**Surfaces with singularities and Osserman-type inequalities.**

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## Motivation.

This a development of my talk on minimal surfaces in $\mathbf{R}^3$ at 26th ENCOUNTER with MATHEMATICS 2003, March 14 (Chyuo Univ.).

$\mathcal{C}$: a class of immersed surfaces in a space form

$\exists_\infty$ Non-trivial examples $\Longrightarrow$ OK

$\exists$ Only a few examples

$\Longrightarrow$ Consider a new class $\tilde{\mathcal{C}} \supset \mathcal{C}$.

$\exists$Osserma-type ineqality $\Longrightarrow \tilde{\mathcal{C}}$ is OK.

More precisely, there are several classes of surfaces which do not admit only a few complete immersed surfaces:

- **Flat surfaces in $H^3$,**
- **Space-like maximal surfaces in $R^3_1$,**
- **space-like CMC-1 surfaces in $S^3_1$,**
- **Flat surfaces in $\mathbf{R}^3$.**

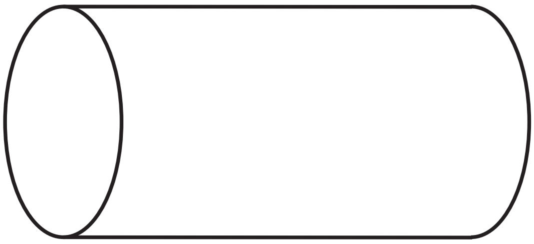
What is the best restrictions of singularities for global study of these such objects?

(Philosophy) A good classes of surfaces (which may have singularities) should satisfy certain kind of Osserman-type inequalities.

**(Cohn-Vossen inequality)**
$(\Sigma^2, ds^2)$ : a complete Riemannian 2-manifold with finite total curvature

$$\frac{1}{2\pi}\int_{\Sigma^2} K dA \leq \chi(\Sigma^2).$$

Example (A cylinder) $\chi(\Sigma^2) = 0$ and $K \equiv 0$.



(A refinement by R. Finn 65 and K. Shiohama 85)

$$\frac{1}{2\pi}\int_{\Sigma^2} K dA = \chi(\Sigma^2) - i(\Sigma^2),$$

where

$$i(\Sigma^2) := \lim_{r\to\infty} \frac{\mathrm{Area}(B_p(r))}{\pi r^2}.$$

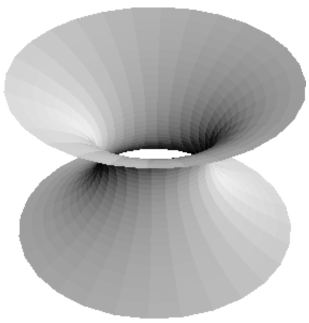
## Osserman inequality

Let $f : \Sigma^2 \to \mathbf{R}^3$ be an immersed minimal surfaces with finite total curvature.

- (Huber 57) $\exists$ a closed Riemann surface $\bar{\Sigma}^2$ such that
  $\Sigma^2 \approx \bar{\Sigma}^2 \setminus \{p_1, ..., p_n\}$ (bi-holomorphic).
- (Osserman 64) $\dfrac{1}{2\pi}\displaystyle\int_{\Sigma^2}(-K)dA \geq -\chi(\Sigma^2) + n$
- (Jorge-Meeks 83) The equality holds iff each end is properly embedded.

If a minimal surface admits branch points, Osserman inequality does not hold in general.

$$(\text{The degree of } G : \bar{\Sigma}^2 \to S^2) = -\frac{1}{4\pi}\int_{\Sigma^2} K dA$$



# Applications of inequality

**(Fact 1.)**(Osserman 64)
If the Gauss map of a complete minimal (immersed) surface of finite total curvature omits more than three directions, it is a plane.

**(Fact 2.)**(Fujimoto 88)
If the Gauss map of a complete minimal (immersed) surface omits more than four directions, it is a plane.

**(Fact 3.)**(Kawakami-R.Kobayashi-Miyaoka)
The Gauss map of a complete minimal (immersed) surface of finite total curvature satisfies

$$D_G \leq \nu_G \leq 2 + \frac{2}{R} < 4,$$

where $D_G$ is the number of exceptional values, and

$$\nu_G := D_G + \sum_{a \in \mathrm{Im}(G)} \left(1 - \frac{1}{1 + \max_{p \in G^{-1}(a)} \mathrm{ord}_p(dG)}\right),$$

$$\frac{1}{R} := \frac{\mathrm{genus} - 1 + \#(\mathrm{ends})/2}{\deg(G)},$$

which are geometric invariants based on Nevanlinna theory.

## The Chern-Osserman inequality

Let $f : \Sigma^2 \to \mathbf{R}^N$ be an orientable complete minimal immersion with finite total curvature. Then it holds that

$$\frac{1}{2\pi}\int_{\Sigma^2}(-K)dA \geq -\chi(\Sigma^2) + \#(\text{ends}).$$

(Kokubu-Yamada-U. 2002) The equality holds iff each end is properly embedded and asymptotic to a plane or a catenoid.

