Introduction to Vertex Algebras and Conformal Field Theory

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References

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- E. Frenkel and Ben-Zvi, "Vertex Algebras and Algebraic Curves," AMS
- Ueno, "Conformal Field Theory with Gauge Symmetry," AMS
- Hotta, Takeuchi and Tanisaki, "D-modules, Perverse Sheaves, and Representation Theory," Birkhäuser
- Dennis Gaitsgory's website

Topics

- Geometric Langlands program
- Alday-Gaiotto-Tachikawa conjecture

Contents

1. Vertex algebras

   (a) Locality of fields

   (b) Definition of vertex algebra

   (c) Current Lie algebras

1.(a) Locality of fields

Notations

M $\mathbb{C}$-vector space

$$M[z] := \{ \sum_{n=0}^{n_0} a_n z^n \mid a_n \in M,\ n_0 \in \mathbb{Z}_{\geq 0} \}$$

$$M[[z]] := \{ \sum_{n=0}^{\infty} a_n z^n \}, \quad M((z)) := \{ \sum_{n=-n_0}^{\infty} a_n z^n \} = M[[z]][z^{-1}]$$

$\mathbb{C}[[z]]$ ring of functions on an infinitesimal disk

$\mathbb{C}((z))$ field of functions on an infinitesimal punctured disk

Fields

$A(z) : M \to M((z))$ **fields** of operators acting on M over a punctured disk Spec $\mathbb{C}((z))$

$$A(z) = \sum_{n \in \mathbb{Z}} A_n z^{-n-1}, \qquad A_n : M \to M$$

$$A_n u = 0 \quad \text{for any} \quad n > n_0(u)$$

In general, $\dim M = \infty$, $A(z) \notin \mathrm{End}(M)((z))$

Equation of motion

$$T : M \to M, \qquad \partial_z = \frac{\partial}{\partial z}$$

$$\partial_z A(z) = [T, A(z)] \quad \text{equation of motion of a field}$$

- Coordinate-free style $\to$ Language of D-modules
  - Kashiwara's category equivalence, de Rham functors

Fourier modes

$$\partial_z \sum A_{n-1} z^{-n} = [T, \sum A_n \, z^{-n-1}]$$

$$-nA_{n-1} = [T, A_n]$$

Essentially,

$$A_0 \leftarrow A_1 \leftarrow A_2 \leftarrow \cdots$$

$$A_{-1} \rightarrow A_{-2} \rightarrow A_{-3} \rightarrow \cdots$$

- Poisson algebras: commutative / Lie bracket
  functions / Hamiltonian vector fields

Composition of fields

For fields $A(z) = \sum A_n z^{-n-1}$, $B(z) = \sum B_n z^{-n-1}$,

$$A(z)B(z)u = \sum_k (\sum_{n \leq n_0} A_{k-n-1} B_n u) z^{-k-1}$$

is not well-defined.

Regularization

$$A(z)B(w) : M \to M((z))((w))$$

$$A(z)B(w)u = \sum_{n \leq n_0} \sum_{m \leq m_0(n)} A_m B_n u \cdot z^{-m-1} w^{-n-1}$$

- Finally, take a limit $z - w \to 0$

Remark

$$M((z))((w)) = \{ \sum_{n \le n_0} \sum_{m \le m_0(n)} u_{m,n} z^{-m-1} w^{-n-1} \}$$

$$M((z, w)) := \{ \sum_{n \le n_0} \sum_{m \le m_0} u_{m,n} z^{-m-1} w^{-n-1} \} = M[[z, w]][(zw)^{-1}]$$

$$M((z, w)) = M((z))((w)) \cap M((w))((z)) \subset M[[z^{\pm 1}, w^{\pm 1}]]$$

Locality

$$A(z),\, B(w) \quad \textbf{mutually local}$$

$$\underset{\text{def}}{\Longleftrightarrow} \quad \exists N \in \mathbb{Z}_{\geq 0}, \quad (z-w)^N [A(z),\, B(w)] = 0$$

- Fundamental idea of vertex algebras and CFT
- It is not so strange because ...

