On Lagrangian submanifolds in complex hyperquadrics obtained from isoparametric hypersurfaces

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1. Lagrangian Submanifolds in Kähler Manifolds

\[ \varphi : L \longrightarrow (M^{2n}, \omega) \text{ immersion} \]

symplectic mfd.

**Definition**

1. \( \varphi^* \omega = 0 \)

“Lagrangian immersion” \( \Leftrightarrow \) \( \varphi : \text{“isotropic”} \)

2. \( \dim L = n \)

\[ \varphi^{-1}(TM/\varphi_* TL) \cong T^*L \]

linear isom.

\[ \alpha_v := \omega(v, \cdot) \]

\[ \alpha_v : \mathbb{R}^n \rightarrow \mathbb{R} \]
\( \varphi_t : L \to (M^{2n}, \omega) \) immersion with \( \varphi_0 = \varphi \)

\[ V_t := \frac{\partial \varphi_t}{\partial t} \in C^\infty(\varphi_t^{-1}TM) \]

"Lagrangian deformation" \( \iff \varphi_t : \text{Lagr. imm. for}^{\forall} t \) 
\( \iff \alpha_{V_t} \in Z^1(L) \text{ for}^{\forall} t \) \( \text{closed} \)

"Hamiltonian deformation" \( \iff \alpha_{V_t} \in B^1(L) \text{ for}^{\forall} t \) \( \text{exact} \)

Hamil. deform. \( \implies \) Lagr. deform.

The difference between Lagr. deform. and Hamil. deform. is equal to \( H^1(L; \mathbb{R}) \cong Z^1(L)/B^1(L) \).
\[ \varphi_t : L \rightarrow M : \text{Lagr. deform.} \]
Suppose \( \frac{1}{2\pi} [\omega] \) integral.
\{\varphi_t\} : Hamil. deform.

\[ 
\begin{array}{c}
\varphi_t^{-1} L \\
\varphi_t^{-1} \nabla \text{ flat}
\end{array} \rightarrow \exists (L, \nabla) \downarrow
\begin{array}{c}
\pi_1(L) \rightarrow U(1) \\
(\text{"isomonodromy deformation"})
\end{array}
\]
[Lagrangian orbits and moment maps]

K : a Lie group with Lie algebra \( \mathfrak{t} \)

Suppose \( K \) has the Hamiltonian group action on a symplectic manifold \((M, \omega)\) with moment map \( \mu : M \to \mathfrak{t}^* \).

1. An orbit \( K \cdot p \) is an isotropic submfd of \( M \)
   \[ \leftrightarrow \quad L = K \cdot p \subset \mu^{-1}(\alpha) \]
   for some \( \alpha \in \mathfrak{z}(\mathfrak{t}^*) \).

   Here
   \[ \mathfrak{z}(\mathfrak{t}^*) := \{ \alpha \in \mathfrak{t}^* \mid \text{Ad}^*(a)\alpha = \alpha \text{ for all } a \in K \}. \]

2. Suppose \( M \) and \( K \) are compact and connected.

   \( L = K \cdot p \) is a Lagr. submfd. in \( M \)
   \[ \leftrightarrow \quad L = \mu^{-1}(\alpha) \]
   for some \( \alpha \in \mathfrak{z}(\mathfrak{t}^*) \cong \mathfrak{c}(\mathfrak{k}) \) : center of \( \mathfrak{k} \).
2. Lagrangian Submanifolds in Kähler Manifolds

\((M, \omega, J, g)\) : Kähler manifold

\(\varphi : L \hookrightarrow M\) Lagr. imm.

\(B\) : the second fundamental form of \(L\) in \((M, g)\).

\(S(X, Y, Z) := \omega(B(X, Y), Z)\) sym. 3-tensor field on \(L\)

\(H\) : mean curvature vector field of \(\varphi\)

\(\uparrow\)

\(\alpha_H\) : “mean curvature form” of \(\varphi\)

\(d\alpha_H = \varphi^* \rho_M\) where \(\rho_M\) : Ricci form of \(M\).

**Definition**

*When \((M, \omega, J, g)\) : Kähler manifold.*

\(K \subset \text{Aut}(M, \omega, J, g)\) : connected Lie subgroup,

\(L = K \cdot x \subset M\) : a Lagrangian orbit

“homogeneous Lagrangian submanifold”
Suppose \( L \) : compact with \( \partial M = \emptyset \).

\( \varphi : \) “Hamiltonian minimal” or shortly “\( H \)-minimal”

\[ \iff \forall \varphi_t : L \rightarrow M \text{ Hamil. deform. with } \varphi_0 = \varphi \]

\[ \iff \frac{d}{dt} \text{Vol} (L, \varphi_t^* g) \bigg|_{t=0} = 0 \]

\[ \iff \delta \alpha_H = 0 \]

**Proposition**

\( L = K \cdot x \subset M \) : compact homogeneous Lagr. submfd.

\[ \implies Hamiltonian minimal \]
Theorem (Urbano, independently O., about 1985)

\( M = \tilde{\mathcal{M}}(c) \): complex space form,
\( L \): compact Lagrangian submfd. in \( M \).

Then
\( L \) is H-minimal and has sectional curvatures \( K_L \geq 0 \)
\iff
\( L \) has parallel second fundamental form, i.e. \( \nabla S = 0 \).