Here, a map $f : \{0 < |z| < 1\} \to \mathbf{R}^N$ is asymptotic to another map $f_0 : \{0 < |z| < 1\} \to \mathbf{R}^3 \subset \mathbf{R}^N$ if

$$\lim_{z \to 0} \frac{f(z) - f_0(z)}{z} = 0.$$

**Example** $f(z) = (z, \dfrac{1}{z^2}) : \mathbf{C} \setminus \{0\} \to \mathbf{C}^2 = \mathbf{R}^4$ has embedded ends but total curvature $-6\pi$, which does not attain the equality.

## CMC-1 surfaces in $H^3(-1)$

$$\mathbf{R}^4_1 \ni (t,x,y,z) \Leftrightarrow \begin{pmatrix} t+z & x+iy \\ x-iy & t-z \end{pmatrix} \in Herm(2),$$

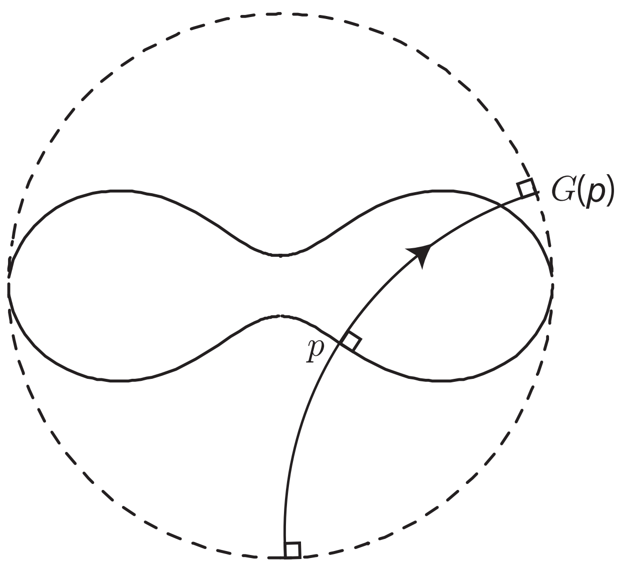
$$\begin{aligned} H^3 &:= \{ (t,x,y,z) \in \mathbf{R}^4_1 \,;\, x^2+y^2+z^2-t^2 = -1 \} \\ &= \{ X \in \mathrm{Herm}(2)\,;\, \mathrm{trace}(X) > 0,\ \det(X) = 1 \} \end{aligned}$$

$\exists_\infty$ complete CMC-1 surfaces in $H^3$.

$\exists$ A Weierstrass-type formula (by Bryant 87).

| Minimal surfaces in $\mathbf{R}^3$ | CMC-1 surfaces in $H^3$ |
|---|---|
| $(1+\lvert G\rvert^2)\lvert\omega\rvert > 0$ | $(1+\lvert G\rvert^2)\lvert\omega\rvert > 0$ |
| $F : \Sigma^2 \to \mathbf{C}^3$ | $F : \Sigma^2 \to SL(2,\mathbf{C})$ |
| $dF = ((1-G^2), i(1+G^2), 2G)\omega/2$ | $dF F^{-1} = \begin{pmatrix} G & -G^2 \\ 1 & -G \end{pmatrix} \omega$ |
| $f := F + \overline{F} : \Sigma^2 \to \mathbf{R}^3$ | $f := F\,{}^t\overline{F} : \Sigma^2 \to H^3$ |

(hyperbolic Gauss map)

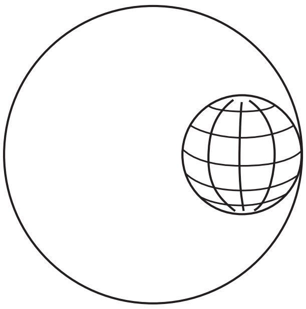


## Osserman-type inequality for CMC-1 surfaces

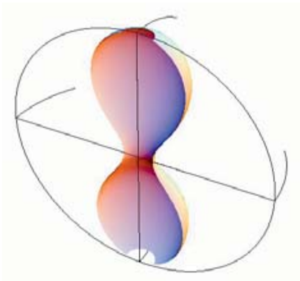
**(Fact 1.)** (Yamada-U. 93)
The equality never holds on Cohn-Vossen ineqality for complete CMC-1 immersions of finite total curvature.

$$\frac{1}{2\pi}\int_{\Sigma^2} K dA < \chi(\Sigma^2).$$

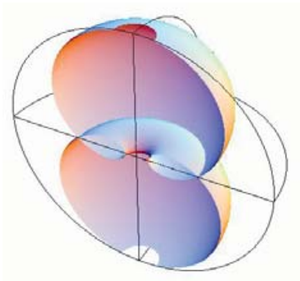
The total curvature tends to 0 for a catenoid cousin.



