Estimates for eigenvalues on complete Riemannian manifolds

Qing-Ming Cheng

Saga University

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1. Introduction

Let M be an n-dimensional complete Riemannian manifold, Ω a bounded domain with a piecewise smooth boundary $\partial\Omega$ in M. We consider

a Dirichlet eigenvalue problem of Laplacian

$$\begin{cases} \Delta u = -\lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Δ denotes the Laplacian on M.

This Dirichlet eigenvalue problem (1.1) is also called a fixed membrane problem.

It is well known that the spectrum of this eigenvalue problem (1.1) is real and discrete.

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \to \infty.$$

where each λ_i has finite multiplicity which is repeated according to its multiplicity.

Furthermore, the following Weyl's asymptotic formula holds:

$$\lambda_k \sim \frac{4\pi^2}{(\omega_n \mathrm{vol}\Omega)^{\frac{2}{n}}} k^{\frac{2}{n}}, \quad k \to \infty,$$

where ω_n is the volume of the unit ball in $\mathbf{R}^n$. From the formula, it is not difficult to infer

$$\frac{1}{k}\sum_{i=1}^{k} \lambda_i \sim \frac{n}{n+2} \frac{4\pi^2}{(\omega_n \mathrm{vol}\Omega)^{\frac{2}{n}}} k^{\frac{2}{n}}, \quad k \to \infty.$$

2. Lower bounds for eigenvalues

For a bounded domain in the Euclidean space $\mathbf{R}^n$, Pólya (1961) proved, for $k = 1, 2, \cdots$,

$$\lambda_k \geq \frac{4\pi^2}{(\omega_n \mathrm{vol}\Omega)^{\frac{2}{n}}} k^{\frac{2}{n}},$$

if Ω is a tiling domain in $\mathbf{R}^n$.

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Theorem 2.1 (Li and Yau, Comm. Math. Phys. 1983). If Ω is a bounded domain in $\mathbf{R}^n$, then eigenvalue λ_k of the Dirichlet eigenvalue problem (1.1) satisfies, for $k = 1, 2, \cdots$,

$$\frac{1}{k}\sum_{i=1}^{k}\lambda_i \geq \frac{n}{n+2}\frac{4\pi^2}{(\omega_n \mathrm{vol}\Omega)^{\frac{2}{n}}}k^{\frac{2}{n}},$$

Remark

According to the Weyl's asymptotic formula, we know that the result of Li and Yau is optimal in the sense of average. From this formula, we have, for $k = 1, 2, \cdots$,

$$\lambda_k \geq \frac{n}{n+2} \frac{4\pi^2}{(\omega_n \mathrm{vol}\Omega)^{\frac{2}{n}}} k^{\frac{2}{n}},$$

which gives a partial solution for the conjecture of Pólya with a factor $\frac{n}{n+2}$.

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- a lemma of Hörmander

Lemma of Hömander

If f is a function defined in $\mathbf{R}^n$ satisfying

$$0 \leq f \leq a_1, \quad \int_{\mathbf{R}^n} |z|^2 f(z) dz \leq a_2$$

then, we have

$$\int_{\mathbf{R}^n} f(z) dz \leq \left(a_1 \omega_n\right)^{\frac{2}{n+2}} a_2^{\frac{n}{n+2}} \left(\frac{n+2}{n}\right)^{\frac{n}{n+2}}.$$

where a_1 and a_2 are constant.

Proof of the theorem of Li and Yau

Let u_i be an eigenfunction corresponding to eigenvalue λ_i such that $\{u_i\}$ becomes an orthonormal basis of $L^2(\Omega)$. By defining a function

$$\varphi(x,y) = \begin{cases} \sum_{i=1}^{k} u_i(x)u_i(y), & (x,y) \in \Omega \times \Omega \\ 0, & \text{the other,} \end{cases}$$

and

$$f(z) = \int_{\mathbf{R}^n} |\hat{\varphi}(z,y)|^2 dy,$$

where $\hat{\varphi}(z,y)$ is the Fourier transform of $\varphi(x,y)$ in x,

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Thus, from the lemma of Hörmander, proof of the theorem of Li and Yau is finished.