Delta function

$$\delta(z-w) := \sum_{n\in\mathbb{Z}} z^n w^{-n-1} \in \mathbb{C}[[z^{\pm1}, w^{\pm1}]]$$

$$(z-w)\cdot\delta(z-w) = 0$$

$$(z-w)^{k+1}\partial_w{}^k\delta(z-w) = 0$$

Fock space (Kashiwara's category equivalence)

$$[z-w, \partial_w] = 1, \quad z-w : \text{annihilation}, \quad \partial_w : \text{creation}$$

$$|0\rangle = \delta(z-w), \qquad (z-w)|0\rangle = 0$$

Two expansions of $(z-w)^{-1}$

$$\begin{aligned}\varepsilon_{w/z}((z-w)^{-1}) &= \varepsilon_{w/z}(z^{-1}(1-\frac{w}{z})^{-1}) \quad (|z|>|w|) \\ &= \sum_{n\geq 0} w^n z^{-n-1} \in \mathbb{C}((z))((w))\end{aligned}$$

$$\begin{aligned}\varepsilon_{z/w}((z-w)^{-1}) &= \varepsilon_{z/w}(-w^{-1}(1-\frac{z}{w})^{-1}) \quad (|z|<|w|) \\ &= -\sum_{n<0} w^n z^{-n-1} \in \mathbb{C}((w))((z))\end{aligned}$$

$$\delta(z-w) = (\varepsilon_{w/z} - \varepsilon_{z/w})((z-w)^{-1})$$

Product

If $(z-w)^N[A(z),\, B(w)] = 0$,

$$(z-w)^N A(z)B(w)u = (z-w)^N B(w)A(z)u \in M((z,\, w)).$$

$\Rightarrow \exists A(z) \circ B(w) \in \mathrm{Hom}(M,\, M((z,\, w)))[(z-w)^{-1}]$ s. t.

$$\varepsilon_{w/z}(A(z) \circ B(w)) = A(z)B(w)$$
$$\varepsilon_{z/w}(A(z) \circ B(w)) = B(w)A(z)$$

Operator product expansion (**OPE**)

$$\begin{aligned}\varepsilon_{z-w/w}(z^{-1}) &= \varepsilon_{z-w/w}(w^{-1}(1+\frac{z-w}{w})^{-1}) \\ &= \sum_{n\geq 0}(-1)^n(z-w)^n w^{-n-1} \\ &\in \mathbb{C}((w))((z-w))\end{aligned}$$

$$\begin{aligned}\varepsilon_{z-w/w}(A(z)\circ B(w)) &= \sum_{n<N,\, m\in\mathbb{Z}} C_{m,\,n} w^{-m-1}(z-w)^{-n-1} \\ &\in \mathrm{Hom}(M,\ M((w)))((z-w))\end{aligned}$$

Borcherds' identity

Put $z = tw$ (blowing up at $z = w = 0$)

$$A(tw) \circ B(w) \in \mathrm{Hom}(M,\, M[t,\, t^{-1}]((w)))[(t-1)^{-1}]$$

The residue theorem for $A(tw) \circ B(w)\, t^k (t-1)^m dt$ implies

$$\sum_{j \geq 0} \binom{k}{j} C_{n+k-j,\, m+j}$$

$$= \sum_{j \geq 0} (-1)^j \binom{m}{j} (A_{k+m-j} B_{n+j} - (-1)^m B_{n+m-j} A_{k+j})$$

1.(b) Definition of vertex algebra

Vertex algebra $\quad (V, Y_z, T, |0\rangle)$

V $\mathbb{C}$-vector space; $T : V \to V$; vacuum $|0\rangle \in V$

$$Y_z : V \otimes V \to V((z)), \quad Y_z(a \otimes b) = a(z)b = \sum_n a_{(n)}b \cdot z^{-n-1}$$

1. $a(z), b(z)$ mutually local fields

2. $\partial_z a(z) = [T, a(z)]$

3. $a(z)|0\rangle - a \in zV[[z]]$, $T|0\rangle = 0$

Example: commutative vertex algebra

- $V = \mathbb{C}[[t]], \quad T = \partial_t, \quad |0\rangle = 1, \quad Y_z(f(t) \otimes g(t)) = f(t+z)g(t)$
- $V = \mathbb{C}((t)), \quad T = \partial_t, \quad |0\rangle = 1, \quad Y_z(f(t) \otimes g(t)) = \varepsilon_{z/t} f(t+z)g(t)$

$\Rightarrow$ Language of sheaves

Example

Heisenberg Lie algebra

$$0 \to \mathbb{C}c \to \mathcal{H} \to \mathbb{C}((t)) \to 0, \quad [f, g] = \mathrm{Res}_{t=0}[f dg]c \quad (f, g \in \mathbb{C}((t)))$$

$$\widetilde{\mathcal{H}} = U(\mathcal{H})_{c=1}, \quad \pi = \widetilde{\mathcal{H}} \otimes_{\widetilde{\mathcal{H}}_+} \mathbb{C}$$

Examples

- Tensor product

- Enveloping algebra of a vertex Lie algebra

- Construction by screening operators

Proposition

1. $a = a(z)|0\rangle|_{z=0} = a_{(-1)}|0\rangle$. $\quad a \mapsto a(z)$ is injective.