(a) horosphere



(b) catenoid cousin 1



(c) catenoid cousin 2

**(Fact 2.)** (Yamada-U. 97) The following inequality holds for complete CMC-1 immersions of finite total curvature in $H^3$.

$$2\deg(G) \geq -\chi(\Sigma^2) + \#(\text{ends}).$$

'$=$' holds iff each end is properly embedded.

## Space-like maximal surfaces in $(\mathbf{R}^3_1, ++-)$

$H \equiv 0 \iff$ Maximal surfaces

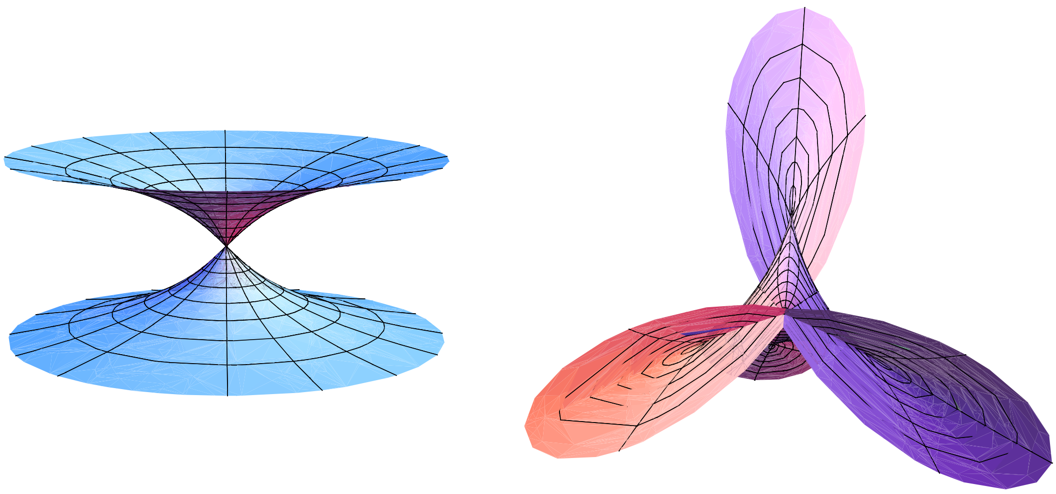
**(Fact 1.)** complete immersed space-like maximal surfaces in $\mathbf{R}^3_1$ is a plane.

**(Fact 2.)** (Osamu Kobayashi 83)

$\exists$ Weierstrass-type representation formula

| | |
|---|---|
| Minimal surfaces in $\mathbf{R}^3$ | Maximal surfaces in $\mathbf{R}^3_1$ |
| $(1+\lvert G\rvert^2)\lvert\omega\rvert > 0$ | $(1-\lvert G\rvert^2)\lvert\omega\rvert > 0$ |
| $F : \Sigma^2 \to \mathbf{C}^3$ | $F : \Sigma^2 \to \mathbf{C}^3$ |
| $dF = ((1-G^2), i(1+G^2), 2G)\omega/2$ | $dF = ((1-G^2), i(1+G^2), 2iG)\omega/2$ |
| $f := F + \overline{F} : \Sigma^2 \to \mathbf{R}^3$ | $f := F + \overline{F} : \Sigma^2 \to \mathbf{R}^3_1$ |

Figure 1. Representation formula for maximal surfaces



Lorentzian catenoid and Lorentzian Enneper surfaces

A conformal maximal immersion $f : \Sigma^2 \to \mathbf{C}^3$ induces a holomorphic immersion $F = (F^1, F^2, F^3) : \tilde{\Sigma}^2 \to \mathbf{C}^3$ such that

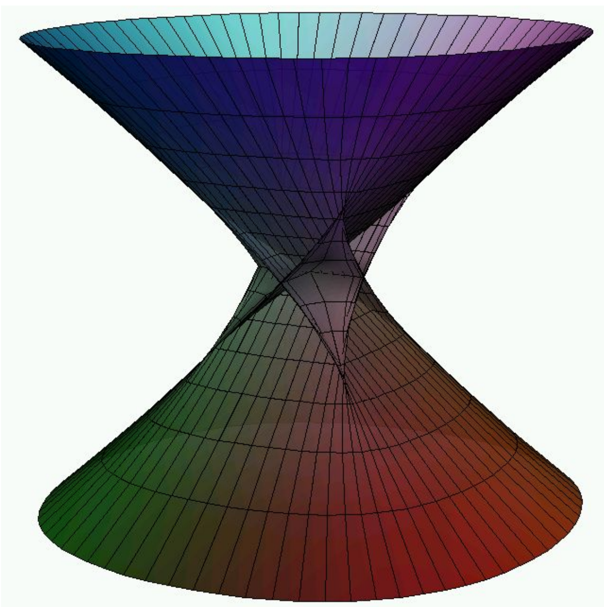
$$f = F + \bar{F}, \quad F^1_z \cdot F^1_z + F^2_z \cdot F^2_z - F^3_z \cdot F^3_z = 0.$$

Such a map $F$ is called a *Lorentzian null immersion*.

