The Poisson-Nernst-Planck (PNP) system for ion transport

Tai-Chia Lin
National Taiwan University

3rd OCAMI-TIMS Workshop in Japan, Osaka, March 13-16, 2011
Background

- Ion transport is crucial in the study of many physical and biological problems, such as
  - Semiconductors,
  - Electro-kinetic fluids,
  - Transport of electrochemical systems and
  - Ion channels in cell membranes
Ion transport (IT)

- Movement of salts and other electrolytes in the form of ions from place to place within living systems
- Ions may travel by themselves or as a group of two or more ions in the same or opposite directions
- The movement of ions across cell membranes through ion channels
Cell Membranes surround all biological cells.
Ion channels are pores in cell membranes and the gatekeepers for cells to control the movement of anions (陰離子) and cations (陽離子) across cell membranes.
Information of ion channels

- Each channel can transport 1 million to 100 million ions per second (10^{-10} to 10^{-12} amperes).
- Close and open within a millisecond.
- Action potential: -70mV to 50mV

Continuous model is reasonable for the open channel
Several kinds of ion channels:

- **Ca$$^{2+}$$**
  - Concentration: 2.5 mM
  - Extracellular to Intracellular movement
  - Site: Cell membrane
  - Effect: Nernst potential (E_{rev}) = +150 mV

- **Na$$^+$$**
  - Concentration: 142 mM
  - Intracellular to Extracellular movement
  - Site: Cell membrane
  - Effect: Nernst potential (E_{rev}) = +70 mV

- **Cl$$^-$$**
  - Concentration: 101 mM
  - Extracellular to Intracellular movement
  - Site: Cell membrane
  - Effect: Nernst potential (E_{rev}) = −30 to −65 mV

- **K$$^+$$**
  - Concentration: 4 mM
  - Intracellular to Extracellular movement
  - Site: Cell membrane
  - Effect: Nernst potential (E_{rev}) = −98 mV

**Control mechanisms**

- **Gating**
  - Voltage
  - Time
  - Direct agonist
  - G protein
  - Calcium

- **Modulation**
  - Increases in phosphorylation
  - Oxidation–reduction
  - Cytoskeleton
  - Calcium
  - ATP
Despite the small differences in their radii, ions rarely go through the “wrong” channel.

For example, sodium or calcium ions rarely pass through a potassium channel.
How to model the flow in ion channels?

- Use EVA to find a PDE system which may describe the flow.
- Total energy consists of
  - Hydrodynamics: incompressible Navier-Stokes equations
  - Ion-exchange: PNP (Poisson-Nernst-Planck) systems
- Finite size effects give compressibility
Model for ion channels

- A complicated PDE model (cf. Chun Liu et al, 2010) including the PNP system which is effective to simulate the ion selectivity of ion channels

**PNP equation:**

\[
\frac{\partial c_i}{\partial t} + \nabla \cdot (\nabla c_i + \frac{x_i e}{k_B T} c_i \nabla \phi) = -\nabla \cdot \left[ \frac{c_i}{k_B T} \int \frac{12 \epsilon_{i,j}(\epsilon_i + \epsilon_j)^2 (x-y)^2}{|x-y|^{14}} c_j(y) \, dy \right] \\
- \nabla \cdot \left[ \int \frac{6 \epsilon_{i,j}(\epsilon_i + \epsilon_j)^2 (x-y)^2}{|x-y|^{14}} c_j(y) \, dy \right] \text{ for } i,j = n,p, \quad i \neq j
\]

\[
\nabla \cdot (\varepsilon \nabla \phi) = -4\pi \left( \rho + \sum_{i=1}^{N} x_i m_i c_i \right).
\]

- Finite size (Equation of state, Lenard-Jones, DFT)
Two basic principles of IT

- Electro-neutrality (EN)
  - The total amounts of the positive charge and the negative charge are the same

- Nonelectro-neutrality (NN)
  - The total positive and negative charge densities are not equal to each other
Motivation

- For almost all biological systems, EN is presumed.
- NN is very rare but exists (cf. Hsu et al ’97, Lee et al ’97, Bazant et al ’05 and Riccardi et al ‘09)
- It is natural to believe that EN is quite stable even under the NN perturbation. Why?
Model of IT