Remark

All Lagrangian submfd's with \( \nabla S = 0 \) in complex space forms were completely classified by Professors Hiroo Naitoh and Masaru Takeuchi about the first half of 1980's.
Assume $\varphi : H$-minimal.

$\forall \{\varphi_t\} : \text{Hamil. deform. of } \varphi_0 = \varphi$

$\varphi : "\text{Hamiltonian stable} " \iff \begin{align*} 
\frac{d^2}{dt^2} \text{Vol} (L, \varphi_t^* g) \bigg|_{t=0} & \geq 0 
\end{align*}$

(The Second Variational Formula)

\[
\begin{align*}
\frac{d^2}{dt^2} \text{Vol} (L, \varphi_t^* g) \bigg|_{t=0} &= \\
\int_L \left( \langle \Delta^1_L \alpha, \alpha \rangle - \langle \overline{R}(\alpha), \alpha \rangle - 2\langle \alpha \otimes \alpha \otimes \alpha_H, S \rangle + \langle \alpha_H, \alpha \rangle^2 \right) dv
\end{align*}
\]

where $V := \frac{\partial \varphi_t}{\partial t} \big|_{t=0} \in C^\infty(\varphi^{-1}TM)$

- $\alpha := \alpha_V \in B^1(L)$
- $\langle \overline{R}(\alpha), \alpha \rangle := \sum_{i,j=1}^n \text{Ric}^M(e_i, e_j) \alpha(e_i) \alpha(e_j)$, $\{e_i\}$: o.n.b. of $T_pL$
- $S(X, Y, Z) := \omega(B(X, Y), Z)$ sym. 3-tensor field on $L$
[The null space of the second variation on Hamiltonian deformations] = [the vector space of all solutions to the linearized Hamiltonian minimal Lagrangian submanifold equation]. The “nullity” of $H$-minimal Lagr. imm. $\varphi$:

$$n(\varphi) := \dim[ \text{the null space}].$$

$X$ : holomorphic Killing vector field of $M$

$$\implies \alpha_X = \omega(X, \cdot) \text{ is closed}$$

$$\implies \alpha_X = \omega(X, \cdot) \text{ is exact, i.e. } X \text{ is a Hamiltonian vector field on } M$$

if $H^1(M, R) = \{0\}$.

Suppose that $\pi_1(M) = \{1\}$, more generally $H^1(M, R) = \{0\}$.

$$\implies \text{Each holomorphic Killing vector field of } M \text{ generates a volume-preserving Hamiltonian deformation of } \varphi.$$

**Definition**

Such a Hamiltonian deformation of $\varphi$ is called **trivial**.

$$n_{hk}(\varphi) := \dim\{\varphi^*\alpha_X \mid X \text{ is a holom. Killing vector field of } M\} \leq n(\varphi).$$
Suppose that $\pi_1(M) = \{1\}$, more generally $H^1(M, \mathbb{R}) = \{0\}$.

Assume $\varphi$ : H-minimal Lagr. imm.

$\varphi$ : “strictly Hamiltonian stable”

$\iff$

\begin{align*}
(1) & \text{ $\varphi$ is Hamiltonian stable } \\
(2) & \text{ The null space of the second variation on Hamiltonian deformations coincides with the vector subspace induced by trivial Hamiltonian deformations of $\varphi$. That is, } n(\varphi) = n_{hk}(\varphi). \end{align*}

Remark.

$\varphi$ : strictly Hamiltonain stable

$\implies$ $\varphi$ has local minimun volume under any Hamil. deform.

$\varphi$ : “Hamiltonian rigid” (Yng-Ing Lee),

$\iff$

\begin{align*}
& n(\varphi) = n_{hk}(\varphi). \\
& \text{def}
\end{align*}
[Hamiltonian stability of min. Lagr. submfd.] Assume $M$: Einstein-Kähler manifold of Einstein constant $\kappa$.

$L \hookrightarrow M$ cpt. minimal Lagr. submfd. (i.e. $\alpha_H \equiv 0$)

Then the second variational formula implies

$L$ is Hamiltonian stable $\iff \lambda_1 \geq \kappa$

Here

$\lambda_1$: the first (positive) eigenvalue of the Laplacian of $L$ on $C^\infty(L)$.

(B. Y. Chen - T. Nagano - P. F. Leung, Y. G. Oh)
[Upper bounds of the first eigenvalue of min. Lagr. submfd. and Hamiltonian stability]

[Fact] (cf. Hajime Ono, Amarzaya - O.)

Assume $M$ : cpt. homog. Einstein - Kähler mfd. with $\kappa > 0$.  
$L \hookrightarrow M$ cpt. minimal Lagr. submfd.

$\implies \lambda_1 \leq \kappa$.

- $L$ is Hamil. stable
  $\iff \lambda_1 = \kappa$
  The restrictions of all first eigenfunctions of $M$ to $L$ are constants + the first eigenfunctions of $L$.

- $L$ is strictly Hamil. stable
  $\iff$ The restrictions of all first eigenfunctions of $M$ to $L$ give constants + all the first eigenfunction of $L$.

Works on Spectral Geometry of Submanifolds by Antonio Ros, F. Urbano, etc.
Examples.