A generalized conjecture of Pólya

For a complete Riemannian manifold M, eigenvalues of the Dirichlet eigenvalue problem (1.1) also satisfy the Weyl's asymptotic formula. Hence, it is natural to try to obtain a lower bound for eigenvalues.

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For a complete Riemannian manifold M, eigenvalues of the Dirichlet eigenvalue problem (1.1) also satisfy the Weyl's asymptotic formula. Hence, it is natural to try to obtain a lower bound for eigenvalues.

In fact, for a complete Riemannian manifold, by making use of the Sobolev constant, Li (1980), Chavel-Feldman (1981) proved

$$\lambda_k \geq \begin{cases} c(n,s)\dfrac{k^{\frac{1}{n-1}}}{(vol\Omega)^{\frac{2}{n}}}, & n > 2 \\ c(n,s)\dfrac{k^{\frac{1}{3}}}{vol\Omega}, & n = 2 \end{cases}$$

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- Whether is it possible for one to consider the same problem as the conjecture of Pólya for a complete Riemannian manifold other than $\mathbf{R}^n$?
- First of all, a difficulty which we will encounter is that there is no the Fourier transform for a complete Riemannian manifold.

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In order to consider the same problem as the conjecture of Pólya, we need to come over the following problems:

- What method will we use to replace the Fourier transform?

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Therefore, our purpose is to study the lower bounds with the optimal order of k for eigenvalues of the Dirichlet eigenvalue problem of Laplacian on a bounded domain in complete Riemannian manifolds.

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First of all, we will propose the following version of the conjecture of Pólya for complete Riemannian manifolds.

The generalized conjecture of Pólya

Let Ω be a bounded domain in an n-dimensional complete Riemannian manifold M. Then, there exists a constant $c(M, \Omega)$, which only depends on M and Ω such that eigenvalues of the Dirichlet eigenvalue problem (1.1) satisfy, for $k = 1, 2, \cdots$,

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$$\frac{1}{k}\sum_{i=1}^{k} \lambda_i + c(M,\Omega) \geq \frac{n}{n+2} \frac{4\pi^2}{(\omega_n \mathrm{vol}\Omega)^{\frac{2}{n}}} k^{\frac{2}{n}}.$$

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Let Ω be a bounded domain in an n-dimensional complete Riemannian manifold M. Then, there exists a constant H_0^2, which only depends on M and Ω such that eigenvalues of the Dirichlet eigenvalue problem (1.1) satisfy, for $k = 1, 2, \cdots$,

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$$\frac{1}{k}\sum_{i=1}^{k}\lambda_i + \frac{n^2}{4}H_0^2 \geq \frac{n}{\sqrt{(n+2)(n+4)}}\frac{4\pi^2}{(\omega_n \mathrm{vol}\Omega)^{\frac{2}{n}}}k^{\frac{2}{n}},$$

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- What method will we use to replace the lemma of Hörmander?
  The answer is a recursion formula of Cheng and Yang.

A recursion formula (Cheng and Yang, Math. Ann. 2007)

Let $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_{k+1}$ be any non-negative real numbers satisfying

$$\sum_{i=1}^{k}(\mu_{k+1}-\mu_i)^2 \leq \frac{4}{t}\sum_{i=1}^{k}\mu_i(\mu_{k+1}-\mu_i).$$

Define

$$G_k = \frac{1}{k}\sum_{i=1}^{k}\mu_i, \quad T_k = \frac{1}{k}\sum_{i=1}^{k}\mu_i^2,$$

$$F_k = \left(1+\frac{2}{t}\right)G_k^2 - T_k.$$

Then, we have the following recursion formula

$$F_{k+1} \leq C(t,k)\left(\frac{k+1}{k}\right)^{\frac{4}{t}} F_k,$$

where t is any positive real number and

$$C(t,k) = 1 - \frac{1}{3t}\left(\frac{k}{k+1}\right)^{\frac{4}{t}} \frac{\left(1+\frac{2}{t}\right)\left(1+\frac{4}{t}\right)}{(k+1)^3} < 1.$$

Remark

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- By making use of the recursion formula of Cheng and Yang, we can not only derive lower bounds for eigenvalues, but also derive upper bounds for eigenvalues.

Universal inequalities for eigenvalues

The case of a Euclidean space

First of all, we consider universal inequalities for eigenvalues of the Dirichlet eigenvalue problem (1.1) for a bounded domain Ω in an n-dimensional Euclidean space.