- Electro-diffusion (Fick’s law)
- Electrophoresis (Kohlrausch’s laws)
- Electrostatic force (Poisson’s law)
- Nernst-Planck equations describe electro-diffusion and electrophoresis
- Poisson’s equation is used for the electrostatic force between ions
Poisson-Nernst-Planck (PNP) system for two ions

\begin{align}
\frac{\partial n}{\partial x} &= -\nabla n, \\
\frac{\partial p}{\partial x} &= -\nabla p, \\
J_n &= -D_n \left(n_x - \frac{z_n e}{k_B T} n \phi_x \right), \\
J_p &= -D_p \left(p_x + \frac{z_p e}{k_B T} p \phi_x \right), \\
\epsilon^2 \phi_{xx} &= \rho + z_n e n - z_p e p,
\end{align}

where $\phi$ is the electrostatic potential, $n$ is the density of anions, $p$ is the density of cations, $\rho$ is the permanent (fixed) charge density in the domain, $z_n$, $z_p$ are the valence of ions, $e$ is the unit charge, $k_B$ is the Boltzmann constant, $T$ is temperature, $J_n, J_p$ are the ionic flux densities and $D_n, D_p$ are their diffusion coefficients. The parameter $\epsilon > 0$ related to the ratio of the Debye length to a characteristic length scale can be assumed as a small parameter tending to zero. Such a hypothesis can
Energy (dissipation) law

As for Fokker-Planck equation, the energy law of PNP is given by

\[
\frac{d}{dt} \int_{-1}^{1} \left( n \log n + p \log p + \epsilon^2 \frac{\left| \nabla \phi \right|^2}{2} \right) dx = - \int_{-1}^{1} \left( n \left| \frac{\nabla n}{n} - \nabla \phi \right|^2 + p \left| \frac{\nabla p}{p} + \nabla \phi \right|^2 \right) dx.
\]

For simplicity, we consider monovalent ions, that is, \( z_n = z_p = 1 \) with \( e/k_B T = 1 \), \( \rho = 0 \), \( D_n = D_p = 1 \). Here we reuse the notation, \( \epsilon \) again. Then the PNP system (1.1)-(1.3) becomes

\[
\begin{align*}
    n_t &= -\partial_x J_n, & p_t &= -\partial_x J_p, \\
    J_n &= -(n_x - n\phi_x), & J_p &= -(p_x + p\phi_x), \\
    \epsilon^2 \phi_{xx} &= n - p, & \text{for } x \in (-1, 1), \ t > 0.
\end{align*}
\]
Known results for PNP

- No small parameter $\varepsilon$

- Existence, uniqueness and long time (i.e. time goes to infinity) asymptotic behaviors (Arnold et al, ’99 and Biler et al, ’00)

- However, in general, bio-systems can not have such a long life

- Nothing to do with NN and EN
The small parameter

\[ \epsilon = \left( \epsilon_0 \frac{U_T}{(d^2 e S)} \right)^{1/2} > 0, \]

- \( \epsilon_0 \) is the dielectric constant of the electrolyte
- \( U_T \) is the thermal voltage
- \( d \) is the length of the domain
- \( S \) is the appropriate concentration scale
Problems and results

- The equilibrium (steady state) of the PNP system using a new Poisson-Boltzmann type of equations (with Chiun-Chang Lee 2010)
- Linear stability of the equilibrium with respect to the PNP system
- We show that near the equilibrium, NN may evolve into EN in an extremely short time
Model steady state PNP

- **Conventional way:** Poisson-Boltzmann Eqn (PB)
  
- **New way:** a new Poisson-Boltzmann type (PB_n) equation

PB: solve $J_n=J_p=0$ directly

PB_n: conservation law of total charges

\[
\begin{align*}
n_t &= -\partial_x J_n, \\
p_t &= -\partial_x J_p, \\
J_n &= -(n_x - n\phi_x), \\
J_p &= -(p_x + p\phi_x), \\
\varepsilon^2 \phi_{xx} &= n - p, & \text{for } x \in (-1,1), t > 0.
\end{align*}
\]
Conservation law of total charges

no-flux boundary conditions

\[
\frac{d}{dt} \int_{-1}^{1} n dx = - \int_{-1}^{1} \partial_x J_n dx = -J_n \bigg|_{x=-1}^{x=1} = 0, \\
\frac{d}{dt} \int_{-1}^{1} p dx = - \int_{-1}^{1} \partial_x J_p dx = -J_p \bigg|_{x=-1}^{x=1} = 0, \quad \text{for } t > 0.
\]

Consequently, we have

\[
\int_{-1}^{1} n dx = \alpha, \quad \int_{-1}^{1} p dx = \beta, \quad \text{for } t > 0
\]

