mathbb{P}^*$. And $\mathcal{L}$ is a sheaf on X, where its fiber $\mathcal{L}_p$ at $p = (\lambda, F) \in X$ is given by $\mathcal{L}_p = H^0(T, E(\lambda)_F)$.

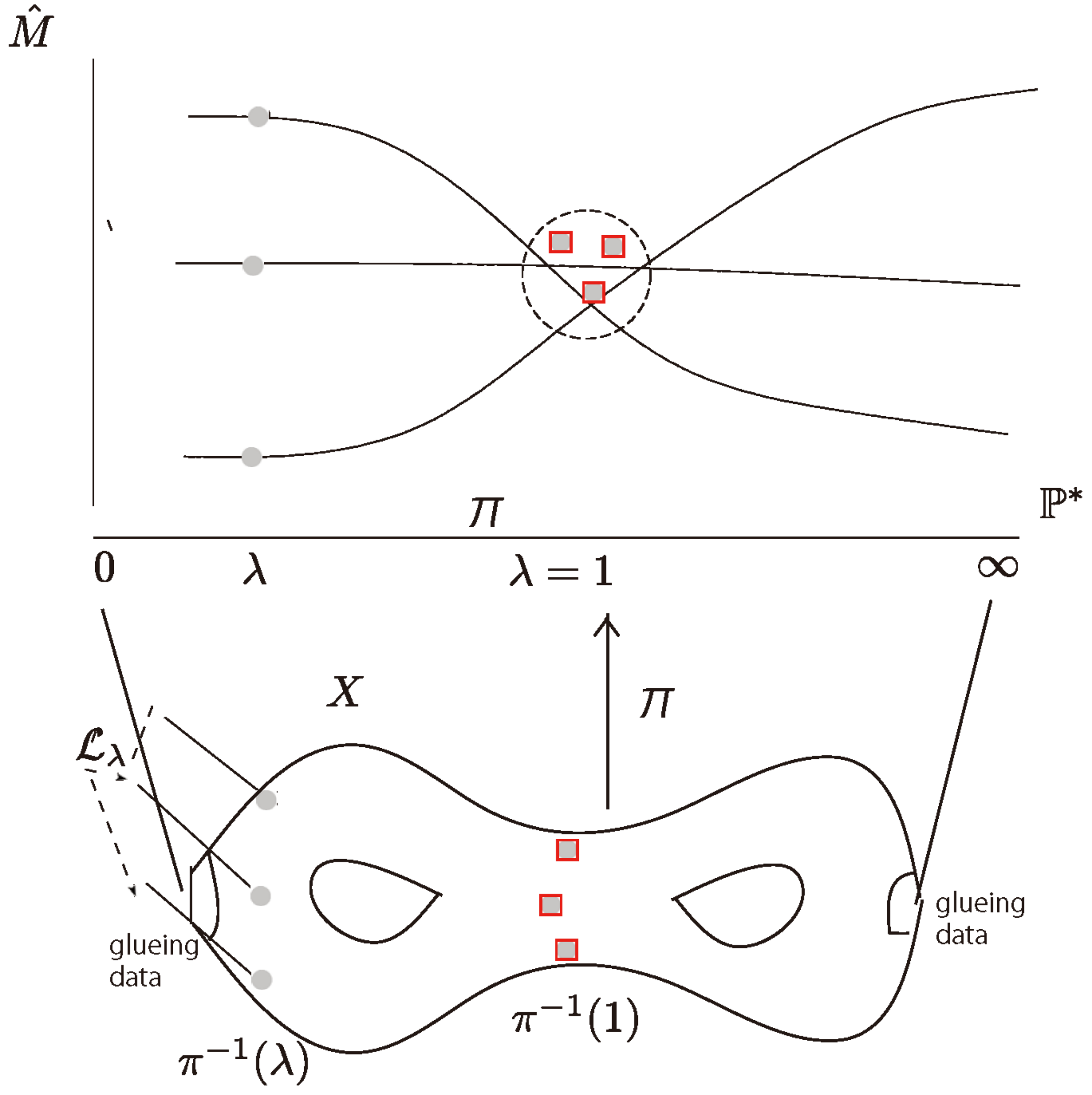


Moreover, we also get a map L_X from $M = T$ to the category of sheaves on X defined by $t \in M \mapsto L_X(t)$, where $L_X(t)$ is the pull-back $\psi^* Alb(t)$ of $Alb(t) \in \widehat{M} = H^1(\hat{M}, \mathbb{C}^*)$ by the composition $\psi = pr_2 \circ \iota_X \colon X \to \hat{T}$, where ι_X is the inclusion map $\iota_X \colon X \hookrightarrow \mathbb{P}^* \times \hat{M}$ and pr_2 is the second projection $pr_2 \colon \mathbb{P}^* \times \hat{M} \to \hat{M}$.



In other words, L_X is considered as a double periodic flow on $Pic(X)$, by using the natural map $i\colon H^1(\hat{T}, \mathbb{C}^*) \to Pic^0(\hat{T})(\subset Pic(\hat{T}) = H^1(\hat{T}, \mathcal{O}^*_{\hat{T}}))$ which sends a flat bundle to the bundle with the same algebraic transition matrices.

3. Converse constructions



$\hat{M}$
π
$\mathbb{P}^*$
0
λ
$\lambda = 1$
∞
X
π
$\mathcal{L}_\lambda$
glueing data
glueing data
$\pi^{-1}(1)$
$\pi^{-1}(\lambda)$

Ex.

X : a Riemann surface

$\pi: X \to CP^1$: a holomorphic map $n+1:1$

$\mathcal{L}$: a line bundle of degree genus$+n$

$(X, \pi, \mathcal{L})$ $\Rightarrow$ $\psi: \mathbb{R}^2 \to \mathbb{C}P^n$: a harmonic map

$X = \mathbb{C}P^1$,

$\pi: X \to \mathbb{C}P^1$ $\zeta \mapsto \lambda = \zeta^4$,

$\mathcal{L}$: a line bundle of degree 3

$\Rightarrow$ $\psi = [\psi_1 : \psi_2 : \psi_3 : \psi_4]$

$$\psi_i = \exp(\eta_i^{-1} z/2 - \eta_i \bar{z}/2)/\sqrt{2}$$

$$\eta_1 = 1, \eta_2 = i, \eta_3 = -1, \eta_4 = -i.$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{-1}{\sqrt{2}i} & 0 & \frac{1}{\sqrt{2}i} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}i} & 0 & \frac{-1}{\sqrt{2}i} & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

The Clifford torus in S^3

$$x_1 = \frac{\cos x}{\sqrt{2}}, x_2 = \frac{\sin x}{\sqrt{2}}, x_3 = \frac{\cos y}{\sqrt{2}}, x_4 = \frac{\sin y}{\sqrt{2}},$$

where $0 \leqq x, y \leqq 2\pi$.

Setting $M = T^2$ and $D =$ a Dirac operator with a potential, Iskander A. Taimanov compute the spectral curve of the Clifford torus. $X = CP^1$. $\infty_+ = (\zeta = \infty)$, $\infty_- = (\zeta = 0)$ with double points obtained by stacking together the points from the following pairs:

$$\left(\frac{1+i}{4}, \frac{-1+i}{4}\right), \left(-\frac{1+i}{4}, \frac{1-i}{4}\right).$$

See “Finite-Gap Theory of the Clliford Torus” International Mathematics Reserch Notices 2005, No.2.

Ref. Fourier-Mukai Transforms in Algebraic Geometry (Oxford Mathematical Monographs)

D. Huybrechts

Thank you very mcuh for kind attention.