2. $Ta = Ta(z)|0\rangle|_{z=0} = \partial_z a(z)|0\rangle|_{z=0} = a_{(-2)}|0\rangle$. $\quad T$ is unique.

3. $a(z)|0\rangle = e^{zT}a$

4. $\varepsilon_{w/z} a(z-w) = e^{-wT} a(z) e^{wT}$

Skew-symmetry

1. $a(z) \circ b(w)|0\rangle = e^{wT} a(z-w) b = e^{zT} b(w-z) a$

2. $a(z) b = e^{zT} b(-z) a$

Proof

$$\varepsilon_{w/z} a(z) \circ b(w)|0\rangle = \varepsilon_{w/z} e^{wT} a(z-w) b$$

Corollary

1. $Y_z(|0\rangle \otimes a) = a$. $|0\rangle$ is unique.

2. $(Ta)(z) = \partial_z a(z)$. In other words, $(Ta)_{(n)} = -na_{(n-1)}$.

Associativity

$$\varepsilon_{z-w/w}(a(z)\circ b(w)) = (a(z-w)b)(w) = \sum_{n\in\mathbb{Z}} (a_{(n)}b)(w)\cdot(z-w)^{-n-1}$$

Proof

$$\begin{aligned} \varepsilon_{w/z}(a(z)\circ b(w))c &= \varepsilon_{w/z}e^{wT}(a(z-w)\circ c(-w))b \\ \therefore \quad (a(z)\circ b(w))c &= e^{wT}(a(z-w)\circ c(-w))b \\ \therefore \quad \varepsilon_{z-w/w}(a(z)\circ b(w))c &= e^{wT}c(-w)a(z-w)b \\ &= (a(z-w)b)(w)c \end{aligned}$$

Borcherds identity

1. $$\sum_{j\geq 0}\binom{k}{j}(a_{(m+j)}b)_{(n+k-j)}$$
$$=\sum_{j\geq 0}(-1)^j\binom{m}{j}(a_{(k+m-j)}b_{(n+j)}-(-1)^m b_{(n+m-j)}a_{(k+j)})$$

2. $$[a_{(k)}, b_{(n)}]=\sum_{j\geq 0}\binom{k}{j}(a_{(j)}b)_{(n+k-j)}$$

3. $[a_{(0)}, b_{(0)}]=(a_{(0)}b)_{(0)}$ Jacobi identity

Remark

- vertex algebra
  = commutative algebra on $z - w \neq 0$, Lie algebra on $z - w = 0$

- If a binary operation satisfies $a(bc) = b(ac)$, $a1 = a$,

$$\Rightarrow \quad \begin{aligned} ba &= b(a1) = a(b1) = ab \\ a(bc) &= a(cb) = c(ab) = (ab)c. \end{aligned}$$

- Can we start from commutativity and associativity?

1.(c) Current Lie algebra

Lie algebra of coinvariants

$$V/TV \otimes V/TV \to V/TV, \quad [a] \otimes [b] \mapsto [\mathrm{Res}\, a(z) b\, dz] = [a_{(0)} b]$$

- well-definedness $\quad \partial_z a(z) = [T, a(z)] = (Ta)(z)$
- $[a_{(0)} b] = -[b_{(0)} a] \quad (\Leftarrow\ a(z)b = e^{zT} b(-z) a)$
- Jacobi identity $\quad [a_{(0)}, b_{(0)}] = (a_{(0)} b)_{(0)}$

Tensor product

V_1, V_2 vertex algebras $\Rightarrow$ $V_1 \otimes V_2$ vertex algebra

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$$Y_z^{12}((a_1 \otimes a_2) \otimes (b_1 \otimes b_2)) = Y_z^1(a_1 \otimes b_1) \otimes_{\mathbb{C}((z))} Y_z^2(a_2 \otimes b_2)$$

$$T_{12} = T_1 \otimes 1 + 1 \otimes T_2, \quad |0\rangle_{12} = |0\rangle_1 \otimes |0\rangle_2$$

Current Lie algebra

V vertex algebra $\Rightarrow$ $V((t)) = V\hat{\otimes}\mathbb{C}((t))$ vertex algebra