Maxface = the projection of a Lorentzian null immersion

**(Completeness)** A maxface $f : \Sigma^2 \to \mathbf{R}^3_1$ is called *complete* if $\exists T$ (a symmetric tensor) such that

- supp$(T)$ is compact,
- $T + ds^2$ is a complete Riemannian metric on $\Sigma^2$.



Kim-Yang maximal torus (by Fujimori)

## Osserman-type inequality for maxfaces

**Fact 3.** (Yamada-Umehara 03) Let $f : \Sigma^2 \to \mathbf{R}^3_1$ be a complete maxface, then

- $\Sigma^2 \approx \bar{\Sigma}^2 \setminus \{p_1, ..., p_n\}$ (bi-holomorphic).
- The Gauss map

$$G : \Sigma^2 \to \overline{H^2_+ \cup H^2_-} = S^2 = \mathbf{C} \cup \{\infty\}$$

  is a meromorphic function on $\bar{\Sigma}^2$.
- An Osserman-type inequality holds:

$$2\deg(G) \geq -\chi(\Sigma^2) + \#(\text{ends}).$$

  '$=$' holds iff all ends are properly embedded.

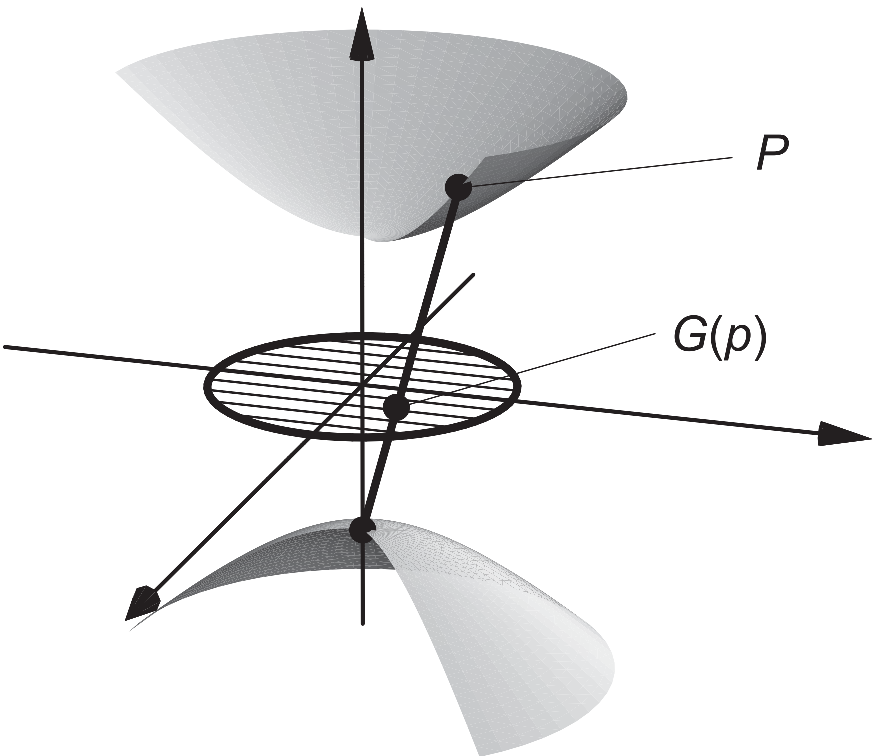
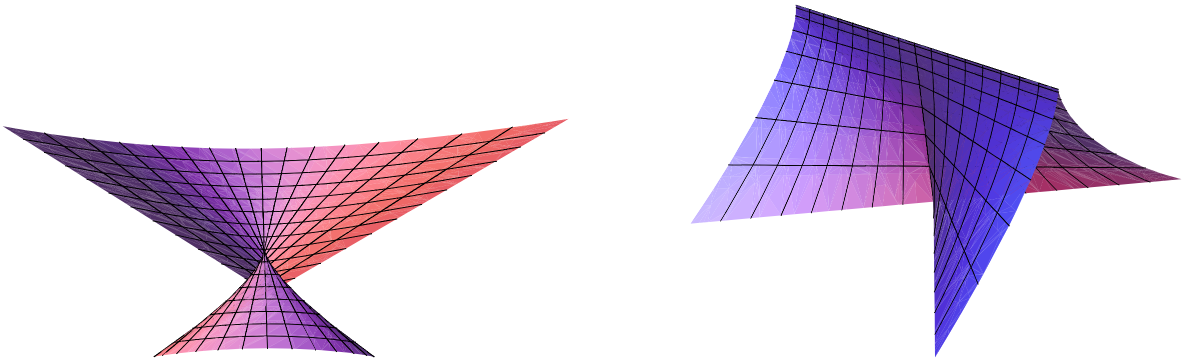