The investigation of universal inequalities for eigenvalues of the Dirichlet eigenvalue problem (1.1) was initiated by Payne, Pólya and Weinberger (1955 and 1956). They proved

$$\lambda_{k+1} - \lambda_k \leq \frac{4}{nk} \sum_{i=1}^{k} \lambda_i.$$

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In fact, Hile and Protter (1980) improved this result of Payne, Pólya and Weinberger to

$$\sum_{i=1}^{k} \frac{\lambda_i}{\lambda_{k+1} - \lambda_i} \geq \frac{nk}{4}.$$

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$$\sum_{i=1}^{k} \frac{\lambda_i}{\lambda_{k+1} - \lambda_i} \geq \frac{nk}{4}.$$

Yang (1991) proved a very sharp inequality:

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq \frac{4}{n} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)\lambda_i.$$

The case of a unit sphere

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Theorem 2.3 (Cheng and Yang, Math Ann. 2005). Let Ω be a domain in an n-dimensional unit sphere. Eigenvalue λ_i's of the Dirichlet eigenvalue problem (1.1) satisfy

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \leq \frac{4}{n}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)(\lambda_i+\frac{n^2}{4}).$$

Remark

The above inequality is best possible since this inequality does not depend on the domain Ω and when Ω tends to the unit sphere, this inequality becomes an equality for all of k.

The case of a hyperbolic space

When M is $H^n(-1)$, although many mathematicians want to derive a universal inequality for eigenvalues, there are no any results on universal inequalities for eigenvalues of the Dirichlet eigenvalue problem (1.1) excepting $n = 2$.

If $n = 2$, by making use of estimates for eigenvalues of the eigenvalue problem of the Schrödinger like operator with a weight, Harrell and Michel (Comm. PDEs, 1994) and Ashbaugh (2002) have obtained several results. In fact, if $n = 2$, the Laplacian on $H^2(-1)$ is like to the Laplacian on $\mathbf{R}^2$ with a weight. But, when $n > 2$, this property does not hold again.

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Recently, Cheng and Yang find an appropriate trial function for $\boldsymbol{H^n(-1)}$. Hence, we can derive a universal inequality for eigenvalues of the Dirichlet eigenvalue problem (1.1), that is, we prove the following:

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Theorem 2.4 (Cheng and Yang, J. Diff. Eqns, 2009)

For a bounded domain Ω in $H^n(-1)$, eigenvalue λ_i's of the Dirichlet eigenvalue problem (1.1) satisfy

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \leq 4\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)(\lambda_i-\frac{(n-1)^2}{4}).$$

An application of theorem 2.4

We consider an application of our universal inequality. Let Ω be an n-disk of radius $r > 0$ in $H^n(-1)$. McKean (J. Diff. Geom. 1970) proved that the first eigenvalue $\lambda_1(r)$ of the Dirichlet eigenvalue problem (1.1) satisfies

$$\lambda_1(r) > \frac{(n-1)^2}{4},$$

$$\lim_{r\to\infty} \lambda_1(r) = \frac{(n-1)^2}{4}.$$

From domain monotonicity of eigenvalues, we have, for any bounded domain Ω in $H^n(-1)$,

$$\lambda_1(\Omega) > \frac{(n-1)^2}{4},$$

$$\lim_{\Omega \to H^n(-1)} \lambda_1(\Omega) = \frac{(n-1)^2}{4}.$$

It is obvious that, for any $k > 1$,

$$\lambda_k(\Omega) > \lambda_1(\Omega) > \frac{(n-1)^2}{4}.$$

It would be interesting to study behaviors of $\lambda_k(\Omega)$, for $k \geq 2$, when Ω tends to $H^n(-1)$. As an application of our universal inequality, we can prove the following:

Proposition 2.1 (Cheng and Yang, J. Diff. Eqns, 2009)

Let Ω be a bounded domain in $H^n(-1)$. Then, eigenvalue $\lambda_k(\Omega)$, for any k, of the Dirichlet eigenvalue problem (1.1) satisfies

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In order to prove this result, we need to investigate upper bounds for eigenvalues.

The general case

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The general case

For a general n-dimensional complete Riemannian manifold, we want to obtain universal inequalities for eigenvalues.

