Intersection of stable and unstable manifolds for invariant Morse functions

Hitoshi Yamanaka

(Osaka City University)

March 14, 2011
Contents

1. Witten’s Morse theory
2. GKM theory
3. Motivation
4. Results
1. Witten’s Morse theory
Terminology from Morse theory

- $M$: Compact Riemannian manifold.

- $\Phi$: Morse function.

- $\text{Cr}(\Phi)$: the set of critical points of $\Phi$.

- $\lambda(p)$: the Morse index of $p \in \text{Cr}(\Phi)$
\( \gamma_x(t) \) : negative gradient flow w.r.t \( \Phi \)
i.e. the solution of
\[
\gamma_x(0) = x, \quad \frac{d}{dt} \gamma_x(t) = - (\text{grad} \, \Phi) \gamma_x(t)
\]

Since \( \Phi \) is a Morse function,
\[
\lim_{t \to \pm \infty} \gamma_x(t) \in \text{Cr}(\Phi)
\]
The unstable manifold $W^u(p)$ and the stable manifold $W^s(p)$ for $p \in \text{Cr}(\Phi)$ are defined by

$$W^u(p) := \left\{ x \in M \mid \lim_{t \to -\infty} \gamma_x(t) = p \right\},$$
$$W^s(p) := \left\{ x \in M \mid \lim_{t \to \infty} \gamma_x(t) = p \right\}.$$ 

- These are submanifolds of $M$.
- $\dim W^u(p) = \lambda(p)$, $\dim W^s(p) = \dim M - \lambda(p)$. 
Definition

\[ \Phi : \text{Morse-Smale} \]

\[ \iff \quad W^u(p) \text{ and } W^s(q) \text{ intersect transversally for all } p, q \in \mathcal{C}(\Phi). \]

- If \( \Phi \) is Morse-Smale,

\[ \tilde{\mathcal{M}}(p, q) := W^u(p) \cap W^s(q) \]

is a \((\lambda(p) - \lambda(q))\)-dim submanifold of \( M \).

- It is known that \( \tilde{\mathcal{M}}(p, q) \cup \{p, q\} \) is compact if \( \lambda(p) - \lambda(q) = 1 \).

In particular, there are only finitely many negative gradient flows which connect \( p \) and \( q \).
Witten’s Morse theory

- We can calculate the homology of $M$ from a picture.
2. GKM theory
Let $T = (S^1)^n$ be a (compact) torus of rank $n$.

**Definition**

Let $M$ be a smooth manifold with an effective $T$-action. Then $M$ is called **GKM manifold** if the following conditions are satisfied:

- $M^T$ is a finite set
- $M$ has an $T$-invariant almost complex structure
- The weights of the isotropy representation of $T$ on $T_pM$ are pairwise linearly independent for all fixed point $p$.

These condition implies the followings:

- The union of 0 and 1-orbits forms a **balloon art** in $M$
- The action of $T$ on a 1-orbit is given by the rotation of sphere
We consider the universal $T$-bundle $ET \to BT$.

**Definition**

We say that $M$ satisfies **equivariant formality** if the Serre spectral sequence for the fibration $ET \times_T M \to BT$ collapses.

- If $H^i(M) = 0$ for all odd integer $i$, $M$ satisfies the equivariant formality.
Let $M$ be an equivariantly formal GKM manifold.

By a theorem of Goresky-Kottwitz-MacPherson, the **equivariant cohomology** of $M$ can be described in a combinatorial way.
We denote by $X(T)$ the character group of $T$.

Since the universal $T$-bundle is a principal $T$-bundle, we can associate the complex line bundle $\gamma_\alpha$ on $BT$ for all $\alpha \in X(T)$.

Using this construction, we have the following isomorphism of $\mathbb{C}$-algebras:

$$\text{Sym}(X(T) \otimes_{\mathbb{Z}} \mathbb{C}) \longrightarrow H^*(BT)$$

In particular, a character $\alpha$ defines an element of $H^*(BT)$.
Theorem (Goresky-Kottwitz-MacPherson)

Let $M$ be an equivariantly formal GKM manifold.

- The inclusion $M^T \hookrightarrow M$ induces the injection:
  $$H_T^*(M) \longrightarrow H_T^*(M^T) = \mathbb{C}[x_1, \ldots, x_n]^\oplus |M^T|$$

- The image of the injection is given by
  $$\{(f_p)_p \in \mathbb{C}[x_1, \ldots, x_n]^\oplus |M^T| | f(p) \equiv f(q) \mod \alpha_i \ (p, q \in S_i)\}$$
GKM theory

- We can calculate the $T$-equivariant cohomology of $M$ from the picture.
3. Motivation
In Witten’s Morse theory, it is important to consider the negative gradient flows which connect two critical points \( p, q \) such that \( \lambda(p) - \lambda(q) = 1 \).