$$\mathrm{Lie}(V) = V((t))/(\partial_t + T)V((t))$$

$$[a\,f(t),\, b\,g(t)] = \mathrm{Res}_{z=0}[a(z)b \cdot f(t+z)g(t)\,dz]$$

$$[a\,t^k,\, b\,t^n] = \sum_{j\geq 0} \binom{k}{j} (a_{(j)}b)\, t^{n+k-j}$$

Current algebra

$$U(V) = U(\mathsf{Lie}(V))/\mathsf{Borcherds\ rel.}$$

<u>Vertex Lie algebra</u> $\quad (L, (T : L \to L), Y_z^-)$

$Y_z^- : L \otimes L \to L((z))/L[[z]] = z^{-1}L[z^{-1}]$

$$Y_z^-(a \otimes b) = a[z]b = \sum_{n \geq 0} (a_{[n]}b)z^{-n-1}$$

- $(Ta)[z] = \partial_z a[z]$
- $a[z]b = e^{zT}b[-z]a \quad \mathrm{mod}\ L[[z]]$
- $[a_{[k]}, b_{[n]}] = \sum_{j=0}^{k} \binom{k}{j} (a_{[j]}b)_{[n+k-j]}$. In other words,

$$\mathrm{Res}_{z=0}[[a[z], b[w]] \cdot f(z)dz] = \mathrm{Res}_{z=w}[a[z-w]b \cdot f(z)dz]$$

Current Lie algebra functor

$$\text{vertex algebras} \underset{\text{forget}}{\longrightarrow} \text{vertex Lie algebras} \underset{\text{Lie}}{\longrightarrow} \text{Lie algebras}$$

Enveloping vertex algebra

L vertex Lie algebra

$$\mathsf{Lie}(L) = L((t))/(\partial_t + T)L((t)), \quad a_{(n)} := a\, t^n$$

$$\mathsf{Lie}_+(L) = L[[t]]/(\partial_t + T)L[[t]]$$

$$\mathsf{Vac}(L) = U(\mathsf{Lie}(L)) \otimes_{U(\mathsf{Lie}_+(L))} \mathbb{C}$$

$$|0\rangle = 1 \otimes 1, \quad (a_{(-1)}|0\rangle)(z) = \sum a_{(n)} z^{-n-1}$$

2. Chiral algebras

   (a) Skew-symmetry and D-modules

   (b) Sheaves and D-modules

   (c) Chiral algebras

2.(a) Skew-symmetry and D-modules

<u>Apparent asymmetry</u>

$$\partial_z Y_z(a \otimes b) = Y_z(Ta \otimes b) = TY_z(a \otimes b) - Y_z(a \otimes Tb)$$

$$Y_z(|0\rangle \otimes a) = a, \quad Y_z(a \otimes |0\rangle) = e^{zT} a$$

$$Y_z(a \otimes b) = e^{zT} Y_{-z}(b \otimes a)$$

Notations

$\mathcal{O} = \mathbb{C}[[z, w]], \quad \Theta = \mathcal{O}\partial_z + \mathcal{O}\partial_w$

$\mathcal{D}$ the ring of differential operators generated by $\mathcal{O} \oplus \Theta$

$\mathcal{O}$ is a left $\mathcal{D}$-module.

Skew-symmetry and D-modules

$(V,\, Y_z,\, T,\, |0\rangle)$ vertex algebra

left $\mathcal{D}$-module $\mathcal{V}^z = V[[z,\, w]]$ by

$$\nabla_z(f \cdot a) = \partial_z f \cdot a + f \cdot Ta, \quad \nabla_w(f \cdot a) = \partial_w f \cdot a$$

left $\mathcal{D}$-module $\mathcal{V}^w = V[[z,\, w]]$ by

$$\nabla_z(f \cdot a) = \partial_z f \cdot a, \quad \nabla_w(f \cdot a) = \partial_w f \cdot a + f \cdot Ta$$

$$(f \in \mathbb{C}[[z,\, w]], \quad a \in V)$$

$e^{(z-w)T} : \mathcal{V}^z \to \mathcal{V}^w$ $\mathcal{D}$-homomorphism:

$$e^{(z-w)T}\nabla_z(a) = e^{(z-w)T}Ta = \partial_z(e^{(z-w)T}a) = \nabla_z(e^{(z-w)T}a)$$

$$e^{(z-w)T}\nabla_w(a) = 0$$

$$\nabla_w(e^{(z-w)T}a) = \partial_w(e^{(z-w)T}a) + Te^{(z-w)T}a = 0$$

The $\mathcal{O}$-homomorphism

$$\mathcal{Y} : \mathcal{V}^z \otimes_{\mathcal{O}} \mathcal{V}^w \to \mathcal{V}^w[(z-w)^{-1}], \quad \mathcal{Y}(a \otimes b) = a(z-w)b \quad (a, b \in V)$$

is a $\mathcal{D}$-homomorphism:

$$\begin{aligned}
\mathcal{Y}(\nabla_z(a \otimes b)) &= \mathcal{Y}(Ta \otimes b) = (Ta)(z-w)b = \partial_z(a(z-w)b) \\
&= \nabla_z \mathcal{Y}(a \otimes b), \\
\mathcal{Y}(\nabla_w(a \otimes b)) &= \mathcal{Y}(a \otimes Tb) = a(z-w)Tb = \partial_w(a(z-w)b) + Ta(z-w)b \\
&= \nabla_w \mathcal{Y}(a \otimes b).
\end{aligned}$$

Similarly, the $\mathcal{O}$-homomorphism

$$\mathcal{Y}' : \mathcal{V}^z \otimes_{\mathcal{O}} \mathcal{V}^w \to \mathcal{V}^z[(z-w)^{-1}], \quad \mathcal{Y}'(a \otimes b) = b(w-z)a$$

is a $\mathcal{D}$-homomorphism.