1. \( L \hookrightarrow M = \mathbb{C}P^n \) embedded cpt. min. Lagr. submfd.

\[ L = \]

- \( \mathbb{R}P^n \) (Y. G. Oh)
- \( SU(p)/SO(p) \cdot \mathbb{Z}_p, SU(p)/\mathbb{Z}_p, SU(2p)/Sp(p) \cdot \mathbb{Z}_{2p} \)
- \( E_6/F_4 \cdot \mathbb{Z}_3 \) (Amarzaya - O., Tohoku Math. J. 2003)

\[ \rho_3(SU(2))[z^3_0 + z^3_1] \subset \mathbb{C}P^3 \]

\[ \implies L \text{ is strictly Hamil. stable.} \]
Examples.

1. \( L \leftrightarrow M = \mathbb{C}P^n \) embedded cpt. min. Lagr. submfd.

\[
L =
\]

- \( \mathbb{R}P^n \) (Y. G. Oh) \( S = 0 \)
- \( SU(p)/SO(p) \cdot \mathbb{Z}_p, SU(p)/\mathbb{Z}_p, SU(2p)/Sp(p) \cdot \mathbb{Z}_{2p} \)
- \( E_6/F_4 \cdot \mathbb{Z}_3 \) (Amarzaya - O., Tohoku Math.J. 2003)
  \( S \neq 0, \nabla S = 0 \)
- \( \rho_3(SU(2))[z_0^3 + z_1^3] \subset \mathbb{C}P^3 \) \( \nabla S \neq 0 \)
  (River Chiang, Inter. Math, Res, Not. 2004,
  O., Osaka J. Math. 2007)

\( \implies L \) is strictly Hamil. stable.

2
Examples.

1. \( L \leftrightarrow M = \mathbb{C}P^n \) embedded cpt. min. Lagr. submfd.
   \[
   L =
   \begin{align*}
   &\mathbb{R}P^n \quad (\text{Y. G. Oh}) \quad S = 0 \\
   &SU(p)/SO(p) \cdot \mathbb{Z}_p, \; SU(p)/\mathbb{Z}_p, \; SU(2p)/Sp(p) \cdot \mathbb{Z}_{2p} \\
   &E_6/F_4 \cdot \mathbb{Z}_3 \quad (\text{Amarzaya - O., Tohoku Math.J. 2003}) \\
   &S \neq 0, \nabla S = 0 \\
   &\rho_3(SU(2))[z_0^3 + z_1^3] \subset \mathbb{C}P^3 \quad \nabla S \neq 0 \\
   \end{align*}
   \]

\[\implies L \text{ is strictly Hamil. stable.}\]

2. \( M \) : cpt. irred. Herm. sym. sp. of rank \( \geq 2 \)

\[
(L, M) \neq \begin{cases}
(Q_{p,q}(\mathbb{R}) = (S^{p-1} \times S^{q-1})/\mathbb{Z}_2, \\
Q_{p+q-2}(\mathbb{C}))(q - p \geq 3) \\
(U(2p)/Sp(p), SO(4p)/U(p))(p \geq 3) \\
(T \cdot E_6/F_4, E_7/T \cdot E_6)
\end{cases}
\]

\[\iff L \text{ is Hamil. stable.}\]
Theorem (Urbano, 1993)

\[ L^2 \leftrightarrow \mathbb{C}P^2: \text{Hamiltonian stable minimal Lagrangian torus} \]
\[ \iff L^2 = T^2 \leftrightarrow \mathbb{C}P^2: \text{Clifford minimal torus} \]
\[ (\Rightarrow \nabla S = 0) \]

Theorem (Amarzaya-O.)

\[ L^n \leftrightarrow \tilde{M}(c) \quad (= \mathbb{C}^n, \mathbb{C}P^n, \mathbb{C}H^n) \]
\[ \text{cplx. sp. form} \]
\[ \text{compact embedded Lagr. submfd. with } \nabla S = 0 \]
\[ \implies \text{Hamil. stable.} \]
Conjecture.

$L \hookrightarrow \mathbb{C}P^n$ compact \textit{embedded} minimal Lagr. submfd.

$\implies \lambda_1 = \kappa$ ?

i.e. Hamil. stable.

(Any counter example is \textbf{not} known yet.)
3. Lagrangian Submanifolds in Complex Hyperquadrics and Hypersurface Geometry in Spheres

**Complex Hyperquadrics**

\[ Q_n(\mathbb{C}) \cong \widetilde{Gr}_2(\mathbb{R}^{n+2}) \cong SO(n + 2)/SO(2) \times SO(n) \]

is a compact Hermitian symmetric space of rank 2

\[ Q_n(\mathbb{C}) := \{ [z] \in \mathbb{C}P^{n+1} \mid z_0^2 + z_1^2 + \cdots + z_{n+1}^2 = 0 \} \]

\[ \widetilde{Gr}_2(\mathbb{R}^{n+2}) := \{ W \mid \text{oriented 2-dimensional vector subspace of } \mathbb{R}^{n+2} \} \]

\[ Q_n(\mathbb{C}) \ni [a + \sqrt{-1}b] \leftrightarrow a \wedge b \in \widetilde{Gr}_2(\mathbb{R}^{n+2}) \]

Here \{a, b\} is an orthonormal basis of \( W \) compatible with its orientation.
[The Gauss Map of Oriented Hypersurfaces in the Unit Sphere]

**Oriented hypersurface in a sphere**

\[ N^n \hookrightarrow S^{n+1}(1) \subset \mathbb{R}^{n+2} \]

\( x : \) the position vector of points of \( N^n \)

\( n : \) the unit normal vector field of \( N^n \) in \( S^{n+1}(1) \)

**“Gauss map”**

\[ G : N^n \ni p \mapsto [x(p) + \sqrt{-1}n(p)] = x(p) \wedge n(p) \in Q_n(\mathbb{C}) \]

is a Lagrangian immersion.