(1.8)

where \( \alpha \) and \( \beta \) are positive constants only determined by the initial conditions.
Steady state PNP

\( \partial_x (n_x - n\phi_x) = 0, \quad \text{for} \quad x \in (-1, 1), \)  
\( \partial_x (p_x + p\phi_x) = 0, \quad \text{for} \quad x \in (-1, 1), \)  
\( \epsilon^2 \phi_{xx} + p - n = 0, \quad \text{for} \quad x \in (-1, 1) \)

no-flux boundary conditions:

\( (n_x - n\phi_x)(\pm 1) = 0, \)  
\( (p_x + p\phi_x)(\pm 1) = 0 \)

\( \int_{-1}^{1} n \, dx = \alpha, \quad \int_{-1}^{1} p \, dx = \beta \)

\( n = n(x) = \tilde{\alpha} e^{\phi(x)}, \quad p = p(x) = \tilde{\beta} e^{-\phi(x)}, \)  
\( \tilde{\alpha} = \frac{\alpha}{\int_{-1}^{1} e^{\phi} \, dx}, \quad \tilde{\beta} = \frac{\beta}{\int_{-1}^{1} e^{-\phi} \, dx} \)
Differential and integral equations with nonlocal terms

Nice variational structure

Asymptotic behaviors for EN and NN
$\Omega$: bounded smooth domain

**PB\_n equation:**

$$-\epsilon^2 \Delta \phi(x) = - \sum_{k=1}^{N_1} a_k \alpha_k e^{a_k \phi(x)} \int_{\Omega} e^{a_k \phi(y)} \, dy + \sum_{l=1}^{N_2} \frac{b_l \beta_l e^{-b_l \phi(x)}}{\int_{\Omega} e^{-b_l \phi(y)} \, dy} \quad \text{in } \Omega$$

**PB equation:**

$$-\epsilon^2 \Delta \phi(x) = - \sum_{k=1}^{N_1} A_k e^{a_k \phi(x)} + \sum_{l=1}^{N_2} B_l e^{-b_l \phi(x)} \quad \text{in } \Omega$$
Boundary Conditions

No-flux boundary condition:  $J_i(\partial \Omega, t) \cdot \vec{v} = 0, \quad t > 0$

Conservation of ionic charge:  $\frac{d}{dt} \int_\Omega c_i \, dx = -\int_\Omega \nabla \cdot J_i \, dx = 0$

Interfacial boundary condition (of electrostatic potential $\phi$)

$$\phi + \eta \frac{\partial \phi}{\partial \nu} \bigg|_\Gamma \approx \phi_S + \eta \frac{\partial \phi_S}{\partial \nu} = \phi_{\text{extra}}$$

$\eta = \epsilon_S / C_S$: Stern layer thickness

$C_S$: capacitance of the Stern layer

$\epsilon_S$: effective permittivity of the Stern layer

[Diagram of interfacial region, intracellular, extracellular]
Existence

Energy:

\[
E[\phi] = \int_{\Omega} \frac{\varepsilon^2}{2} |\nabla \phi|^2 + \sum_k \alpha_k \log \int_{\Omega} e^{a_k \phi} + \sum_l b_l \log \int_{\Omega} e^{-b_l \phi} \\
+ \frac{\varepsilon^2}{2 \eta} \int_{\partial \Omega} (\phi - \phi_0)^2 dS, \quad \phi \in H^1(\Omega)
\]

* Friedrichs’ inequality: \( \int_{\Omega} |u|^2 \leq C (\int_{\Omega} |\nabla u|^2 + \int_{\partial \Omega} u^2 dS) \)

Direct Method \(\Rightarrow\) Weak Solution \(\phi^*\)

\(\Rightarrow\ \phi^* \in L^\infty(\Omega) + \text{Elliptic Regularity} \Rightarrow \text{Classical Solution} \)
Uniqueness

\[ \varepsilon^2 \Delta \phi(x) = \sum_{k=1}^{N_1} \frac{a_k \alpha_k e^{a_k \phi(x)}}{\int_{\Omega} e^{a_k \phi(y)} \, dy} - \sum_{l=1}^{N_2} \frac{b_l \beta_l e^{-b_l \phi(x)}}{\int_{\Omega} e^{-b_l \phi(y)} \, dy} \]

\[ \phi + \eta \varepsilon \frac{\partial \phi}{\partial \nu} \bigg|_{\partial \Omega} = \phi_0 \]