In the cases of a Euclidean space and a unit sphere, one can make use of the coordinate functions to construct the trial functions. Thus, one can derive universal inequalities for eigenvalues according to the Rayleigh-Ritz inequality.

But for the hyperbolic space, it has been very difficult to construct an appropriate trial function.

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Nash's Theorem

Each complete Riemannian manifold can be isometrically immersed in a Euclidean space.

Theorem 2.5 (Chen and Cheng, J. Math. Soc. Japan, 2008)

Let Ω be a bounded domain in an n-dimensional complete Riemannian manifold M. Then, there exists a constant H_0^2, which only depends on M and Ω such that eigenvalues λ_j of the Dirichlet eigenvalue problem (1.1) satisfy, for any k,

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \leq \frac{4}{n}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)(\lambda_i+\frac{n^2}{4}H_0^2).$$

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We should remark that El Soufi, Harrell and Ilias (Trans. Amer. Math. Soc. 2009) have also proved a similar result, independently.

In particular, when M is a complete minimal submanifold in $\mathbf{R}^N$, we have

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Corollary 2.3

Let Ω be a bounded domain in an n-dimensional complete minimal submanifold M^n in $\mathbf{R}^N$. Then, we have

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \leq \frac{4}{n}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\lambda_i.$$

Remark

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- There exist many complete minimal submanifolds in $\mathbf{R}^N$.
- The universal inequality for eigenvalues of Yang does not only holds for bounded domains in $\mathbf{R}^n$, but also for bounded domains in any complete minimal submanifold in $\mathbf{R}^N$.

A general inequality

Proposition 2.2

Let M be a complete Riemannian manifold and Ω a bounded domain with a piecewise smooth boundary $\partial\Omega$. For any function $f \in C^3(\Omega) \cap C^2(\partial\Omega)$ eigenvalues λ_i of the Dirichlet eigenvalue problem (1.1) satisfy

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\|u_i\nabla f\|^2 \leq \sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\|2\nabla f\cdot\nabla u_i+u_i\Delta f\|^2$$

where $\|f\|^2 = \int_M f^2$ and $\nabla f \cdot \nabla u_i = g(\nabla f, \nabla u_i)$ and u_i is an orthonormal eigenfunction corresponding to λ_i.

Proof. Assume that u_i is an orthonormal eigenfunction corresponding to the ith eigenvalue λ_i, i.e. u_i satisfies

$$\begin{cases} \Delta u_i = -\lambda_i u_i, \text{ in } \Omega, \\ u_i|_{\partial\Omega} = 0, \\ \displaystyle\int_\Omega u_i u_j = \delta_{ij}. \end{cases}$$

Defining φ_i, a_{ij} and b_{ij}, for $i, j = 1, \cdots, k$, by

$$a_{ij} = \int_M f u_i u_j, \quad b_{ij} = \int_M u_j(\nabla u_i \cdot \nabla f + \frac{1}{2} u_i \Delta f),$$

$$\varphi_i = f u_i - \sum_{j=1}^{k} a_{ij} u_j.$$

we have

$$a_{ij} = a_{ji}, \quad \int_M \varphi_i u_j = 0, \text{ for } j = 1, 2, \cdots, k.$$

Thus, we have, from Rayleigh-Ritz formula,

$$\lambda_{k+1} \|\varphi_i\|^2 \leq \int_M |\nabla \varphi_i|^2.$$

By a direct computation, we infer

$$(\lambda_{k+1} - \lambda_i)\|\varphi_i\|^2 \leq -\int_M \varphi_i (2\nabla f \cdot \nabla u_i + u_i \Delta f) \equiv w_i,$$

On the other hand, we have

$$w_i = -\int_M \varphi_i(2\nabla f \cdot \nabla u_i + u_i \Delta f - 2\sum_{j=1}^{k} b_{ij} u_j).$$

From Schwarz inequality, we obtain

$$\begin{aligned}
&(\lambda_{k+1} - \lambda_i) w_i^2 \\
&\leq (\lambda_{k+1} - \lambda_i) \|\varphi_i\|^2 \|2\nabla f \cdot \nabla u_i + u_i \Delta f - 2\sum_{j=1}^{k} b_{ij} u_j\|^2 \\
&\leq w_i \|2\nabla f \cdot \nabla u_i + u_i \Delta f - 2\sum_{j=1}^{k} b_{ij} u_j\|^2.
\end{aligned}$$