**Proposition**

\( N_1^n, N_2^n \subset S^{n+1}(1) \) are parallel \iff \( G(N_1^n) = G(N_2^n) \)
**Oriented hypersurface in a sphere**

\[ N^n \hookrightarrow S^{n+1}(1) \subset \mathbb{R}^{n+2} \]

- \( x \) : the position vector of points of \( N^n \)
- \( n \) : the unit normal vector field of \( N^n \) in \( S^{n+1}(1) \)

**“Gauss map”**

\[ G : N^n \ni p \longmapsto [x(p) + \sqrt{-1}n(p)] = x(p) \wedge n(p) \in Q_n(\mathbb{C}) \]

is a Lagrangian immersion.

**Proposition**

*Deformation of \( N^n = \) Hamiltonian deformation of \( G \)*
Remark. \((2n + 1)\)-dimensional real Stiefel manifold

\[ V_2(R^{n+2}) := \{ (a, b) | a, b \in R^{n+2} \text{ orthonormal} \} \cong SO(n+2)/SO(n) \]

the standard Einstein-Sasakian manifold over \(Q_n(C)\).

The natural projections

\[ p_1 : V_2(R^{n+2}) \ni (a, b) \mapsto a \in S^{n+1}(1), \]
\[ p_2 : V_2(R^{n+2}) \ni (a, b) \mapsto a \wedge b \in Q_n(C). \]

\[ \tilde{N}^n \xrightarrow{\psi} UTS^{n+1} = V_2(R^{n+2}) \]
\[ \text{Legend.} \]
\[ \text{ori.hypsurf.} \]

Here the Legendrian life \(\tilde{N}^n\) of \(N^n \hookrightarrow S^{n+1}(1)\) to \(V_2(R^{n+2})\) is defined by \(N^n \ni p \mapsto (x(p), n(p)) \in V_2(R^{n+2}).\)
(Conormal bundle construction)
More generally for a given $N^m \subset S^{n+1}(1)$ submanifold,

In the case $m = n$ and $N^n$ oriented, it coincides with our Gauss map construction.
Lemma (Mean Curvature Form Formula (Palmer, 1997))

\[ \alpha_H = -d \left( \sum_{i=1}^{n} \arccot \kappa_i \right) \]

\[ = d \left( \text{Im} \left( \log \prod_{i=1}^{n} (1 + \sqrt{-1} \kappa_i) \right) \right), \]

where \( H \) : mean curvature vector field of \( G \),
\( \kappa_i \ (i = 1, \cdots, n) \) : principal curvatures of \( N^n \subset S^{n+1}(1) \).

\( N^2 \subset S^3(1) \) min. surf.
\[ \implies G : N^2 \to Q_2(\mathbb{C}) \cong S^2 \times S^2 \] min. Lagr. imm.
See [Castro-Urbano, 2007].

\( N^n \subset S^{n+1}(1) \) austere min. hypersurf.
\[ \implies G : N^n \to Q_n(\mathbb{C}) \] min. Lagr. imm.
Oriented hypersurface in a sphere

\[ N^n \hookrightarrow S^{n+1}(1) \subset \mathbb{R}^{n+2} \] with constant principal curvatures ("isoparametric hypersurface")

"Gauss map"

\[ \mathcal{G} : N^n \ni p \overset{\text{Larg. imm.}}{\mapsto} [x(p) \wedge n(p)] \in Q_n(\mathbb{C}) \]

\[ N^n \twoheadrightarrow \mathcal{G}(N^n) \cong N^n/\mathbb{Z}_g \hookrightarrow Q_n(\mathbb{C}) \]

cpt. embedded minimal Lagr. submfd

Here \( g := \# \{\text{distinct principal curvatures of } N^n\} \),
\( m_1 \leq m_2 \) : multiplicities of the principal curvatures.
Oriented hypersurface in a sphere

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cpt. embedded minimal Lagr. submfd

Here \( g := \# \{ \text{distinct principal curvatures of } N^n \} \),

\( m_1, m_2 \) : multiplicities of the principal curvatures.
Isoparametric Hypersurface Theory in $S^{n+1}(1)$]