However there are too many examples of Morse functions which have no such pairs.
Example (Flag manifold)

- $G$: compact semisimple Lie group
- $T$: maximal torus
- $G_C, T_C$: complexification
- $B$: Borel subgroup s.t. $T_C \subset B$
- $B^-$: opposite Borel subgroup
- $W$: Weyl group
\begin{itemize}
\item $G = SU(n)$
\item $T = \{ \text{diag}(t_1, \cdots, t_n) \mid t_1, \cdots, t_n \in S^1 \}$
\item $G_{\mathbb{C}} = SL_n(\mathbb{C})$, $T_{\mathbb{C}} = \{ \text{diag}(t_1, \cdots, t_n) \mid t_1 \cdots t_n = 1 \}$
\item $B = \begin{bmatrix} \mathbb{C}^* & * & \cdots & * \\ 0 & \mathbb{C}^* & \cdots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & \mathbb{C}^* \end{bmatrix}$, $B^- = \begin{bmatrix} \mathbb{C}^* & 0 & \cdots & 0 \\ * & \mathbb{C}^* & \cdots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ * & \cdots & * & \mathbb{C}^* \end{bmatrix}$ (det = 1)
\item $W \cong S_n$
\end{itemize}
The inclusion $G \hookrightarrow G_\mathbb{C}$ induces the diffeomorphism

$$G_\mathbb{C}/B \rightarrow G/T$$

$G/T$ can be identified with a coadjoint orbit $\mathcal{O} \subset \text{Lie}(T)^*$ (moment map)

$\xi \in \text{Lie}(T)$ defines a function $\Phi_\xi : G_\mathbb{C}/B \rightarrow \mathbb{R}$
Fact

- For generic \( \xi \), \( \Phi_\xi \) is a Morse-Smale function.
- \( \text{Cr}(\Phi_\xi) = W \)
- \( W^u(p_w) = BwB/B, \; W^s(p_w) = B^{-w}B/B \) (Bruhat cell)

In particular, we have

\[
\lambda(p_w) = \dim W^u(p_w) = \dim BwB/B \in 2\mathbb{Z}.
\]
This phenomenon leads us to the study of the structure of \( \widetilde{\mathcal{M}}(p, q) \) for \( p, q \in C^r(\Phi), \lambda(p) - \lambda(q) = 2 \).
Analogy

- Witten’s Morse theory $\leftrightarrow$ GKM theory
- Critical point $\leftrightarrow$ Fixed point
- Negative gradient flow $\leftrightarrow$ 2-sphere
- Homology $\leftrightarrow$ equivariant cohomology

I want to understand the Morse theoretic aspects of GKM theory
4. Results
Critical points

**Proposition**

Let $G$ be a compact connected Lie group, $M$ be a compact smooth $G$-manifold, and $\Phi : M \to \mathbb{R}$ be a $G$-invariant Morse function on $M$. Assume that there exist only finitely many $G$-fixed points on $M$. Then we have $\text{Cr}(\Phi) = M^G$. 
Corollary

Let $p_0$ be a point of $M$ and $H$ be its stabilizer. Assume the following three conditions:

- $H$ is connected
- $W_H := N_G(H)/H$ is a finite group
- The Fixed point set of the $H$-action on $M$ is contained in the $G$-orbit of $p_0$.

Then, we have

$$\text{Cr}(\Phi) = W_H \cdot p_0$$

for any $H$-invariant Morse function $\Phi : M \rightarrow \mathbb{R}$. 
We apply the above corollary to flag manifolds.

**Corollary**

Let $G$ be a compact Lie group and $T$ be a maximal torus. Then, the critical point set of any $T$-invariant Morse function on the flag manifold $G/T$ is given by its Weyl group.
Structure of $\tilde{\mathcal{M}}(p, q)$

**Theorem**

Let $\Phi$ be a $G$-invariant Morse-Smale function on $\mathcal{M}$. Let $p, q$ be critical points of $\Phi$ such that $\lambda(p) - \lambda(q) = 2$.

If $\mathcal{M}^G$ is a finite set, every connected component of $\tilde{\mathcal{M}}(p, q)$ is diffeomorphic to $S^1 \times \mathbb{R}$.

- We have the same picture which appeared in GKM theory
In the setting of the above theorem, let $C$ be a connected component of $\widetilde{M}(p, q)$.

We consider the group action on $C$. 
Let $G$ be a compact connected Lie group which acts smoothly on $S^1$.

We consider the following commutative diagram:

$$
\begin{array}{ccc}
\text{Lie}(G) & \longrightarrow & \Gamma(TS^1) \\
\downarrow & & \downarrow \\
G & \longrightarrow & \text{Diff}(S^1)
\end{array}
$$

Plante gave the classification of the image of the map $\text{Lie}(G) \longrightarrow \Gamma(TS^1)$. 
Lemma (Plante)

Let $G$ be a Lie group. If $G$ acts smoothly and transitively on $S^1$, the image of $\text{Lie}(G) \to \Gamma(TS^1)$ is conjugate via a diffeomorphism to one of the following sub Lie algebras of $\Gamma(TS^1)$

\[
\begin{align*}
\left\langle \frac{\partial}{\partial x} \right\rangle \\
\left\langle (1 + \cos x) \frac{\partial}{\partial x}, (\sin x) \frac{\partial}{\partial x}, (1 - \cos x) \frac{\partial}{\partial x} \right\rangle
\end{align*}
\]
Theorem

There is a surjective group homomorphism \( \alpha : G \to S^1 \) and a diffeomorphism \( C \to S^1 \times \mathbb{R} \) which satisfies the following:

The action of \( G \times \mathbb{R} \) on \( C \cong S^1 \times \mathbb{R} \) is given by

\[
(g, t) \cdot (x, s) = (\alpha(g)x, t + s)
\]

for all \((g, t) \in G \times \mathbb{R}, (x, s) \in S^1 \times \mathbb{R}\).

This theorem says that

the action of \( T \) on \( C \) is given by the rotation of sphere
Corollary

The stabilizer of $c \in C$ is independent of choice of $c$ and is a codimension 1 closed normal Lie subgroup of $G$. 
Thank you!