Hence,

$$(\lambda_{k+1} - \lambda_i)w_i \leq \|2\nabla f \cdot \nabla u_i + u_i \Delta f\|^2 - 4\sum_{j=1}^{k} b_{ij}^2.$$

Multiplying the above inequality by $(\lambda_{k+1} - \lambda_i)$ and taking sum on i from 1 to k, we have

$$\sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)^2 w_i \leq -4\sum_{i,j=1}^{k}(\lambda_{k+1} - \lambda_i)b_{ij}^2$$

$$+ \sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)(\|2\nabla f \cdot \nabla u_i + u_i \Delta f\|^2.$$

Since

$$2b_{ij} = (\lambda_i - \lambda_j)a_{ij}$$

and

$$w_i = \|u_i \nabla f\|^2 + \sum_{j=1}^{k} (\lambda_i - \lambda_j) a_{ij}^2,$$

we obtain

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \|u_i \nabla f\|^2$$

$$\leq \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \|2\nabla f \cdot \nabla u_i + u_i \Delta f\|^2.$$

Proof of theorem 2.5

Let M be an n-dimensional complete Riemannian manifold with metric g. From Nash's theorem, there exists an isometrical immersion from M into $\mathbf{R}^N$. For any point P in M, assuming that y with components y^α is the position vector of P in $\mathbf{R}^N$, we have, for any function $u \in C^\infty(M)$,

$$\sum_{\alpha=1}^{N} g(\nabla y^\alpha, \nabla y^\alpha) = n, \quad \sum_{\alpha=1}^{N} (\Delta y^\alpha)^2 = n^2|H|^2$$

$$\sum_{\alpha=1}^{N} \Delta y^\alpha \nabla y^\alpha = 0, \quad \sum_{\alpha=1}^{N} g(\nabla y^\alpha, \nabla u)^2 = |\nabla u|^2,$$

where H is the mean curvature vector field of this immersion.
Letting $f = y^\alpha$ in the proposition 2.2 and taking sum on α from 1 to N, we obtain

$$\sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)^2 \leq \frac{4}{n}\sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)(\lambda_i + \frac{n^2}{4}\sup_{\Omega}|H|^2).$$

Since eigenvalues of the Dirichlet eigenvalue problem of the Laplacian are invariant under isometric transformations. Defining $H_0^2 = \inf_{\phi\in\Phi}\sup_{\Omega}|H|^2$, where Φ is the set of all isometric immersions from M into Euclidean spaces, we prove our result.

Theorem 2.2 (Cheng and Yang, J. Diff. Eqns., 2009)

Let Ω be a bounded domain in an n-dimensional complete Riemannian manifold M. Then, there exists a constant H_0^2, which only depends on M and Ω such that eigenvalues of the Dirichlet eigenvalue problem (1.1) satisfy, for $k = 1, 2, \cdots$,

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$$\frac{1}{k}\sum_{i=1}^{k}\lambda_i + \frac{n^2}{4}H_0^2 \geq \frac{n}{\sqrt{(n+2)(n+4)}}\frac{4\pi^2}{(\omega_n \mathrm{vol}\Omega)^{\frac{2}{n}}}k^{\frac{2}{n}},$$

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Proof of the theorem 2.2

From the theorem 2.5, we have

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \leq \frac{4}{n}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)(\lambda_i+\frac{n^2}{4}H_0^2).$$

Letting $\mu_i = \lambda_i + \frac{n^2}{4}H_0^2$, we have

$$\sum_{i=1}^{k}(\mu_{k+1}-\mu_i)^2 \leq \frac{4}{n}\sum_{i=1}^{k}(\mu_{k+1}-\mu_i)\mu_i.$$

From the recursion formula of Cheng and Yang with $t = n$, we infer

$$\frac{F_{k+\ell}}{(k+\ell)^{\frac{4}{n}}} \leq \frac{F_k}{k^{\frac{4}{n}}}.$$

According to the Weyl's asymptotic formula, we derive, for any positive integer k,

$$\frac{F_k}{k^{\frac{4}{n}}} \geq \frac{2n}{(n+2)(n+4)} \frac{16\pi^4}{(\omega_n \mathrm{vol}\Omega)^{\frac{4}{n}}}.$$

