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Sol-solitons on Lorentzian manifolds

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1. Introduction
2. Ricci solitons
3. Sol-solitons
4. Examples

1 Introduction

M : (C^∞-)manifold

g : (pseudo-)Riemannian metric

Question

What is the best, or most distinguished metrics on (pseudo-)Riemannian manifold M?

- The Einstein metric

$$\mathrm{Rc}[g_0] = \alpha g_0$$

where $\alpha \in \mathbb{R}$.

- The Ricci soliton

$$-2\mathrm{Rc}[g_0] = L_X g_0 + \alpha g_0$$

where $X \in \chi(M)$, $\alpha \in \mathbb{R}$.

- The Sol-soliton on a solvable Lie group

$$\mathrm{Rc} = cI + D$$

where $D \in \mathrm{Der}(\mathfrak{g}) = \{D \in \mathfrak{gl}(\mathfrak{g}) | D[X,Y] = [DX,Y] + [X,DY]\}$, $c \in \mathbb{R}$.

Theorem 1 (Rahmani 1992). The three-dimensional Heisenberg group admit only three left-invariant Lorentzian metrics up to isometry and scaling, given by

$$g_1 = - dx^2 + dy^2 + (x\ dy + dz)^2 , \qquad (center > 0)$$
$$g_2 = dx^2 + dy^2 - (x\ dy + dz)^2 , \qquad (center < 0)$$
$$g_3 = dx^2 + (x\ dy + dz)^2 - \big((1 - x)dy - dz\big)^2 \quad (center = 0$$

g_2 : negative constant curvature

g_3 : flat

$\Rightarrow$ metric g_1 ?

Theorem 2 (Rahmani 1992, O 2010)**.** The 3-dimensional Heisenberg group admits only three left-invariant Lorentzian metrics up to isometry and scaling. These are a shrinking non-gradient Ricci soliton, a negative constant curvature metric and a flat metric.

g_1 g_2 g_3

Theorem 3 (O11)**.** The 3-dimensional Heisenberg group H_3, the group of rigid motions of Minkowski 2-space $E(1,1)$, the group of rigid motions of Euclidean 2-space $E(2)$ have Lorentzian Ricci solitons which are Lorentzian sol-solitons.

and
N-dim Heisenberg grou H_N $(N \geq 3)$

2 Ricci solitons

Definition 4. Let g_0 be a pseudo-Riemannian metric on manifold M^n. If g_0 satisfies

$$-2\mathrm{Rc}[g_0] = L_X g_0 + \alpha g_0 \ ,$$

where X is some vector field and α is some constant, then (M^n, g_0, X, α) is called a *Ricci soliton structure* and g_0 is called *the Ricci soliton.*

- the vector field X satisfies $X = \nabla f$ where f is some function $\Rightarrow$ g_0 *gradient Ricci soliton*
- $X \neq \nabla f$ for any function $f \Rightarrow g_0$ is a *non-gradient Ricci soliton*
- $\alpha < 0, \ = 0, \ > 0 \Rightarrow g$ is called a shrinking, steady, or expanding Ricci soliton, respectively.

Definition 6. $N^n(c)$:Einstein manifold.
Then, $N^n(c) \times \mathbb{R}^k$ has the (gradient) Ricci soliton structure. This Ricci soliton is called *a rigid Ricci soliton.* ($X = \nabla f$, where $f = \frac{\alpha}{2}|x|^2$)

Theorem 7 (Petersen, Wylie 2009)**.** Homogeneous Riemannian Gradient Ricci solitons are rigid.

Theorem 8 (Batat, Zazquez, Rio, Fernardez 2010)**.** Lorentzian version of Theorem 8 doesn't hold.

Example 9. • Egorov space $(\mathbb{R}^{n+2}, g_f)$

$$g_f(u, v, x_1, \cdots, x_n) = dudv + f(u)\sum_{i=1}^{n}(dx^i)^2$$

• ε-space $(\mathbb{R}^{n+2}, g_\varepsilon)$ $(\varepsilon \neq 0)$

$$g_\varepsilon(u, v, x_1, \cdots, x_n) = \varepsilon \cdot \sum_{i=1}^{n} x_i^2 du^2 + dudv + \sum_{i=1}^{n}(dx^i)^2$$

have Ricci soliton structures.

Those are steady gradient non-rigid Ricci solitons. In particular, the Lorentzian metric on ε-space is homogeneous Lorentzian non-rigid gradient Ricci soliton.

- Ricci solitons are the fixed point of the Ricci flow

$$\frac{\partial}{\partial t} g(t)_{ij} = -2\mathrm{Rc}[g(t)]_{ij} \ , \ g(0) = g_0$$

- $\{\textit{Ricci solitons}\} \supset \{\textit{Einstein metrics}\}$

Example 5. Koiso soliton $\mathbb{C}P^2 \# (-\mathbb{C}P^2)$,
The cigar soliton $\mathbb{R}^2$
(Riemannian metrics)

Example 12. H_3 and $E(1,1)$ have sol-solitons.