Skew-symmetry

$$\mathcal{Y} = e^{(z-w)T}\mathcal{Y}' : \mathcal{V}^z \otimes_{\mathcal{O}} \mathcal{V}^w \to \mathcal{V}^w[(z-w)^{-1}]$$

Vacuum

$$|0\rangle : \mathbb{C}[[z]] \to V[[z]]$$

$$\begin{array}{ccc} \mathcal{O} \otimes \mathcal{V}^w & \overset{|0\rangle \otimes id}{\longrightarrow} & \mathcal{V}^z \otimes \mathcal{V}^w \\ \downarrow & \curvearrowright & \downarrow \mathcal{Y} \\ \mathcal{V}^w & \subset & \mathcal{V}^w[(z-w)^{-1}] \end{array}$$

2.(b) Sheaves and D-modules

Sheaves: examples

X complex manifold

$$\mathcal{O}_X : U \longmapsto \mathcal{O}_X(U) = \{\text{holo. functions on } U\}$$
$$\text{open sets} \mapsto \text{vector spaces}$$

$\mathcal{O}_X$ **structure sheaf** of X (analytic/algebraic)

Θ_X sheaf of vector fields on X

Ω_X^k sheaf of k-forms on X

$\mathcal{D}_X$ sheaf of the ring generated by $\mathcal{O}_X \oplus \Theta_X$

Definition of sheaves

1. (presheaf) $(U_1 \supset U_2 \supset U_3) \mapsto (\mathcal{F}(U_1) \to \mathcal{F}(U_2) \to \mathcal{F}(U_3))$

2. $0 \to \mathcal{F}(\bigcup U_\alpha) \to \prod \mathcal{F}(U_\alpha) \rightrightarrows \prod \mathcal{F}(U_\alpha \cap U_\beta)$ exact

Sheaf cohomology

$$0 \to \mathcal{F}' \to \mathcal{F} \quad \text{exact} \quad \Longleftrightarrow \quad \forall U;\ \mathcal{F}'(U) \subset \mathcal{F}(U)$$

$$(\mathcal{F}/\mathcal{F}')(U) \neq \mathcal{F}(U)/\mathcal{F}'(U)$$

$$H^0(X, \mathcal{F}) := \mathcal{F}(X), \quad \mathcal{F}'' = \mathcal{F}/\mathcal{F}',$$

$$0 \to H^0\mathcal{F}' \to H^0\mathcal{F} \to H^0\mathcal{F}'' \to H^1\mathcal{F}' \to H^1\mathcal{F} \to H^1\mathcal{F}'' \to H^2\mathcal{F}' \to \cdots$$

Direct images

$f : X \to Y$ continuous

$\mathcal{F}$ sheaf on X, $f_*\mathcal{F}$ sheaf on Y

$$f_*\mathcal{F}(U) := \mathcal{F}(f^{-1}(U))$$

- $i : S \subset X$ $i_*\mathcal{O}_S$ skyscraper sheaf

Locally free sheaf

$$\mathcal{F} \quad \textbf{locally free} \quad \Longleftrightarrow \quad \mathcal{F}|U_\alpha \cong {\mathcal{O}_X}^{\oplus n}|U_\alpha \quad (X = \bigcup_\alpha U_\alpha)$$

- vector bundle

$\mathcal{D}_X$-modules

$\mathcal{O}_X$ left $\mathcal{D}_X$-modules

$\omega_X = \Omega_X^n$ right $\mathcal{D}_X$-module ($n = \dim X$)

$$\sigma \cdot \xi = L_\xi \sigma = d(\iota_\xi \sigma), \quad \sigma \in \omega_X,\ \xi \in \Theta_X$$

Left and right D-modules

$\pi : X \to S, \quad X = \mathrm{Spec}\,\mathbb{C}[[z]], \quad S = \mathrm{Spec}\,\mathbb{C}$

$\mathcal{V}_X = \mathcal{O}_X \otimes_{\mathbb{C}} V$ left $\mathcal{D}_X$-module