FIGURE 2. The streo-graphic projection

## The duality between swallowtails and cuspidal cross caps

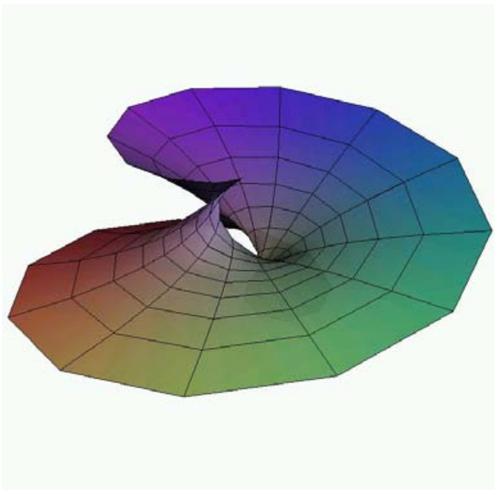


swallowtail and cuspidal cross-cap

**Fact 4.**(Fujimori-Saji-Yamada-U. 07)
Let $f = \mathrm{Re}(F) : \Sigma^2 \to \mathbf{R}^3_1$ be a maxface,
and $f^{\perp} := \mathrm{Im}(F)$ its conjugate surface. Then

a swallowtail of $f \iff$ a cuspidal cross-cap of $f^{\perp}$.

**Fact 5.**(Fujimori-Lopez 08)
$\exists$ non-orientable complete maxfaces with one end of genus zero or genus one.



non-orientable maxface (by Fujimori)

**Spacelike CMC-1 surfaces in $S^3_1$**

$S^3_1 := \mathrm{SL}(2,\mathbf{C})/\mathrm{SU}(1,1)$: de Sitter space-time

$$\mathbf{R}^4_1 \ni (t,x,y,z) \Leftrightarrow \begin{pmatrix} t+z & x_1+iy \\ x-iy & t-z \end{pmatrix} \in Herm(2),$$

$$\begin{aligned} S^3_1 &= \{(t,x,y,z)\in \mathbf{R}^4_1\,;\, -t^2+x^2+y^2+z^2=1\}, \\ &= \{ae_3{}^t\bar{a} \in Herm(2)\,;\, a\in \mathrm{SL}(2,\mathbf{C})\}, \end{aligned}$$

where $e_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

**(Fact 1.)** A complete space-like CMC-1 immersion in $S^3_1$ is totally umbilical.

**(Fact 2.)** (Aiyama-Akutagawa 98)

$\exists$ Weierstrass-type representation formula

| CMC-1 surfaces in $S^3_1$ | CMC-1 surfaces in $H^3$ |
| --- | --- |
| $(1+\lvert G\rvert^2)\lvert\omega\rvert>0$ | $(1+\lvert G\rvert^2)\lvert\omega\rvert>0$ |
| $F:\Sigma^2\to SL(2,\mathbf{C})$ | $F:\Sigma^2\to SL(2,\mathbf{C})$ |
| $dFF^{-1}=\begin{pmatrix} G & -G^2 \\ 1 & -G\end{pmatrix}\omega$ | $dFF^{-1}=\begin{pmatrix} G & -G^2 \\ 1 & -G\end{pmatrix}\omega$ |
| $f:=Fe_3{}^t\overline{F}:\tilde{\Sigma}^2\to S^3_1$ | $f:=F{}^t\overline{F}:\tilde{\Sigma}^2\to H^3$ |

FIGURE 3. Representation formulas

**space-like CMC-1 surface with singularities.**

A conformal CMC-1 immersion $f : \Sigma^2 \to S^3_1$ induces a holomorphic immersion $F : \tilde{\Sigma}^2 \to \mathrm{SL}(2, \mathbf{C})$ such that

$$\det(F_z) = 0.$$

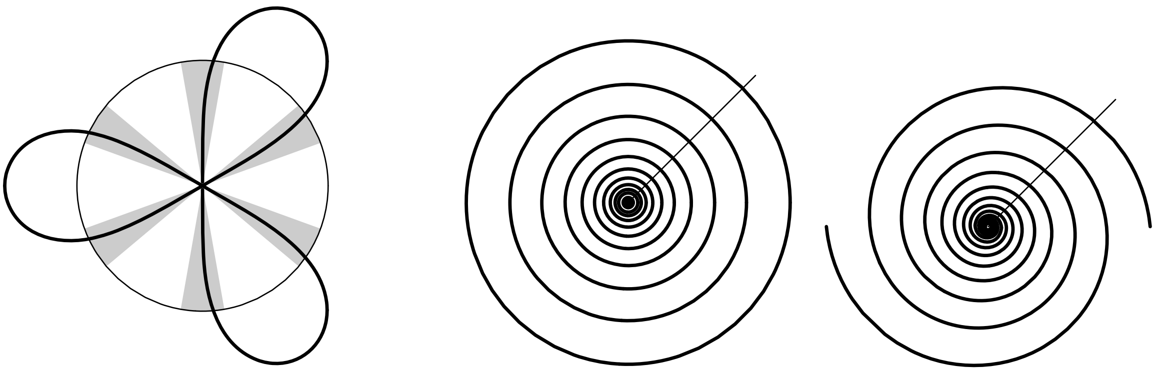
$F$ is called the *null holomorphic immersion*.