Since

$$F_k \leq \frac{2}{n} G_k^2,$$

we infer

$$\frac{1}{k}\sum_{i=1}^{k}\lambda_i + \frac{n^2}{4}H_0^2 \geq \frac{n}{\sqrt{(n+2)(n+4)}}\frac{4\pi^2}{(\omega_n \mathrm{vol}\Omega)^{\frac{2}{n}}}k^{\frac{2}{n}}.$$

3. Upper bounds for eigenvalues

First, for a bounded domain in the Euclidean space $\mathbf{R}^n$, according to the partial solution of the conjecture of Pólya due to Li and Yau, we have, for $k = 1, 2, \cdots$,

$$\lambda_k \geq \frac{n}{n+2} \frac{4\pi^2}{(\omega_n \mathrm{vol}\Omega)^{\frac{2}{n}}} k^{\frac{2}{n}}$$

and from the Weyl's asymptotic formula, we know

$$\lambda_k \sim \frac{4\pi^2}{(\omega_n \mathrm{vol}\Omega)^{\frac{2}{n}}} k^{\frac{2}{n}}.$$

Hence, it is very important to obtain upper bounds for eigenvalues with the optimal order of k.

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A recursion formula (Cheng and Yang, Math. Ann. 2007)

Let $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_{k+1}$ be any non-negative real numbers satisfying

$$\sum_{i=1}^{k}(\mu_{k+1}-\mu_i)^2 \leq \frac{4}{t}\sum_{i=1}^{k}\mu_i(\mu_{k+1}-\mu_i).$$

Define

$$G_k = \frac{1}{k}\sum_{i=1}^{k}\mu_i, \quad T_k = \frac{1}{k}\sum_{i=1}^{k}\mu_i^2,$$

$$F_k = \left(1+\frac{2}{t}\right)G_k^2 - T_k.$$

Then, we have the following recursion formula

$$F_{k+1} \leq C(t,k)\left(\frac{k+1}{k}\right)^{\frac{4}{t}} F_k,$$

where $t > 0$ is any positive number.

By making use of the recursion formula, we have proved

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Theorem 3.1 (Cheng and Yang, Math. Ann. 2007)

For a bounded domain $\Omega \subset \mathbf{R}^n$, eigenvalues of the Dirichlet eigenvalue problem (1.1) satisfy, for any k,

$$\lambda_{k+1} \leq C_0(n) k^{\frac{2}{n}} \lambda_1,$$

where

$$C_0(n) \leq 1 + \frac{2.6}{n}$$

if $k > 1$.

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our upper bound is best possible in the sense of order of k and it is universal inequality.

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Theorem 3.2 (Chen and Cheng, 2008)

For a bounded domain Ω in an n-dimensional complete Riemannian manifold, there exists a constant H_0^2 such that eigenvalues of the Dirichlet eigenvalue problem (1.1)satisfy

$$\lambda_{k+1} \leq C_0(n)k^{\frac{2}{n}}(\lambda_1 + \frac{n^2}{4}H_0^2),$$

Next, we consider the lower order eigenvalues of the Dirichlet eigenvalue problem of the Laplacian.

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Conjecture of PPW

For a bounded domain $\Omega \subset \mathbf{R}^n$, eigenvalues of the Dirichlet eigenvalue problem (1.1) satisfy

- (1) $\dfrac{\lambda_2}{\lambda_1} \leq \dfrac{\lambda_2}{\lambda_1}\Big|_{n\text{–ball}}$,

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- (1) $\dfrac{\lambda_2}{\lambda_1} \leq \dfrac{\lambda_2}{\lambda_1}\Big|_{n\text{–ball}},$
- (2) $\dfrac{\lambda_2 + \lambda_3 + \cdots + \lambda_{n+1}}{\lambda_1} \leq n\dfrac{\lambda_2}{\lambda_1}\Big|_{n\text{–ball}},$

For the conjecture (1) of Payne, Pólya and Weinberger, many mathematicians studied it. For examples, Payne, Pólya and Weinberger, Brands, de Vries, Chiti, Hile and Protter, Marcellini and so on. Finally, Ashbaugh and Benguria (1992, Ann. of Math.) solved this conjecture.