All isoparametric hypersurfaces in $S^{n+1}(1)$ are classified into

- **Homogeneous** ones (Hsiang-Lawson, R. Takagi-T. Takahashi) can be obtained as principal orbits of the isotropy representations of Riemannian symmetric pairs $(U, K)$ of rank 2.
  - $g = 1 : N^n = S^n$, a great or small sphere;
  - $g = 2, N^n = S^{m_1} \times S^{m_2}, (n = m_1 + m_2, 1 \leq m_1 \leq m_2)$, the Clifford hypersurfaces;
  - $g = 3, N^n$ is homog., $N^n = \frac{SO(3)}{\mathbb{Z}_2 + \mathbb{Z}_2}, \frac{SU(3)}{T^2}, \frac{Sp(3)}{Sp(1)^3}, \frac{F_4}{Spin(8)}$;
  - $g = 6$: Only homog. examples are known now.
    - $g = 6, m_1 = m_2 = 1$: homog. (Dorfmeister-Neher, R. Miyaoka)
    - $g = 6, m_1 = m_2 = 2$: homog. (R. Miyaoka)

- **Non-homogenous** ones exist (H. Ozeki- M. Takeuchi, 1975) and are almost classified (Ferus-Karcher-Münzner, Cecil-Chi-Jensen, Immervoll).
  - $g = 4$: except for $(m_1, m_2) = (4, 5), (6, 9), (7, 8)$, either homog. or OT-FKM type.
4. The Gauss Images of Isoparametric Hypersurfaces

**Oriented hypersurface in a sphere**

\[ N^n \hookrightarrow S^{n+1}(1) \subset \mathbb{R}^{n+2} \] with constant principal curvatures ("isoparametric hypersurface")

**“Gauss map”**

\[ G : N^n \ni p \quad \mapsto \quad x(p) \wedge n(p) \in Q_n(\mathbb{C}) \]

\[ N^n \xrightarrow{\mathbb{Z}_g} L^n = G(N^n) \cong N^n/\mathbb{Z}_g \hookrightarrow Q_n(\mathbb{C}) \]

cpt. embedded minimal Lagr. submfd

**Lemma (Hui Ma-O.)**

\[ L^n = G(N^n) \text{ is orientable} \iff \frac{2n}{g} \text{ is even.} \]

Note that \[ \frac{2n}{g} = \begin{cases} m_1 + m_2 & \text{if } g \text{ is even,} \\ 2m_1 & \text{if } g \text{ is odd.} \end{cases} \]
4. The Gauss Images of Isoparametric Hypersurfaces

**Oriented hypersurface in a sphere**

\[ N^n \hookrightarrow S^{n+1}(1) \subset \mathbb{R}^{n+2} \text{ with constant principal curvatures} \]

(“isoparametric hypersurface”)

**“Gauss map”**

\[ G : N^n \ni p \mapsto x(p) \wedge n(p) \in Q_n(\mathbb{C}) \]

\[ N^n \twoheadrightarrow L^n = G(N^n) \cong N^n / \mathbb{Z}_g \hookrightarrow Q_n(\mathbb{C}) \]

cpt. embedded minimal Lagr. submfd

**Theorem (Hui Ma-O.)**

\[ L^n = G(N^n) \text{ is a monotone and cyclic Lagrangian submanifold whose minimal Maslov number is equal to} \]

\[ \Sigma_L = \frac{2n}{g} \]
5. Classification of Compact Homogeneous Lagr. Submfd.s. in $Q_n(\mathbb{C})$.

**Oriented hypersurface in a sphere**

$N^n \hookrightarrow S^{n+1}(1) \subset \mathbb{R}^{n+2}$ with constant principal curvatures ("isoparametric hypersurface")

**“Gauss map”**

\[ G : N^n \ni p \quad \overset{\text{Larg. imm.}}{\longrightarrow} \quad x(p) \wedge n(p) \in Q_n(\mathbb{C}) \]

\[ N^n \overset{\mathbb{Z}_g}{\longrightarrow} L^n = G(N^n) \cong N^n/\mathbb{Z}_g \hookrightarrow Q_n(\mathbb{C}) \]

cpt. embedded minimal Lagr. submfd

**Proposition (Hui Ma-O.)**

$N^n$ is homogeneous $\iff L^n = G(N^n)$ is homogeneous
Classification of Homogeneous Lagr. submfds. in $\mathbb{C}P^n$ (Bedulli and Gori, math.DG/0604169)

16 examples of minimal Lagr. orbits in $\mathbb{C}P^n$

$= [5 \text{ examples with } \nabla S = 0] + [11 \text{ examples with } \nabla S \neq 0]$

$K \subset SU(n + 1)$: cpt. simple subgroup

Lemma (Bedulli-Gori)

Suppose $M$: cpt. Kähler mfd. with $\dim H^{1,1}(M; \mathbb{C}) = 1$.

$L = K \cdot [v] \subset M$ Lagr. submfd.

$\hookrightarrow$

complexified orbit (Zariski open) $K^c \cdot [v] \subset M$ is Stein.
Classification of Homogeneous Lagr. submfds. in $\mathbb{C}P^n$ (Bedulli and Gori, Comm. Anal. Geom. (2008))

16 examples of minimal Leagr. orbits in $\mathbb{C}P^n$

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Lemma (Bedulli-Gori)

Suppose $M$: cpt. Kähler mfd. with $\text{dim } H^{1,1}(M; \mathbb{C}) = 1$.

$L = K \cdot [v] \subset M$ Lagr. submfd.