CMC-1 faces $\Longleftrightarrow$ Projection of null immersions

**We can define the completeness like as the case of maxfaces.**

$\exists$ three type of complete ends according to the monodromy matrix of the lift $F$ of $f$.

- **elliptic ends** (accumulate to one of $\partial_\pm S^3_1$)
- **parabolic ends** (accumulate to one of $\partial_\pm S^3_1$)
- **hyperbolic ends** (accumulate both of $\partial_\pm S^3_1$)



Singular sets of an incomplete elliptic end and of hyperbolic ends

## Osserman-type inequality for CMC-1 faces

**Fact 3.** (Fujimori-Rossman-Yamada-S.D.Yang-U.) Let $f : \Sigma^2 \to S^3_1$ be a complete CMC-1 face, then

- $\Sigma^2 \approx \bar{\Sigma}^2 \setminus \{p_1, ..., p_n\}$ (bi-holomorphic),
- All ends are elliptic or parabolic

An end $p_j$ is called *regular* if it is at most pole of the Gauss map

$$G : \Sigma^2 \to \mathbf{C} \cup \{\infty\}.$$

**Fact 4.** (Fujimori-Rossman-Yamada-S.D.Yang-U.) Let $f : \Sigma^2 \to S^3_1$ be a complete CMC-1 face, then the following Osserman-type inequality holds

$$2\deg(G) \geq -\chi(\Sigma^2) + \#(\text{ends}).$$

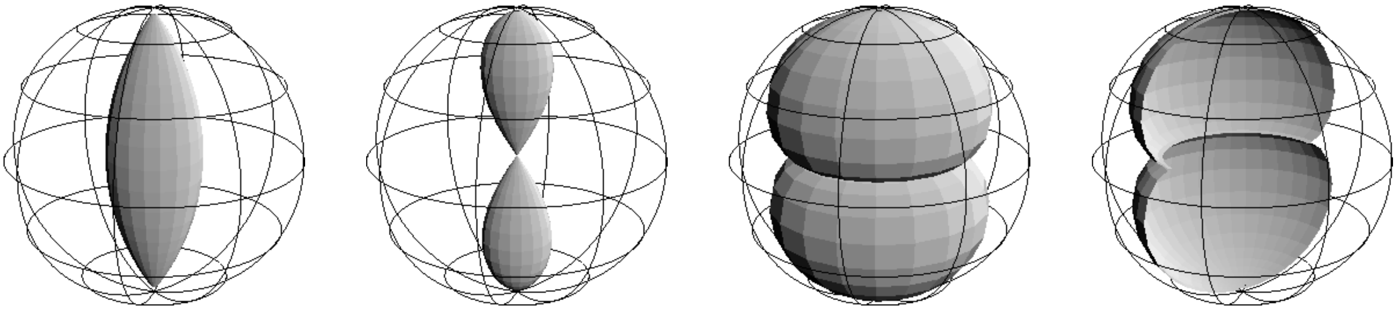
'$=$' holds if and only if all ends are regular and properly embedded.

## **Flat surfaces in $H^3$**

**(Fact 1.)** (Volkov& Vladimirova 71, S.Sasaki 73)
A complete flat (immersed) surface in $H^3$ is a horosphere or a parallel surface of a geodesic.

**(Fact 2.)** Parallel surfaces of flat surfaces are also flat.

**(Fact 3.)** $\exists$ Weierstrass-type representation formula
(The complex structure is induced from the second fundamental form.)
[Galvez-Martinez-Milan 00]



rotationally symmetric examples

## Wave fronts

Let $M^3(c)$ be a space form. A $C^\infty$-map

$$f : \Sigma^2 \to M^3(c)$$

is called a *frontal map* if there exists a $C^\infty$-map

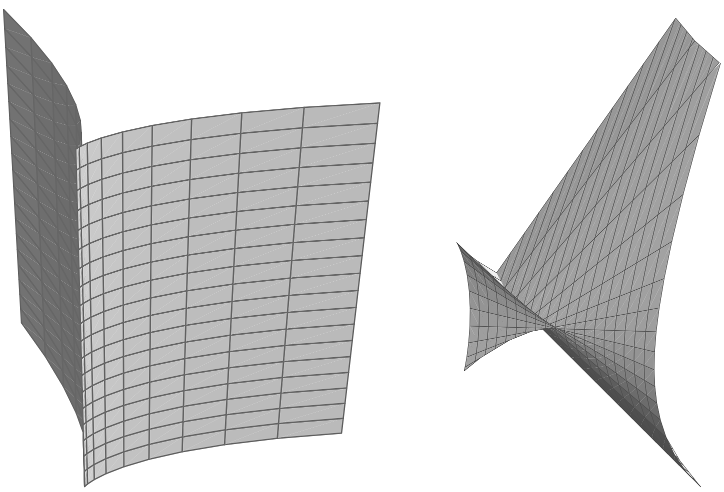
$$L : \Sigma^2 \to T_1 M^3(c)$$

such that $L$ is the unit normal vector field along $f$. Moreover, if $L$ is an immersion, $f$ is called a *wave front* (or a *front*).