* Subtracting PB_n for \( \phi_2 \) from that for \( \phi_1 \) and multiplying by \( \phi_1 - \phi_2 \)

* \( A_i(x) = a_k \phi_i(x) - \log \int_{\Omega} e^{a_k \phi_i} \)

\[ a_k \alpha_k \int_{\Omega} (e^{A_1(x)} - e^{A_2(x)})(\phi_1(x) - \phi_2(x)) \]

\[ = \alpha_k \int_{\Omega} (e^{A_1(x)} - e^{A_2(x)}) \left( A_1(x) - A_2(x) + \log \frac{\int_{\Omega} e^{\phi_1}}{\int_{\Omega} e^{\phi_2}} \right) \]

\[ \geq \alpha_k \int_{\Omega} (e^{A_1(x)} - e^{A_2(x)}) \log \frac{\int_{\Omega} e^{\phi_1}}{\int_{\Omega} e^{\phi_2}} = 0 \]
Main Result: Electroneutrality

**Theorem:** Assume \( \sum_{k=1}^{N_1} a_k \alpha_k = \sum_{l=1}^{N_2} b_l \beta_l \) and \( \phi_0(1) = -\phi_0(-1) > 0 \) then

\[
\lim_{\epsilon \downarrow 0} \phi(\pm 1) = \pm t \quad \text{and} \quad \lim_{\epsilon \downarrow 0} \phi(x) = c \quad \text{for} \quad x \in (-1, 1)
\]

(i) If \( \lim_{\epsilon \downarrow 0} \frac{\eta_\epsilon}{\epsilon} = \infty \), then \( c = t = 0 \) and \( \lim_{\epsilon \downarrow 0} \eta_\epsilon \phi'(\pm 1) = \phi_0(1) \).

(ii) If \( \lim_{\epsilon \downarrow 0} \frac{\eta_\epsilon}{\epsilon} = \gamma < \infty \), then \( |c| < t \leq \phi_0(1) \)

\[
\begin{cases}
\phi_0(1) - t = \gamma(f(t - c) - f(0))^{1/2}, \\
f(t - c) = f(-t - c)
\end{cases}
\]

and \( \lim_{\epsilon \downarrow 0} \epsilon \phi''(\pm 1) = (f(t - c) - f(0))^{1/2} \)

where \( f(s) = \sum_{k=1}^{N_1} \alpha_k e^{a_k s} + \sum_{l=1}^{N_2} \beta_l e^{-b_l s} \)
Debye screening Length

Debye screening length

\[ \lambda_D := \varepsilon \left( \sum_{i=1}^{N} \frac{z_i^2 e^2 c_i^\infty}{k_B T} \right)^{-1/2} \]

\[ \frac{\eta_\varepsilon}{\varepsilon} \sim \frac{\text{Stern layer}}{\text{Debye length}} \frac{\delta}{\lambda_D} \]

Diagram: Stern layer, Debye length, Electrolyte, Diffuse layer, Distance.
Main Result: Non-electroneutrality

**Theorem:** Assume that
\[ \sum_{k=1}^{N_1} a_k \alpha_k < \sum_{k=1}^{N_2} b_k \beta_1. \]

Then for all \( x \in K \subseteq (-1, 1) \)

\[ \phi(x) - \phi(\pm 1) = \frac{2}{k\varepsilon} \log \frac{1}{\varepsilon} + O(1) \quad \text{as} \quad 0 < \varepsilon \ll 1 \]

where \( b_1 \leq k \varepsilon \leq b_{N_2} \).
Main Result: Non-electroneutrality

Assume that \( N_1 = N_2 = 1 \) and \( a_1 \alpha_1 < b_1 \beta_1 \)

(i) If \( \lim_{\epsilon \downarrow 0} \frac{n_\epsilon}{\epsilon^2} = 0 \), then \( \lim_{\epsilon \downarrow 0} \phi(1) = \phi_0(1) \) and \( \lim_{\epsilon \downarrow 0} \phi(-1) = \phi_0(-1) \) and

\[
\lim_{\epsilon \downarrow 0} \epsilon^2 \phi'(1) = \frac{e^{b_1 \phi_0(-1)/2}}{e^{b_1 \phi_0(1)/2} + e^{b_1 \phi_0(-1)/2}} (a_1 \alpha_1 - b_1 \beta_1),
\]

\[
\lim_{\epsilon \downarrow 0} \epsilon^2 \phi'(-1) = -e^{b_1 \phi_0(1)/2} (a_1 \alpha_1 - b_1 \beta_1).
\]