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Classification Theory of “Prehomogeneous Vector Spaces” (Mikio Sato and Tatsuo Kimura)
Classification of Homogeneous Lagrangian submanifolds in $Q_n(\mathbb{C})$ (Hui Ma and O., Math. Z. 2009)

Suppose

$$G \subset SO(n + 2) : \text{cpt. subgroup},$$

$$L = G \cdot [W] \subset Q_n(\mathbb{C}) \quad \text{Lagr. submfd.}$$

There exists

$$N^n \subset S^{n+1}(1) \subset \mathbb{R}^{n+2} : \text{cpt. homog. isopara. hypersurf.}$$

such that

(a) $L = G(N)$ and $L$ is a cpt. minimal Lagr. submfd., or
(b) $L$ is a Lagrangian deformation of $G(N)$. 
W.Y.Hsiang-H.B.Lawson’s theorem (1971)

There is a compact Riemannian symmetric pair \((U, K)\) of rank 2 such that

\[ N = \text{Ad}(K)v \subset S^{n+1}(1) \subset \mathbb{R}^{n+2} = \mathfrak{p}, \]

where \(u = \mathfrak{k} + \mathfrak{p}\) is the canonical decomposition of \((U, K)\).

Moment map

The moment map of the induced action of \(K\) on \(Q_n(\mathbb{C}) = \text{Gr}_2(\mathfrak{p})\) is given by

\[ \mu : Q_n(\mathbb{C}) = \text{Gr}_2(\mathfrak{p}) \ni [W] \mapsto [a, b] \in \mathfrak{k} \cong \mathfrak{k}^* \]

where \([a, b] : \text{orthonormal basis of } W \text{ compatible with the orientation of } [W].\)
The case (b) happens only when \((U, K)\) is one of

1. \((S^1 \times SO(3), SO(2))\),
2. \((SO(3) \times SO(3), SO(2) \times SO(2))\),
3. \((SO(3) \times SO(n + 1), SO(2) \times SO(n))\) \((n \geq 3)\),
4. \((SO(m + 2), SO(2) \times SO(m))\) \((n = 2m - 2, m \geq 3)\).

In the first two cases, it is elementary and well-known to describe all Lagrangian orbits of the natural actions of \(K = SO(2)\) on \(Q_1(C) \cong S^2\) and \(K = SO(2) \times SO(2)\) on \(Q_2(C) \cong S^2 \times S^2\). Also in the last two cases there exist one-parameter families of Lagrangian \(K\)-orbits in \(Q_n(C)\) and each family contains Lagrangian submanifolds which can NOT be obtained as the Gauss image of any homogeneous isoparametric hypersurface in a sphere. The fourth one is a new family of Lagrangian orbits.
If \((U, K)\) is \((S^1 \times SO(3), SO(2))\),
then \(L\) is a small or great circle in \(Q_1(C) \cong S^2\).

If \((U, K)\) is \((SO(3) \times SO(3), SO(2) \times SO(2))\),
then \(L\) is a product of small or great circles of \(S^2\) in
\(Q_2(C) \cong S^2 \times S^2\).
If \((U, K)\) is \((SO(3) \times SO(n + 1), SO(2) \times SO(n))\) \((n \geq 2)\), then

\[
L = K \cdot [W_\lambda] \subset Q_n(C) \quad \text{for some} \ \lambda \in S^1 \setminus \{\pm \sqrt{-1}\},
\]

where \(K \cdot [W_\lambda] \ (\lambda \in S^1)\) is the \(S^1\)-family of Lagr. or isotropic \(K\)-orbits satisfying

1. \(K \cdot [W_1] = K \cdot [W_{-1}] = G(N^n)\) is a tot. geod. Lagr. submfd. in \(Q_n(C)\).

2. For each \(\lambda \in S^1 \setminus \{\pm \sqrt{-1}\},\)

\[
K \cdot [W_\lambda] \cong (S^1 \times S^{n-1})/\mathbb{Z}_2 \cong Q_{2,n}(\mathbb{R})
\]

is a Lagr. orbit in \(Q_n(C)\) with \(\nabla S = 0\).

3. \(K \cdot [W_{\pm \sqrt{-1}}]\) are isotropic orbits in \(Q_n(C)\) with \(\dim K \cdot [W_{\pm \sqrt{-1}}] = 0\).
If \((U, K)\) is \((SO(m + 2), SO(2) \times SO(m))\) \((n = 2m - 2)\), then

\[
L = K \cdot [W_\lambda] \subset Q_n(C) \quad \text{for some } \lambda \in S^1 \setminus \{\pm \sqrt{-1}\},
\]

where \(K \cdot [W_\lambda] \ (\lambda \in S^1)\) is the \(S^1\)-family of Lagr. or isotropic orbits satisfying

1. \(K \cdot [W_1] = K \cdot [W_{-1}] = G(N^n)\) is a minimal (NOT tot. geod.) Lagr. submfd. in \(Q_n(C)\).

2. For each \(\lambda \in S^1 \setminus \{\pm \sqrt{-1}\}\),

\[
K \cdot [W_\lambda] \cong (SO(2) \times SO(m))/(Z_2 \times Z_4 \times SO(m - 2))
\]

is a Lagr. orbit in \(Q_n(C)\) with \(\nabla S \neq 0\).