$\mathcal{A}_X = \mathcal{V}_X \otimes \omega_X$ right $\mathcal{D}_X$-module

Vertex algebra as a $\mathcal{D}_X$-module

$$\varDelta : X \to X \times X, \quad j : (X \times X) \smallsetminus \varDelta \to X \times X$$

$$\mathcal{Y} : j_* j^* \mathcal{V}_X \boxtimes \mathcal{V}_X \to \varDelta_+ \mathcal{V}_X = \frac{j_* j^* \mathcal{O}_X \boxtimes \mathcal{V}_X}{\mathcal{O}_X \boxtimes \mathcal{V}_X}$$

- Why "Algebra"? $A \otimes A \to A$

Kashiwara's category equivalence

$i : Y \to X$ closed embedding

$\Rightarrow \quad i_+ : \mathcal{D}_Y\text{-mod} \overset{\sim}{\to} \mathcal{D}_X\text{-mod}_Y$ equivalence of categories

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- $\mathcal{M} \in \mathcal{D}_X\text{-mod}$

$$\mathcal{M} \in \mathcal{D}_X\text{-mod}_Y \iff (U \subset X \setminus Y \Rightarrow \mathcal{M}(U) = 0)$$

Proof   Fock space argument

$$i : Y = \mathrm{Spec}\,\mathbb{C} \underset{x=0}{\longrightarrow} X = \mathrm{Spec}\,\mathbb{C}[x]$$

$$[\partial_x, x] = 1, \quad \partial_x \text{ creation}, \quad x \text{ annihilation}$$

$$M \in \mathcal{D}_Y\text{-mod}(= \mathbb{C}\text{-mod}) \mapsto i_+ M = M \otimes \mathbb{C}[\partial_x] \in \mathcal{D}_X\text{-mod}_Y$$

$$M' \in \mathcal{D}_X\text{-mod}_Y \ (x \text{ nilpotent }) \mapsto i^+ M' = \mathrm{Ker}\, x \in \mathcal{D}_Y\text{-mod}$$

Our case

$$\Delta : X \to X \times X, \quad j : (X \times X) \smallsetminus \Delta \to X \times X$$

$$\mathcal{Y} : j_* j^* \mathcal{V}_X \boxtimes \mathcal{V}_X \to \Delta_+ \mathcal{V}_X$$

$$\Delta_+ : \mathcal{D}_X\text{-mod} \xrightarrow{\sim} \mathcal{D}_{X \times X}\text{-mod}_\Delta$$

Theorem $\mathcal{O}_X$-coherent $\mathcal{D}_X$-modules are locally free.

Idea of proof

$$X = \mathrm{Spec}\,\mathbb{C}[t], \quad Y = \mathrm{Spec}\,\mathbb{C}, \quad i : Y \underset{t=0}{\longrightarrow} X$$

Why $i_*\mathcal{O}_Y$ is not a $\mathcal{D}_X$-module?

$[\partial_t, t] = 1 \;\Rightarrow\;$ no finite dim. rep. of $\mathcal{D}_X(X) = \langle t, \partial_t \rangle$

2.(c) Chiral algebras

Lie* algebra

$\mathcal{A}$ right $\mathcal{D}_X$-module on a curve X

$\mu^- : \mathcal{A} \boxtimes \mathcal{A} \to \Delta_+ \mathcal{A}$

- Skew-symmetry $\mu^- = -\tau \circ \mu^- \circ \tau$
- Jacobi identity $\mu^-_{1\{23\}} = \mu^-_{\{12\}3} + \mu^-_{2\{13\}} : \mathcal{A} \boxtimes \mathcal{A} \boxtimes \mathcal{A} \to \Delta_+ \mathcal{A}$

Chiral algebra

$\mu : j_* j^* \mathcal{A} \boxtimes \mathcal{A} \to \Delta_+ \mathcal{A}$

- Skew-symmetry $\quad \mu = -\tau \circ \mu \circ \tau$

- Jacobi identity $\quad \mu_{1\{23\}} = \mu_{\{12\}3} + \mu_{2\{13\}}$

- unit $\quad \omega_X \to \mathcal{A}$ with

$$\begin{array}{ccc} \omega_X \boxtimes \mathcal{A} & \stackrel{|0\rangle \otimes id}{\to} & \mathcal{A} \boxtimes \mathcal{A} \\ \downarrow & \curvearrowright & \downarrow \mu \\ \Delta_+ \mathcal{A} & = & \Delta_+ \mathcal{A} \end{array}$$