(ii) If \( \lim_{\epsilon \downarrow 0} \frac{n_\epsilon}{\epsilon^2} = \infty \), then \( \lim_{\epsilon \downarrow 0} (\phi(-1) - \phi(1)) = 0 \) and

\[
\lim_{\epsilon \downarrow 0} \epsilon^2 \phi'(-1) = -\lim_{\epsilon \downarrow 0} \epsilon^2 \phi'(1) = -\frac{1}{2} (a_1 \alpha_1 - b_1 \beta_1).
\]
Main Result: Non-electroneutrality

(iii) If $\lim_{\epsilon \downarrow 0} \frac{n_\epsilon}{\epsilon^2} = \gamma$, $0 < \gamma < \infty$, then $\lim_{\epsilon \downarrow 0} \phi(1) = \phi_1^*$ and $\lim_{\epsilon \downarrow 0} \phi(-1) = \phi_2^*$

\[
\begin{align*}
\phi_1^* + \phi_2^* &= \phi_0(1) + \phi_0(-1) + \gamma \left( b_1 \beta_1 - a_1 \alpha_1 \right), \\
(\phi_0(1) - \phi_1^*)e^{b_1 \phi_1^*/2} + (\phi_2^* - \phi_0(-1))e^{b_1 \phi_2^*/2} &= 0, \\
\phi_1^* &> \phi_0(1), \quad \phi_2^* > \phi_0(-1),
\end{align*}
\]

\[
\lim_{\epsilon \downarrow 0} \epsilon^2 \phi'(1) = \frac{\phi_0(1) - \phi_1^*}{\gamma}, \quad \lim_{\epsilon \downarrow 0} \epsilon^2 \phi'(-1) = -\frac{\phi_0(-1) - \phi_2^*}{\gamma}.
\]
Idea 1: Pohozaev’s identity

1. Pohozaev’s identity

\[
\sum_{k=1}^{N_1} \alpha_k \left( e^{a_k \phi(1)} + e^{a_k \phi(-1)} \right) \int_{-1}^{1} e^{a_k \phi(y)} \, dy + \sum_{l=1}^{N_2} \beta_l \left( e^{-b_l \phi(1)} + e^{-b_l \phi(-1)} \right) \int_{-1}^{1} e^{-b_l \phi(y)} \, dy \\
+ \frac{\epsilon^2}{2} \int_{-1}^{1} \phi'^2(x) \, dx = \frac{\epsilon^2}{2} \left( \phi'^2(1) + \phi'^2(-1) \right) + f(0)
\]

2. For any \( x^* \in (-1, 1) \),

\[
\sum_{k=1}^{N_1} \alpha_k e^{a_k \phi(x^*)} \int_{-1}^{1} e^{a_k \phi(y)} \, dy + \sum_{l=1}^{N_2} \beta_l e^{-b_l \phi(x^*)} \int_{-1}^{1} e^{-b_l \phi(y)} \, dy \\
+ \frac{\epsilon^2}{4} \left( \int_{-1}^{1} \phi'^2(x) \, dx - 2\phi'^2(x^*) \right) = \frac{1}{2} f(0)
\]
Idea 2: Inverse Hölder’s type inequality

1. \( \exists 1 \leq \bar{k} \leq N_1 \) and \( 1 \leq \bar{l} \leq N_2 \) s.t.

\[
\sup_{\epsilon > 0} \left( \int_{-1}^{1} e^{a_k \phi} \right)^{1/a_k} \left( \int_{-1}^{1} e^{-b_l \phi} \right)^{1/b_l} < \infty.
\]

2. \( k = 1, \ldots, N_1 \)

\[
\sup_{\epsilon > 0} \left( \int_{-1}^{1} e^{a_k \phi} \right)^{1/a_k} \left( \int_{-1}^{1} e^{-b_{N_2} \phi} \right)^{1/b_1} e^{\frac{b_{N_2} - b_1}{b_1} \phi(1)} < \infty
\]

- \( \min\{\phi_0(1), \phi_0(-1)\} \leq \phi(x) \leq M^* \), where

\[
M^* = \max_{1 \leq k \leq N_1, 1 \leq l \leq N_2} \left\{ \frac{1}{a_k + b_l} \log \frac{N_2 b_l \beta_l}{N_1 a_k \alpha_k} \int_{-1}^{1} e^{a_k \phi}, \phi_0(1), \phi_0(-1) \right\}
\]
Asymptotic behavior of boundary layer