(Examples of frontals)

- Maxfaces in $\mathbf{R}^3_1$,
- CMC-1 faces in $S^3_1$.

A front $f$ is called *flat* if the regular set $R$ of $f$ is open dense and $f|_R$ has zero Gaussian curvature.



**(Global Results)**

**(Fact 4.)** (Kokubu-Rossman-Yamada-U. 07)
Flat fronts are all orientable.

**Completeness:** A flat front $f : \Sigma^2 \to H^3$ is called complete if $\exists T$ such that

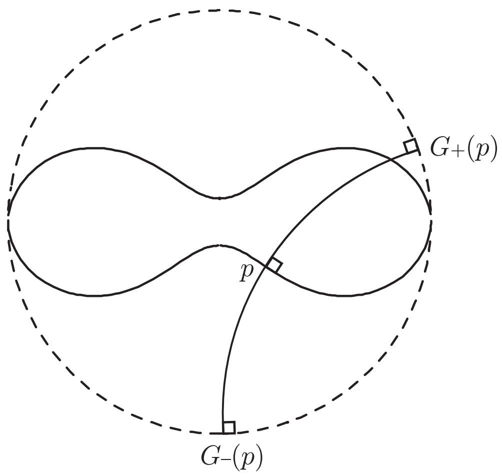
- supp$(T)$ is compact and,
- $ds^2 + T$ : a complete Riemannian metric on $\Sigma^2$.

**(Fact 5.)** (Kokubu-Rossman-Yamada-U.07) A flat front $f : \Sigma^2 \to H^3$ is complete iff

- The singlar set is comapct.
- The metric $|df|^2 + |d\nu|^2$ is complete and of finite total curvature.

**(Fact 6.)** (Kokubu-Yamada-U. 04) Let $f : \Sigma^2 \to H^3$ be a complete flat front. Then

- $\Sigma^2 \approx \bar{\Sigma}^2 \setminus \{p_1, ..., p_n\}$ (bi-holomorphic),
- the hyperbolic Gauss maps $(G_+, G_-)$ are meromorphic functions on $\Sigma^2$.



