Intersection of stable and unstable manifolds for invariant Morse functions

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March 14, 2011

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1. Witten's Morse theory

Terminology from Morse theory

- M : Compact Riemannian manifold.
- Φ : Morse function.
- $\mathrm{Cr}(\Phi)$: the set of critical points of Φ.
- $\lambda(p)$: the Morse index of $p \in \mathrm{Cr}(\Phi)$

- $\gamma_x(t)$: negative gradient flow w.r.t Φ
  i.e. the solution of

$$\gamma_x(0) = x, \qquad \frac{d}{dt}\gamma_x(t) = -(\mathrm{grad}\,\Phi)_{\gamma_x(t)}$$

- Since Φ is a Morse function,

$$\lim_{t\to\pm\infty} \gamma_x(t) \in \mathrm{Cr}(\Phi)$$

Definition

The unstable manifold $W^u(p)$ and the stable manifold $W^s(p)$ for $p \in \mathrm{Cr}(\Phi)$ are defined by

$$W^u(p) := \left\{ x \in M \,\middle|\, \lim_{t \to -\infty} \gamma_x(t) = p \right\},$$
$$W^s(p) := \left\{ x \in M \,\middle|\, \lim_{t \to \infty} \gamma_x(t) = p \right\}$$

- These are submanifold of M.
- $\dim W^u(p) = \lambda(p), \dim W^s(p) = \dim M - \lambda(p)$.

Definition

Φ : Morse-Smale

$\overset{\text{def}}{\Longleftrightarrow}$ $W^u(p)$ and $W^s(q)$ intersect transversally for all $p, q \in \mathrm{Cr}(\Phi)$.

- If Φ is Morse-Smale,

$$\widetilde{\mathcal{M}}(p,q) := W^u(p) \cap W^s(q)$$

is a $(\lambda(p) - \lambda(q))$-dim submanifold of M.

- It is known that $\widetilde{\mathcal{M}}(p,q) \cup \{p,q\}$ is compact if $\lambda(p) - \lambda(q) = 1$.

In particular, there are only finitely many negative gradient flows which connect p and q.

Witten's Morse theory

- We can calculate the homology of M from a picture.

2. GKM theory

Let $T = (S^1)^n$ be a (compact) torus of rank n.

Definition

Let M be a smooth manifold with an effective T-action. Then M is called GKM manifold if the following conditions are satisfied :

- M^T is a finite set
- M has an T-invariant almost complex structure
- The weights of the isotropy representation of T on T_pM are pairwise linearly independent for all fixed point p.

- These condition implies the followings :

  The union of 0 and 1-orbits forms a balloon art in M

  The action of T on a 1-orbit is given by the rotation of sphere

- We consider the universal T-bundle $ET \longrightarrow BT$.

Definition

We say that M satisfies equivariant formality if the Serre spectral sequence for the fibration $ET \times_T M \longrightarrow BT$ collapses,

- If $H^i(M) = 0$ for all odd integer i, M satisfies the equivariant formality

- Let M be an equivariantly formal GKM manifold
- By a theorem of Goresky-Kottwitz-MacPherson, the equivariant cohomology of M can be described in a combinatorial way.

- We denote by $X(T)$ the character group of T.
- Since the universal T-bundle is a principal T-bundle, we can associate the complex line bundle γ_α on BT for all $\alpha \in X(T)$.
- Using this construction, we have the following isomorphism of $\mathbb{C}$-algebras :

$$\mathrm{Sym}(X(T) \otimes_{\mathbb{Z}} \mathbb{C}) \longrightarrow H^*(BT)$$

- In particular, a character α defines an element of $H^*(BT)$.

Theorem (Goresky-Kottwitz-MacPherson)

Let M be an equivariantly formal GKM manifold.

- The inclusion $M^T \hookrightarrow M$ induces the injection :

$$H_T^*(M) \longrightarrow H_T^*(M^T) = \mathbb{C}[x_1, \cdots, x_n]^{\oplus |M^T|}$$

- The image of the injection is given by

$$\{(f_p)_p \in \mathbb{C}[x_1, \cdots, x_n]^{\oplus |M^T|} | f(p) \equiv f(q) \bmod \alpha_i \ (p, q \in S_i)\}$$

GKM theory

- We can calculate the T-equivariant cohomology of M from the picture.

3. Motivation

- In Witten's Morse theory, it is important to consider the negative gradient flows which connect two critical points p, q such that $\lambda(p) - \lambda(q) = 1$.

- However there are too many examples of Morse functions which have no such pairs.