3. \(K \cdot [W_\pm \sqrt{-1}] \cong SO(m)/S(O(1) \times O(m - 1)) \cong \mathbb{RP}^{m-1}\) are isotropic orbits in \(Q_n(C)\) with \(\dim K \cdot [W_{\pm \sqrt{-1}}] = m - 1\).
6. Hamiltonian Stability of Gauss Images of Homogeneous Isoparametric Hypersurfaces

Suppose

\[ N^n \hookrightarrow S^{n+1}(1) \subset \mathbb{R}^{n+2} \text{ cpt. embedded isopara. hypersurf.} \]

\[ G : N^n \xrightarrow{\mathbb{Z}_g} G(N^n) \cong N^n/\mathbb{Z}_g \xhookrightarrow{\subset} Q_n(C) = \widetilde{\text{Gr}}_2(\mathbb{R}^{n+2}) \subset \bigwedge^2 \mathbb{R}^{n+2} \]

\[ \mathfrak{v} \subset \mathfrak{o}(n + 2) : \text{Lie algebra of all Killing vector fields tangent to } N^n \text{ or } G(N^n). \]

\[ \bigwedge^2 \mathbb{R}^{n+2} = \mathfrak{o}(n + 2) = \mathfrak{i} + \mathcal{V} \quad (\text{orthog. direct sum}). \]

Lemma

\[ G(N^n) \subset \widetilde{\text{Gr}}_2(\mathbb{R}^{n+2}) \cap \mathcal{V} \text{ and } G(N^n) \text{ is a cpt. minimal submfd. in the unit hypersphere of } \mathcal{V}. \]
Suppose

\[ N^n \hookrightarrow S^{n+1}(1) \subset \mathbb{R}^{n+2} \text{ cpt. embedded isopara. hypersurf.}\]

\[ \mathcal{G} : N^n \overset{\mathbb{Z}_g}{\longrightarrow} \mathcal{G}(N^n) \cong N^n / \mathbb{Z}_g \hookrightarrow Q_n(C) = \widetilde{Gr}_2(\mathbb{R}^{n+2}) \subset \bigwedge^2 \mathbb{R}^{n+2} \]

\( \tilde{\mathfrak{k}}: \) Lie algebra of all Killing vector fields tangent to \( N^n \) or \( \mathcal{G}(N^n) \).

\[ \bigwedge^2 \mathbb{R}^{n+2} = \mathfrak{o}(n + 2) = \tilde{\mathfrak{k}} + \mathcal{V} \quad \text{(orthog. direct sum)}.\]

Proposition

\( N^n \) is homogeneous \( \iff \mathcal{G}(N^n) = \widetilde{Gr}_2(\mathbb{R}^{n+2}) \cap \mathcal{V} \)
Suppose

\[ N^n \hookrightarrow S^{n+1}(1) \subset \mathbb{R}^{n+2} \text{ cpt. embedded isopara. hypersurf.} \]

\[ \mathcal{G} : N^n \rightarrow \mathcal{G}(N^n) \cong N^n/\mathbb{Z}_g \hookrightarrow Q_n(C) = \widetilde{\text{Gr}}_2(\mathbb{R}^{n+2}) \subset \bigwedge^2 \mathbb{R}^{n+2} \]

\[ \tilde{\mathfrak{k}}: \text{Lie algebra of all Killing vector fields tangent to } N^n \text{ or } \mathcal{G}(N^n). \]

\[ \bigwedge^2 \mathbb{R}^{n+2} = \mathfrak{o}(n + 2) = \tilde{\mathfrak{k}} + \mathcal{V} \text{ (orthog. direct sum).} \]

Note that \( n_{hk}(\mathcal{G}(N^n)) = \text{dim} \mathcal{V}. \)

**Corollary**

\( \mathcal{G}(N^n) \) is Hamil. stable. \( \iff \) Each coordinate function of \( \mathcal{V} \) restricted to \( \mathcal{G}(N^n) \) is the first eigenfunction of \( \mathcal{G}(N^n) \).

\( \mathcal{G}(N^n) \) is strictly Hamil. stable. \( \iff \) \( \dim \mathcal{V} \) is equal to the multiplicity of the first eigenvalue \( n \).
$N^n \leftrightarrow S^{n+1}(1)$: cpt. embedded isopara. hypersurf.

**Hamiltonian stability of the Gauss map (Palmer, 1997)**

Its Gauss map $G : N \rightarrow Q_n(\mathbb{C})$ is Hamiltonian stable
$\iff N^n = S^n \subset S^{n+1} (g = 1)$.

**Question**

Hamiltonian stability of its Gauss image $L = G(N^n) \subset Q_n(\mathbb{C})$?

**Main result**

We determine the Hamiltonian stability of Gauss images of **ALL** homogeneous isoparametric hypersurfaces.
\( g = 1 \) : \( L \) is strictly Hamilton. stable

\( g = 2 \) : \( L \) is Hamilton. stable \( \Longleftrightarrow \) \( m_2 - m_1 < 3 \)
\[ \Rightarrow L = Q_{p,q}(\mathbb{R}) \text{ tot. geod.} \]

\( m_2 - m_1 \geq 3 \Rightarrow \) the spherical harmonics of degree 2 on the sphere \( S^{m_1} \subset \mathbb{R}^{m_1+1} \) of smaller dimension give volume-decreasing Hamilton. deformations of \( G(N^n) \).