3. Conformal blocks

(a) Broken symmetry of a formal punctured disk

(b) Vertex operator algebras

(c) Vertex algebra bundles

(d) Conformal blocks

3.(a) Broken symmetry of a formal punctured disk

<u>A formal disk and a formal punctured disk</u>

$$\mathcal{O} = \mathbb{C}[[z]], \quad \mathcal{K} = \mathbb{C}((z)) : \text{fraction field of } \mathcal{O}$$

$$\mathbb{C} \leftarrow \mathcal{O} \rightarrow \mathcal{K} \quad \sim \quad \mathbb{F}_p \leftarrow \mathbb{Z}_p \rightarrow \mathbb{Q}_p$$

Broken symmetry of a formal punctured disk

$\text{Aut}\,\mathcal{O} = \text{Aut}\,\mathcal{K} = z\mathcal{O} \setminus z^2\mathcal{O} \curvearrowright \mathcal{O}, \mathcal{K}$ by

$$(z\mathcal{O} \setminus z^2\mathcal{O}) \times \mathcal{K} \to \mathcal{K}, \quad (f(z), g(z)) \mapsto f * g(z) = g(f(z))$$

$$\text{Lie}(\text{Aut}\,\mathcal{K}) = \text{Der}_0\,\mathcal{O} := \mathcal{O}\,z\partial_z \subsetneq \text{Der}\,\mathcal{O} = \mathcal{O}\partial_z \subsetneq \text{Der}\,\mathcal{K} = \mathcal{K}\partial_z$$

Central extension

$$0 \to \mathbb{C} \to \text{Vir}_c \to \text{Der}\,\mathcal{K} \to 0$$

Harish-Chandra pairs

The **Harish-Chandra pair** $(\mathfrak{g}, K)$ is a pair of a complex Lie group K and a K-equivariant Lie algebra $\mathfrak{g}$ with a K-equivariant embedding $\operatorname{Lie} K \subset \mathfrak{g}$.

A $(\mathfrak{g}, K)$-**space** $P \to X$ is a principal K-bundle with a compatible infinitesimally transitive $\mathfrak{g}$-action on P.

A $(\mathfrak{g}, K)$-**module** is a $\mathbb{C}$ vector space with compatible actions of $\mathfrak{g}$, K.

localization functor $(\mathfrak{g}, K)$-mod $\to \mathcal{D}_X$-mod

Example: solution of the Kazhdan-Lusztig conjecture

G simple algebraic group $/\mathbb{C}$

$\mathfrak{g} = \mathrm{Lie}\, G$, $B \subset G$ Borel subgroup

$P = G \to X = G/B$

$$(\mathfrak{g},\, B)\text{-mod} \to \mathcal{D}_X\text{-mod}$$

- ∞-dim. version?

3.(b) Vertex operator algebras

Virasoro algebra

$$0 \to \mathbb{C}c \to \mathrm{Vir}_c \to \mathbb{C}[t,\, t^{-1}]\partial_t \to 0$$

$$[L_n,\, L_m] = (n-m)L_{n+m} + \frac{n^3-n}{12}\delta_{n,\,-m}c$$

$$L_n \longmapsto -t^{n+1}\partial_t$$

Vertex operator algebra

$$V = \bigoplus_{n \geq 0} V[n], \quad \dim V[n] < \infty, \quad V[0] = \mathbb{C}|0\rangle$$

$$\omega \in V[2] \quad \text{Virasoro element}, \quad \omega(z) = \sum L_n z^{-n-2}$$

$$L_{-1} = T, \quad L_0 | V[n] = n$$

$$(\mathrm{Vir}_c,\ \mathrm{Aut}\,\mathcal{K}) \curvearrowright V$$

3.(c) Vertex algebra bundles

<u>Curves with formal coordinates</u>

$X \to S$ smooth (= submersion)

$\widetilde{X} \to X$ formal coordinates of fibers

$$\mathcal{V}_X := \widetilde{X} \times_{\mathsf{Aut}\,\mathcal{K}} V \to X,$$
$$\mathcal{A}_X := \mathcal{V}_X \otimes \omega_{X/S} \quad \text{chiral algebra}$$

3.(d) Conformal blocks

<u>V-modules</u>

V vertex algebra

$$Y_z^M : V \otimes M \to M((z)), \quad Y_z^M(a \otimes u) = a^M(z)u = \sum (a_{(n)}^M u) z^{-n-1}$$

- $|0\rangle^M(z) = \mathrm{id}_M$
- $a^M(z),\ b^M(z)$ mutually local for any $a, b \in V$
- Associativity $\varepsilon_{z-w/w} a^M(z) \circ b^M(w) = (a(z-w)b)^M(w)$