Asymptotic Behaviors: \((N_1 = 1, \ N_2 = 2, \ a_1 = b_1 = 1, \ b_2 = 2)\)

\[
\phi(x) \sim c + \ln \left\{ 1 + B_{i,\epsilon}^+ \text{csch}^2 \left[ \frac{C_{i,\epsilon}^+}{\epsilon} (1 - x) + \ln D_{i,\epsilon}^+ \right] \right\}, \ x \in (y_{\epsilon}^+, 1)
\]

\[
\phi(x) \sim c + \ln \left\{ 1 - B_{i,\epsilon}^- \text{sech}^2 \left[ \frac{C_{i,\epsilon}^-}{\epsilon} (1 + x) + \ln D_{i,\epsilon}^- \right] \right\}, \ x \in (-1, y_{\epsilon}^-)
\]

where \(-1 < y_{\epsilon}^- < y_{\epsilon}^+ < 1\) satisfy \(\lim_{\epsilon \downarrow 0} \phi(y_{\epsilon}^\pm) = c\), and

\[
B_{i,\epsilon}^\pm \to 1 + \frac{\beta_2}{\alpha_1}, \quad C_{i,\epsilon}^\pm \to \sqrt{\alpha_1 + \beta_2},
\]

\[
D_{i,\epsilon}^\pm \to \frac{\sqrt{\alpha_1 e^{\pm \frac{\tau - c}{\epsilon}} + \beta_2 + \sqrt{\alpha_1 + \beta_2}}}{\pm \sqrt{\alpha_1 e^{\pm \frac{\tau - c}{\epsilon}} + \beta_2 \mp \sqrt{\alpha_1 + \beta_2}}} \quad \text{as } \epsilon \text{ goes to zero.}
\]
Remark

\[ \phi(1 - \epsilon y) - c \xrightarrow{\epsilon \downarrow 0} 4 \left( 1 + \frac{\beta_2}{\alpha_1} \right) \frac{\sqrt{\alpha_1 e^{t-c} + \beta_2} - \sqrt{\alpha_1 + \beta_2}}{\sqrt{\alpha_1 e^{t-c} + \beta_2} + \sqrt{\alpha_1 + \beta_2}} e^{\sqrt{\alpha_1 + \beta_2} y} \]

uniformly in \( K \subset \subset (0, \infty) \)
Linear stability of PNP

Small perturbations

\[
\begin{align*}
n^0 &= \frac{e^\psi}{\int_{-1}^{1} e^\psi \, dx}, \\
p^0 &= \frac{e^{-\psi}}{\int_{-1}^{1} e^{-\psi} \, dx}.
\end{align*}
\]

To observe EN and NN, we set

\[
\begin{align*}
\delta &= \tilde{n} - \tilde{p}, \\
\eta &= \tilde{n} + \tilde{p},
\end{align*}
\]
Linearized problem and result

Then the linearized problem becomes

\[
\begin{aligned}
\bar{\delta}_t &= \bar{\delta}_{xx} - (\eta^0\bar{\psi}_x)_x - (\bar{\eta}\bar{\psi}_x)_x, \\
\bar{\eta}_t &= \bar{\eta}_{xx} - (\delta^0\bar{\psi}_x)_x - (\bar{\delta}\bar{\psi}_x)_x, \\
\varepsilon\bar{\psi}_{xx} &= \bar{\delta}.
\end{aligned}
\]

We prove that

\(\bar{\delta}\) may tend to zero weakly in an extremely short time as the small parameter \(\varepsilon\) goes to zero.

\(\bar{\eta}\) is governed by the standard heat equation.
Main difficulty

- Due to the existence of boundary layer, spectrum analysis becomes very difficult to get the positive lower bound.
- We use the energy method to get the weak convergence.
- From the experimental data.
- We may believe that the weak convergence is reasonable.
Main ideas for the proof

- **Method I**: Projection (Galerkin) method with a specific orthonormal basis
- Estimate the infinite dimensional system of ordinary differential equations
- **Method II**: Find the energy law of the linearized problem (the idea may come from Method I)
- Derive the associated estimates from the energy law
Summary

- Asymptotic behaviors of 1D PB_n
- Steady state solutions with EN have linear stability
- NN perturbation may tend to EN in an extremely short time