The first explicit examples of Ricci soliton structures on Lie groups were constructed by Baird and Danielo (2007) and independently by Lott(2007). (H_3, and $E(1,1)$)

3 Sol-solitons

Definition 10. G : solvable Lie group
g : left-invariant Riemannian metric
$c \in \mathbb{R}$,
$D \in \mathrm{Der}(\mathfrak{g}) = \{D \in \mathfrak{gl}(\mathfrak{g}) | D[X,Y] = [DX,Y] + [X,DY]\}$

$$\mathrm{Rc} = cI + D$$

$\Rightarrow g$: sol-soliton

Theorem 11 (Lauret 2001)**.** g : left-invariant Riemannian metric
g : sol-soliton $\Rightarrow g$: Ricci soliton.

Proof. g : sol-soliton $\Rightarrow \mathrm{Rc} = cI + D$.
We define φ_t, given by

$$d\varphi_t|_e := e^{-tD}$$
$$X_D = \frac{d}{dt}|_0 \varphi_t(p).$$

Then, $L_{X_D} g = \frac{d}{dt}|_0 \varphi_t^* g = -2g(D\cdot, \cdot)$.
Therefore $-2\mathrm{Ric}(\cdot,\cdot) = -2g(\mathrm{Rc}(\cdot),\cdot) = -2g((cI + D)\cdot,\cdot) = -2cg + L_{X_D} g$ □

Definition 13. G : A solvable Lie group
g : A left-invariant pseudo-Riemannian metric
$c \in \mathbb{R}$, $D \in \mathrm{Der}(\mathfrak{g})$

$$\mathrm{Rc} = cI + D$$

$\Rightarrow g$: sol-soliton

Theorem 14 (O 11)**.** g : A left-invariant pseudo-Riemannian metric
g : sol-soliton $\Rightarrow g$: Ricci soliton.

Proof. g : sol-soliton $\Rightarrow \mathrm{Rc} = cI + D$.
We define φ_t, given by

$$d\varphi_t|_e := e^{-tD}$$
$$X_D = \frac{d}{dt}|_0 \varphi_t(p).$$

Then, $L_{X_D} g = \frac{d}{dt}|_0 \varphi_t^* g = -2g(D\cdot, \cdot)$.
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$\underline{H_3}$: The three-dimensional Heisenberg group

The Heisenberg group H_3 defined as a group 3×3 upper triangular matrices

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix},$$

where $x, y, z \in \mathbb{R}$. Topologically, H_3 is diffeomorphic to $\mathbb{R}^3$ under the map

$$H_3 \ni \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mapsto (x, y, z) \in \mathbb{R}^3 .$$

Under this identification, left multiplication by (a, b, c) corresponds to the map

$$L_{(a,b,c)}(x, y, z) = (a + x, b + y, c + z + ay) .$$

Then the Lie algebra of H_3 has a basis consisting of

$$F_1 = \frac{\partial}{\partial z}, \ F_2 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}, \ F_3 = \frac{\partial}{\partial x}, \ [F_2, F_3] = F_1.$$

The dual coframe field is

$$\theta^1 = x\ dy + dz, \ \theta^2 = dy, \ \theta^3 = dx.$$

$$\Rightarrow g_1 = (\theta^1)^2 + (\theta^2)^2 - (\theta^3)^2,$$

$[F_2, F_3] = F_1$.

The Levi-Civita connection of the metric g_1

$$(\nabla_{F_i} F_j) = \frac{1}{2} \begin{pmatrix} 0 & F_3 & F_2 \\ F_3 & 0 & F_1 \\ F_2 & -F_1 & 0 \end{pmatrix}, \qquad (1)$$

Ricci operator

$$(\mathrm{Rc}) = \frac{1}{2} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Remark

$a^1_2 = -1 \Leftrightarrow D = \mathrm{ad} F_3$

$a^1_3 = 1 \Leftrightarrow D = \mathrm{ad} F_2$

$$DF_i = a_i^j F_j, \qquad \begin{pmatrix} a_2^2 + a_3^3 & a_2^1 & a_3^1 \\ 0 & a_2^2 & a_3^2 \\ 0 & a_2^3 & a_3^3 \end{pmatrix} \in \mathrm{Der}(\mathfrak{g})$$

$$\mathrm{Rc} = cI + D \Longleftrightarrow \begin{cases} c & = \frac{3}{2} \\ a_2^2 & = a_3^3 = -1 \\ a_j^i & = 0 \end{cases}$$

g_1: sol-soliton.

Technique1

Let X be a vector field given by

$$X = f^1 F_1 + f^2 F_2 + f^3 F_3.$$

Ricci soliton equations

$$\begin{aligned} -1 + 2f^1_z + \alpha = 0 \ , \\ f^2_z + f^3 + f^1_y - \frac{x}{2}(1-\alpha) = 0 \ , \\ f^1_x - f^3_z - f^2 = 0 \ , \\ 2f^2_y - 2xf^2_z + 1 + \alpha = 0 \ , \\ -f^3_y + xf^3_z + f^2_x = 0 \ , \\ 1 + 2f^3_x + \alpha = 0 \ . \end{aligned}$$

Solve the above equations.