Modules over chiral algebras

$\pi : X \to S$ proper, smooth curves

$s : S \to X$ section

$\mathcal{M}_X = s_+(\mathcal{O}_S \otimes M)$ module over $\mathcal{A}_X$

de Rham functor

$$h : \mathcal{D}_X\text{-mod}^{\mathsf{right}} \to \mathbb{C}_X\text{-mod}, \quad h(\mathcal{M}) = \mathcal{M}/\mathcal{M} \cdot \Theta_X$$

$$DR = Lh \quad \text{left derived functor}$$

Resolutions

$$0 \to \mathcal{D}_X \otimes \bigwedge^n \Theta_X \to \cdots \to \mathcal{D}_X \otimes \bigwedge^0 \Theta_X \to \mathcal{O}_X \to 0,$$

$$0 \to \mathcal{D}_X \otimes \Omega_X^0 \to \cdots \to \mathcal{D}_X \otimes \Omega_X^n \to \omega_X \to 0,$$

Conformal blocks

$\pi : X \to S$ proper, smooth curve

$$R\pi \circ Lh(\mathcal{A}_X \boxtimes \mathcal{M}_X \to \mathcal{M}_X) \in D^b(\mathcal{D}_S\text{-mod})$$

- When is it $\mathcal{O}_S$-coherent?

C_2-finiteness

V vertex algebra

$$C_n(V) := \mathrm{span}\{a_{(-n)}b \mid a,\, b \in V\}$$

$$V \quad C_n\text{-finite} \quad \underset{\text{def}}{\Longleftrightarrow} \quad \dim V/C_n(V) < \infty$$

M V-module

$$C_n(M) := \mathrm{span}\{a_{(-n)}u \mid a \in V,\ u \in M\}$$

$$M \quad C_n\text{-finite} \quad \underset{\text{def}}{\Longleftrightarrow} \quad \dim M/C_n(M) < \infty$$

Remark

$$(Ta)_{(-n)} = na_{(-n-1)} \quad \Rightarrow \quad C_n(M) \supset C_{n+1}(M) \quad (n \geq 1)$$

$$a = a_{(-1)}|0\rangle \quad \Rightarrow \quad C_1(V) = V$$

Fermionic property

$$a_{(-n)}(b_{(-n)}c),\ (a_{(-n)}b)_{(-n)}c \in C_{n+1}(V) \quad (n \geq 2)$$

Corollary V C_2-finite $\iff$ V C_n-finite $(n \geq 2)$

Zhu's theorem

$V/C_2(V)$ is a commutative Poisson algebra.

$$[a][b] := [a_{(-1)}b], \qquad \{[a],\, [b]\} := [a_{(0)}b]$$

Tower of smooth curves

$$\begin{array}{ccccccccc} \dots & \to & X^3 & \to & X^2 & \to & X^1 & = X \\ & & \downarrow & \square & \downarrow & \square & \downarrow & \\ \dots & \to & X^2 & \to & X^1 & \to & X^0 & = S \end{array}$$

- n-point functions

4. Factorization

(a) Stable curves

(b) Log D-modules

(c) Factorization

4.(a) Stable curves

Stable curves

- $\pi : X \to S$ proper curve with nodes $\Sigma \subset X$
- $\pi(\Sigma) = D$ normal crossing divisor
- discrete symmetry

Tower of stable curves

$$\begin{array}{ccccccccc}
\cdots & \to & X^{(3)} & \to & X^{(2)} & \to & X^{(1)} & = X \\
 & & \downarrow & \square' & \downarrow & \square' & \downarrow & \\
\cdots & \to & X^{(2)} & \to & X^{(1)} & \to & X^{(0)} & = S
\end{array}$$

- Deligne-Mumford stacks (= orbifold)

4.(b) log D-modules

<u>Algebras</u>

- D-modules $[\partial, x] = 1$
- log D-modules $[x\partial, x] = x$

Conformal blocks

$$R\pi \circ Lh(\mathcal{A}_X \boxtimes \mathcal{M}_X \to \mathcal{M}_X) \in D^b(\mathcal{D}_S(-\log D)\text{-mod})$$

4.(c) Factorization

Normalization of stable curves

$\pi : (X, \Sigma) \to (S, D)$ stable curve

$D' \to D$ normalization

X' normalization of $\pi^{-1}(D) \times_D D'$

Factorization theorem

$$(R\pi \circ Lh(\mathcal{A}_X \boxtimes \mathcal{M}_X \to \mathcal{M}_X))_{\text{free}}|D'$$

$$= R\pi \circ Lh(\mathcal{A}_{X'} \boxtimes \mathcal{M}_{X'} \boxtimes \mathcal{T} \boxtimes \mathcal{T} \to \mathcal{M}_{X'} \boxtimes \mathcal{T} \boxtimes \mathcal{T})_{\text{free}}$$

$\mathcal{T} = \sum \mathcal{U}[h_L, h_R]$ $(\mathcal{U}, \mathcal{U})$-module, $\mathcal{U}$ current algebra

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- Locally free?