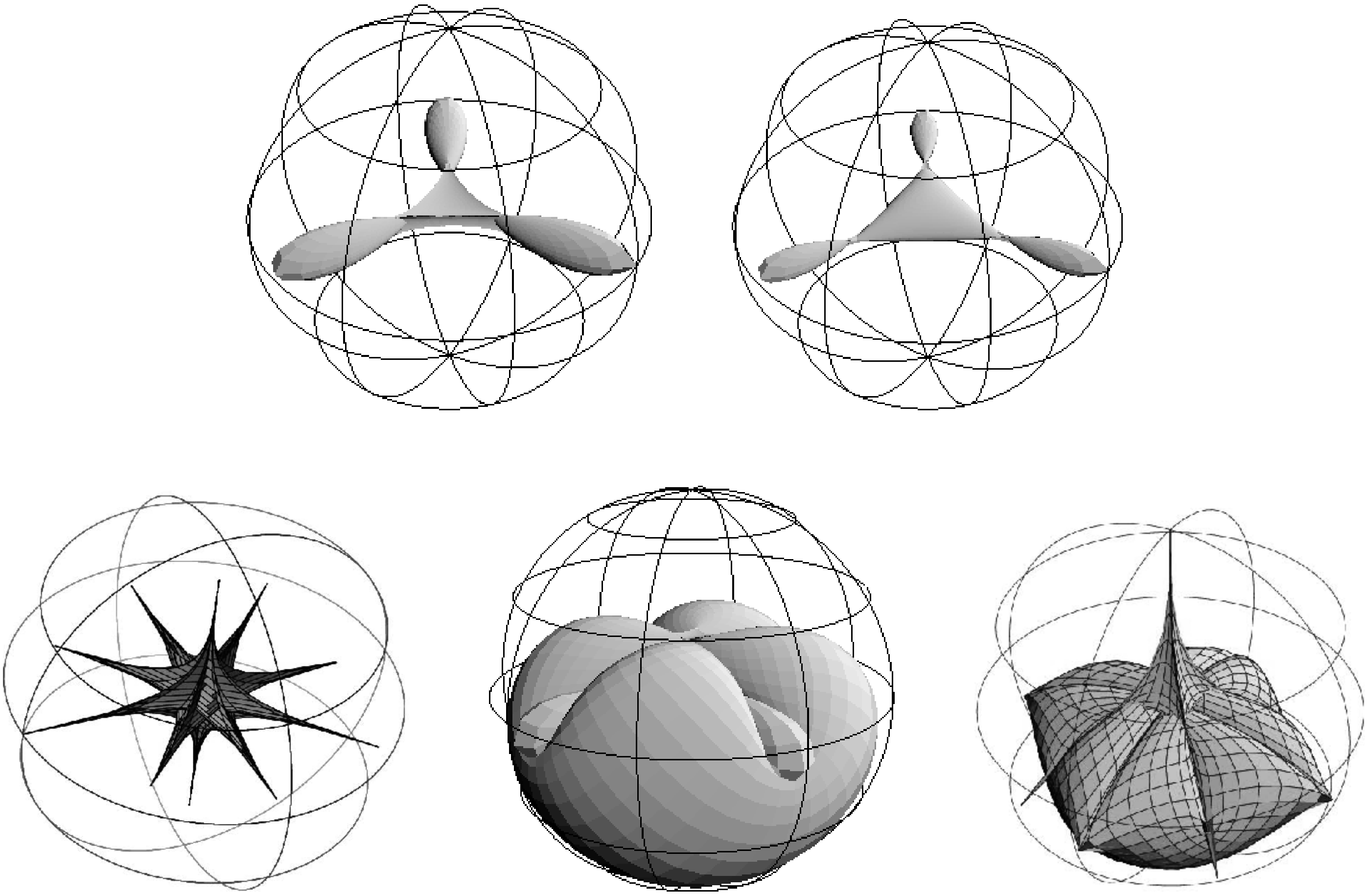

If $G_+, G_-$ have at most pole at $p_j$, an end $p_j$ is called a *regular end*.

**(Fact 7.)** (Kokubu-Yamada-U.04) For a complete flat fronts in $H^3$, it holds that

$$\deg(G_+) + \deg(G_-) \geq \#(\text{Ends}).$$

The equality holds iff all ends are regular and properly embedded.

# (Examples of flat fronts in $H^3$)



# Flat surfaces in $\mathbf{R}^3$

**(Fact 1.)**(Hartman-Nirenberg 59) A complete flat immersed surface is a plane or a cylinder over a planar curve.

A $C^\infty$-map $f : \Sigma^2 \to \mathbf{R}^3$ is called a *wave front* (or *front*) if there exists a $C^\infty$-map $\nu : \Sigma^2 \to S^2$ such that

$$df(T_p\Sigma^2) \perp \nu(p) \qquad (p \in \Sigma^2),$$

and $L := (f, \nu) : \Sigma^2 \to \mathbf{R}^3 \times S^2$ is an immersion. The map $\nu$ is called the *Gauss map* of $f$.

**$f$ is called a flat front $\Leftrightarrow$ $\nu$ is degenerate**

**(Completeness)** A wave front $f : \Sigma^2 \to \mathbf{R}^3$ is called *complete* if $\exists T$ (a symmetric tensor) such that

- supp($T$) is compact,
- $T + ds^2$ is a complete Riemannian metric on $\Sigma^2$.

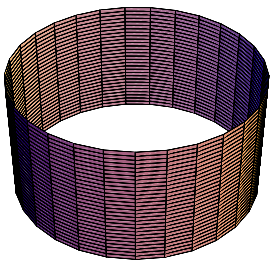
**(Fact 2.)**(Murata-U.) A flat front $f : \Sigma^2 \to \mathbf{R}^3$ is complete iff the singular set is compact and the metric

$$|df|^2 + |d\nu|^2$$

gives a complete Riemannian metric.

**(Fact 3.)**(Murata-U.) If a complete flat front admits singular points, then it does not admit umbilics.

The converse is not true. $\exists$ a circular cylinder.



## A representation formula for complete flat fronts

$$S^1 := \mathbf{R}/(2\pi\mathbf{Z})$$

**(Fact 4.)**(Murata-U.)

Let $\xi : S^1 \to S^2$ be a regular curve without inflections. Let $\alpha$ be a 1-form on $S^1$ such that $\xi\alpha$ is exact, namely $\displaystyle\int_0^{2\pi} \xi\alpha = 0$. Then

$$f : S^1 \times \mathbf{R} \ni (t,u) \mapsto \gamma(t) + u\xi(t) \in \mathbf{R}^3,$$

$$\left(\gamma(t) := \int_0^t \xi\alpha\right)$$

gives a complete flat front with singularity. Conversely, any complete flat fronts with singular points are given in this manner.

**(Cor.)** Let $f : \Sigma^2 \to \mathbf{R}^3$ be a complete flat front. Then $\Sigma^2$ is diffeomorphic to $S^1 \times \mathbf{R}$.

[An Osserman-type inequality]

**(Fact 5.)**(Murata-U.) The image $\mathrm{Im}(\nu)$ of the Gauss map of a complete flat front is a regular spherical closed curve without inflection point. Moreover,

$$\mathrm{Im}(\nu)\text{: convex curve} \Longleftrightarrow f\text{ : embedded ends},$$

that is

$$\#(\text{the crossings of } \mathrm{Im}(\nu)) \geq 0$$

holds.

[An analogue of the four vetex theorem]

(Fact 6.)(Murata-U.) $\exists$ at least four singular points other than cupidal edges on a complete flat front with embedded ends.