Technique 2

g_1 : Sol-soliton

$$D = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\Longrightarrow e^{-tD} = \begin{pmatrix} e^{2t} & 0 & 0 \\ 0 & e^{t} & 0 \\ 0 & 0 & e^{t} \end{pmatrix} = d\varphi_t|_e$$

$$\Longrightarrow \begin{cases} d\varphi_t|_e(F_1) & = e^{2t}F_1 \\ d\varphi_t|_e(F_2) & = e^{t}F_2 \\ d\varphi_t|_e(F_3) & = e^{t}F_3 \end{cases}$$

$$\Longrightarrow \frac{\partial\varphi_t^1}{\partial x} = e^t, \frac{\partial\varphi_t^2}{\partial y} = e^t, \frac{\partial\varphi_t^3}{\partial z} = e^{2t}.$$

$$\Longrightarrow \varphi_t(p) = (e^t x \ , \ e^t y \ , \ e^{2t} z)$$

$$\Longrightarrow X_D = \frac{d}{dt}|_{t=0}\varphi_t(p) = (xy+2z)F_1+yF_2+xF_3$$

$$\Longrightarrow -2\mathrm{Rc}[g_1] = L_{X_D}g_1 - 3g_1$$

$X \neq \nabla f$.

$\Rightarrow g_1$: the non-gradient Ricci soliton.

$\underline{E(1,1)}$: The group of rigid motions of Minkowski 2-space

$$\begin{pmatrix} \cosh x_1 & \sinh x_1 & x_2 \\ \sinh x_1 & \cosh x_1 & x_3 \\ 0 & 0 & 1 \end{pmatrix} \in E(1,1) \approx \mathbb{R}^3.$$

$$\Longrightarrow \begin{cases} F_1 = \frac{\partial}{\partial x_1} , \\ F_2 = \cosh x_1 \frac{\partial}{\partial x_2} + \sinh x_1 \frac{\partial}{\partial x_3} , \\ F_3 = \sinh x_1 \frac{\partial}{\partial x_2} + \cosh x_1 \frac{\partial}{\partial x_3} . \end{cases}$$

$[F_1, F_2] = F_3, [F_2, F_3] = 0, [F_3, F_1] = -F_2.$

We consider the left-invariant Lorentzian metric g_1 given by

$$g_1 = -(\theta^1)^2 + (\theta^2)^2 + (\theta^3)^2.$$

$$(\mathrm{Rc}) = \frac{1}{2} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & 0 & 0 \\ a_1^2 & a_2^2 & a_3^2 \\ a_1^3 & a_3^2 & a_3^3 \end{pmatrix} \in \mathrm{Der}(\mathfrak{g})$$

$$\mathrm{Rc} = cI + D \Longleftrightarrow \begin{cases} c &= 2 \\ a_2^2 &= a_3^3 = -2 \\ a_j^i &= 0 \end{cases}$$

g_1 : sol-soliton.

The vector field X

$$X = 2(ye^{-x} + ze^{x})F_2 + 2(ye^{-x} - ze^{x})F_3$$

$$\Longrightarrow -2\mathrm{Rc}[g_1] = L_{X_D} g_1 - 4g_1$$

$X \neq \nabla f$.

$\Rightarrow g_1$: the non-gradient Ricci soliton.

$\underline{E(2)}$: The group of rigid motions of Euclidean 2-space.

$$\begin{pmatrix} \cos x & -\sin x & y \\ \sin x & \cos x & z \\ 0 & 0 & 1 \end{pmatrix} \in E(2).$$

$E(2)$ with standard coordinates (x, y, z)

$$\Longrightarrow \begin{cases} F_1 = \frac{\partial}{\partial x} , \\ F_2 = \cos x \frac{\partial}{\partial y} + \sin x \frac{\partial}{\partial z} , \\ F_3 = -\sin x \frac{\partial}{\partial y} + \cos x \frac{\partial}{\partial z} . \end{cases}$$

$[F_1, F_2] = F_3, [F_2, F_3] = 0, [F_3, F_1] = F_2.$

We consider the left-invariant Lorentzian metric g_1 given by

$$g_1 = (\theta^1)^2 + (\theta^2)^2 - (\theta^3)^2.$$

$$\Longrightarrow (\mathrm{Rc}) = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & 0 & 0 \\ a_1^2 & a_2^2 & a_3^2 \\ a_1^3 & -a_3^2 & a_3^3 \end{pmatrix} \in \mathrm{Der}(\mathfrak{g})$$

$$\mathrm{Rc} = cI + D \Longleftrightarrow \begin{cases} c & = 2 \\ a_2^2 & = a_3^3 = -2 \\ a_j^i & = 0 \end{cases}$$

g_1 : sol-soliton.

The vector field X

$$X = 2(y\cos x + z\sin x)F_2 + 2(-y\sin x + z\cos x)F_3$$

$$\Longrightarrow -2\mathrm{Rc}[g_1] = L_{X_D} g_1 - 4g_1$$

$X \neq \nabla f.$

$\Rightarrow g_1$: the non-gradient Ricci soliton.