\( m_2 - m_1 = 2 \Rightarrow \) Hamilton. stable but not strictly Hamilton. stable.

\( m_2 - m_1 = 0 \) or \( 1 \Rightarrow \) strictly Hamilton. stable.

\( g = 3 \) : \( L \) is strictly Hamilton. stable \( \Rightarrow \) homog. (E. Cartan)


Homog. case ?

Non-homog. case ??

\( g = 4 \) : 

\begin{align*}
\text{Theorem (Hui Ma and O.)} \\
g = 6 \ : & \ L = SO(4)/(\mathbb{Z}_2 + \mathbb{Z}_2) \cdot \mathbb{Z}_6 \quad (m_1 = m_2 = 1) \\
& L = G_2/T^2 \cdot \mathbb{Z}_6 \quad (m_1 = m_2 = 2) \text{ homog.} \\
\Rightarrow & L \text{ is strictly Hamilton. stable.}
\end{align*}
Theorem (Hui Ma and O.)

\[ g = 4, \text{ homogeneous :} \]

1. \[ L = SO(5)/T^2 \cdot \mathbb{Z}_4 \quad (m_1 = m_2 = 2) \text{ is Hamil. stable.} \]
2. \[ L = U(5)/(SU(2) \times SU(2) \times U(1)) \cdot \mathbb{Z}_4 \]
   \[ (m_1 = 4, m_2 = 5) \text{ is Hamil. stable.} \]
3. \[ L = (SO(2) \times SO(m))/\mathbb{Z}_2 \times SO(m - 2) \cdot \mathbb{Z}_4 \]
   \[ (m_1 = 1, m_2 = m - 2, m \geq 3) \]
   \[ m_2 - m_1 \geq 3 \iff L \text{ is NOT Hamil. stable.} \]
   \[ m_2 - m_1 = 2 \implies L \text{ is Hamil. stable but not strictly Hamil. stable.} \]
   \[ m_2 - m_1 = 1 \text{ or } 0 \implies L \text{ is strictly Hamil. stable.} \]
4. \[ L = S(U(2) \times U(m))/S(U(1) \times U(1) \times U(m - 2))) \cdot \mathbb{Z}_4 \]
   \[ (m_1 = 2, m_2 = 2m - 3, m \geq 2) \]
   \[ m_2 - m_1 \geq 3 \iff L \text{ is NOT Hamil. stable.} \]
   \[ m_2 - m_1 = 1 \text{ or } -1 \implies L \text{ is strictly Hamil. stable.} \]
5. \[ L = Sp(2) \times Sp(m)/(Sp(1) \times Sp(1) \times Sp(m - 2))) \cdot \mathbb{Z}_4 \]
   \[ (m_1 = 4, m_2 = 4m - 5, m \geq 2) \]
   \[ m_2 - m_1 \geq 3 \iff L \text{ is NOT Hamil. stable.} \]
   \[ m_2 - m_1 = -1 \implies L \text{ is strictly Hamil. stable.} \]
Theorem (Hui Ma and O.)

\( g = 4 \), homogeneous:

\[
(6) \ L = U(1) \cdot Spin(10)/(S^1 \cdot Spin(6)) \cdot \mathbb{Z}_4
\]

\( (m_1 = 6, m_2 = 9, \text{ thus } m_2 - m_1 = 3!) \)

\( \implies L \) is strictly Hamil. stable!
Theorem (Hui Ma and O.)

\[ g = 4, \ \text{homogeneous:} \]
\[ (6) \ L = U(1) \cdot \text{Spin}(10)/(S^1 \cdot \text{Spin}(6)) \cdot \mathbb{Z}_4 \]
\[ (m_1 = 6, m_2 = 9, \text{thus } m_2 - m_1 = 3!) \]
\[ \implies L \text{ is strictly Hamil. stable!} \]

In a summary, we obtain a Hamiltonian stability result on the Gauss images of ALL homogeneous isoparametric hypersurfaces in spheres as follows:

Theorem (Hui Ma-O.)

Suppose that \((U, K)\) is not of type \(\text{EIII}\), that is, \((U, K) \neq (E_6, U(1) \cdot \text{Spin}(10))\). Then \(L = G(N)\) is NOT Hamiltonian stable if and only if \(m_2 - m_1 \geq 3\). Moreover if \((U, K)\) is of type \(\text{EIII}\), that is, \((U, K) = (E_6, U(1) \cdot \text{Spin}(10))\), then \((m_1, m_2) = (6, 9)\) but \(L = G(N)\) is strictly Hamiltonian stable.
Further questions

1. Investigate the Hamiltonian stability and other properties of the Gauss images of compact non-homogenous isoparametric hypersurfaces, particularly OT-FKM type, embedded in spheres with $g = 4$.

2. Investigate the relation between our Gauss image construction and Karigiannis-Min-Oo’s results.

3. Are there similar constructions of Lagrangian subamnifolds in compact Hermitian symmetric spaces other than $\mathbb{CP}^n$, $Q_n(\mathbb{C})$, generalized flag manifolds or the spaces of oriented geodesics?
Many Thanks !