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For conjecture (2) of Payne, Pólya and Weinberger,

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For conjecture (2) of Payne, Pólya and Weinberger, when $n = 2$, Payne, Pólya and Weinberger proved $\frac{\lambda_2 + \lambda_3}{\lambda_1} \leq 6$. Brands improved on the bound onto $\frac{\lambda_2 + \lambda_3}{\lambda_1} \leq 3 + \sqrt{7}$. Furthermore, Hile and Protter obtained $\frac{\lambda_2 + \lambda_3}{\lambda_1} \leq 5.622$.

Marcellini proved $\dfrac{\lambda_2 + \lambda_3}{\lambda_1} \leq (15 + \sqrt{345})/6$. Very recently, Chen and Zheng (J. Diff. Eqns., 2011) have proved $\dfrac{\lambda_2 + \lambda_3}{\lambda_1} \leq 5.3507$.

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For a general dimension $n \geq 2$, Ashbaugh and Benguria proved

$$\frac{\lambda_2+\lambda_3+\cdots+\lambda_{n+1}}{\lambda_1} \leq n+4.$$

Furthermore, Ashbaugh and Benguria (1994) (cf. Hile and Protter) improved the above result to

$$\frac{\lambda_2+\lambda_3+\cdots+\lambda_{n+1}}{\lambda_1} \leq n+3+\frac{\lambda_1}{\lambda_2}.$$

For a general complete Riemannian manifold M^n, Chen and Cheng (J. Math. Soc. Japan 2008) have proved that there exists a non-negative constant H_0 such that

$$\frac{\lambda_2 + \lambda_3 + \cdots + \lambda_{n+1}}{\lambda_1} \leq n + 4 + \frac{n^2 H_0^2}{\lambda_1}.$$

Recently, by making use of the fact that eigenfunctions form an orthonormal basis of $L^2(\Omega)$ in place of the Rayleigh-Ritz formula, Cheng and Qi have improved the above result to the following:

Theorem 3.3 (Cheng and Qi)

Let M^n be an n-dimensional Riemannian manifold, $\Omega \subset M^n$ a bounded domain with a piecewise smooth boundary $\partial\Omega$. Then, the lower order eigenvalues of the Dirichlet eigenvalue problem of the Laplacian satisfy

$$\frac{\lambda_2 + \lambda_3 + \cdots + \lambda_{n+1}}{\lambda_1} \leq n + \sqrt{\left(\frac{n^2 H_0^2}{\lambda_1} + 4\right) Q(H_0, \lambda_1, \lambda_2)}$$

where H_0 is a non-negative constant depending on M and Ω only, and

$$Q(H_0, \lambda_1, \lambda_2) = \frac{(2 - \frac{\lambda_1}{\lambda_2})\frac{n^2 H_0^2}{\lambda_1} + 3 + \frac{\lambda_1}{\lambda_2}}{2}$$

$$+ \frac{\sqrt{(3 + \frac{\lambda_1}{\lambda_2} + \frac{n^2 H_0^2}{\lambda_2})^2 + 4(1 - \frac{\lambda_1}{\lambda_2})\frac{n^2 H_0^2}{\lambda_2}}}{2}.$$

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$$+ \frac{\sqrt{(3 + \frac{\lambda_1}{\lambda_2} + \frac{n^2 H_0^2}{\lambda_2})^2 + 4(1 - \frac{\lambda_1}{\lambda_2})\frac{n^2 H_0^2}{\lambda_2}}}{2}.$$

It is easy to prove

$$Q(H_0, \lambda_1, \lambda_2) < \frac{n^2 H_0^2}{\lambda_1} + 4$$

In particular, when M^n is a complete minimal submanifold in the Euclidean space $\mathbf{R}^N$, we have

Corollary 3.1

Let Ω be a bounded domain in an n-dimensional complete minimal submanifold M^n in $\mathbf{R}^N$. Then, we have

$$\frac{\lambda_2 + \lambda_3 + \cdots + \lambda_{n+1}}{\lambda_1} \leq n + 2\sqrt{3 + \frac{\lambda_1}{\lambda_2}}.$$

Thank you!