Example (Flag manifold)

- G : compact semisimple Lie group
- T : maximal torus
- $G_{\mathbb{C}}, T_{\mathbb{C}}$: complexification
- B : Borel subgroup s.t. $T_{\mathbb{C}} \subset B$
- B^{-} : opposite Borel subgroup
- W : Weyl group

- $G = SU(n)$
- $T = \{\mathrm{diag}(t_1, \cdots, t_n) \mid t_1, \cdots, t_n \in S^1\}$
- $G_{\mathbb{C}} = SL_n(\mathbb{C}),\ T_{\mathbb{C}} = \{\ \mathrm{diag}(t_1, \cdots, t_n) \mid t_1 \cdots t_n = 1\ \}$
- $B = \begin{bmatrix} \mathbb{C}^* & * & \cdots & * \\ 0 & \mathbb{C}^* & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & \mathbb{C}^* \end{bmatrix}, \quad B^- = \begin{bmatrix} \mathbb{C}^* & 0 & \cdots & 0 \\ * & \mathbb{C}^* & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ * & \cdots & * & \mathbb{C}^* \end{bmatrix}$

  $(\det = 1)$
- $W \cong S_n$

- The inclusion $G \hookrightarrow G_{\mathbb{C}}$ induces the diffeomorphism

$$G_{\mathbb{C}}/B \longrightarrow G/T$$

- G/T can be identified with a coadjoint orbit $\mathcal{O} \subset \mathrm{Lie}(T)^*$ (moment map)

- $\xi \in \mathrm{Lie}(T)$ defines a function $\Phi_\xi : G_{\mathbb{C}}/B \longrightarrow \mathbb{R}$

Fact

- For generic ξ, Φ_ξ is a Morse-Smale function.
- $\mathrm{Cr}(\Phi_\xi) = W$
- $W^u(p_w) = BwB/B,\ W^s(p_w) = B^- wB/B$ (Bruhat cell)

In particular, we have

$$\lambda(p_w) = \dim W^u(p_w) = \dim BwB/B \in 2\mathbb{Z}.$$

- This phenomenon leads us to the study of the structure of $\widetilde{\mathcal{M}}(p,q)$ for $p,q \in \mathrm{Cr}(\Phi), \lambda(p) - \lambda(q) = 2$

Analogy

- Witten's Morse theory $\longleftrightarrow$ GKM theory
- Critical point $\longleftrightarrow$ Fixed point
- Negative gradient flow $\longleftrightarrow$ 2-sphere
- Homology $\longleftrightarrow$ equivariant cohomology

I want to understand the Morse theoretic aspects of GKM theory

4. Results

Critical points

Proposition

Let G be a compact connected Lie group, M be a compact smooth G-manifold, and $\Phi : M \longrightarrow \mathbb{R}$ be a G-invariant Morse function on M.
Assume that there exist only finitely many G-fixed points on M.
Then we have $\mathrm{Cr}(\Phi) = M^G$

Corollary

Let p_0 be a point of M and H be its stabilizer. Assume the following three conditions:

- H is connected
- $W_H := N_G(H)/H$ is a finite group
- The Fixed point set of the H-action on M is contained in the G-orbit of p_0.

Then, we have

$$\mathrm{Cr}(\Phi) = W_H \cdot p_0$$

for any H-invariant Morse function $\Phi : M \longrightarrow \mathbb{R}$.

- We apply the above corollary to flag manifolds.

Corollary

Let G be a compact Lie group and T be a maximal torus. Then, the critical point set of any T-invariant Morse function on the flag manifold G/T is given by its Weyl group.

Structure of $\widetilde{\mathcal{M}}(p,q)$

Theorem

Let Φ be a G-invariant Morse-Smale function on M. Let p, q be critical points of Φ such that $\lambda(p) - \lambda(q) = 2$.
If M^G is a finite set, every connected component of $\widetilde{\mathcal{M}}(p,q)$ is diffeomorphic to $S^1 \times \mathbb{R}$.

- We have the same picture which appeared in GKM theory

- In the setting of the above theorem, let C be a connected component of $\widetilde{\mathcal{M}}(p,q)$.
- We consider the group action on C.

- Let G be a compact connected Lie group which acts smoothly on S^1.
- We consider the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Lie}(G) & \longrightarrow & \Gamma(TS^1) \\ \downarrow & & \downarrow \\ G & \longrightarrow & \mathrm{Diff}(S^1) \end{array}$$

- Plante gave the classification of the image of the map $\mathrm{Lie}(G) \longrightarrow \Gamma(TS^1)$.

Lemma (Plante)

Let G be a Lie group. If G acts smoothly and transitively on S^1, the image of $\mathrm{Lie}(G) \longrightarrow \Gamma(TS^1)$ is conjugate via a diffeomorphism to one of the following sub Lie algebras of $\Gamma(TS^1)$

- $\left\langle \dfrac{\partial}{\partial x} \right\rangle$
- $\left\langle (1+\cos x)\dfrac{\partial}{\partial x}, (\sin x)\dfrac{\partial}{\partial x}, (1-\cos x)\dfrac{\partial}{\partial x} \right\rangle$

Theorem

There is a surjective group homomorphism $\alpha : G \longrightarrow S^1$ and a diffeomorphism $C \longrightarrow S^1 \times \mathbb{R}$ which satisfies the following :
The action of $G \times \mathbb{R}$ on $C \cong S^1 \times \mathbb{R}$ is given by

$$(g,t) \cdot (x,s) = (\alpha(g)x, t+s)$$

for all $(g,t) \in G \times \mathbb{R}, (x,s) \in S^1 \times \mathbb{R}$.

- This theorem says that

  the action of T on C is given by the rotation of sphere

Corollary

The stabilizer of $c \in C$ is independent of choice of c and is a codimension 1 closed normal Lie subgroup of G.

Thank you !