The Poisson-Nernst-Planck (PNP) system for ion transport

Tai-Chia Lin

National Taiwan University

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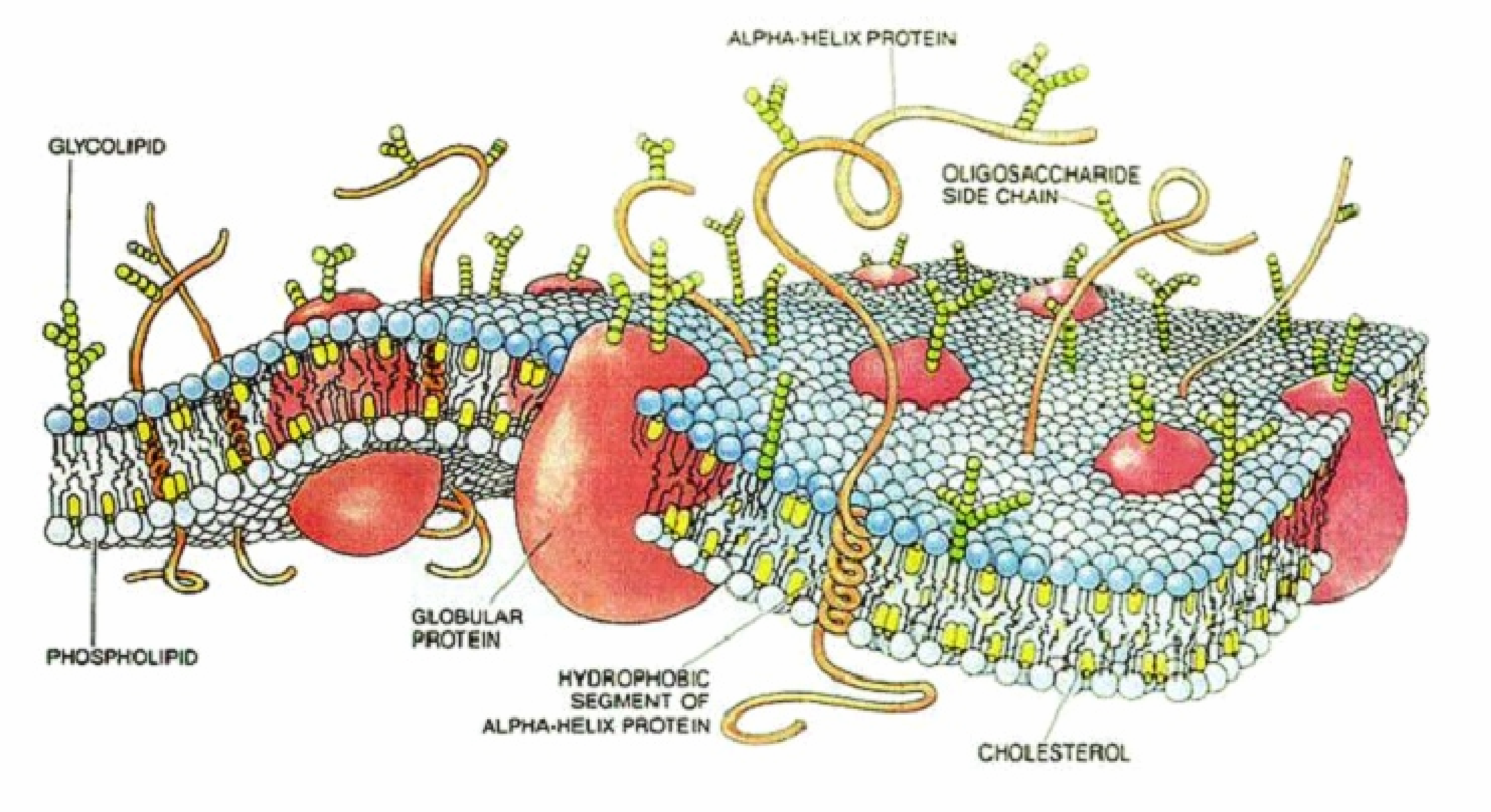
Background

- Ion transport is crucial in the study of many physical and biological problems, such as
- Semiconductors,
- Electro-kinetic fluids,
- Transport of electrochemical systems and
- Ion channels in cell membranes

Ion transport (IT)

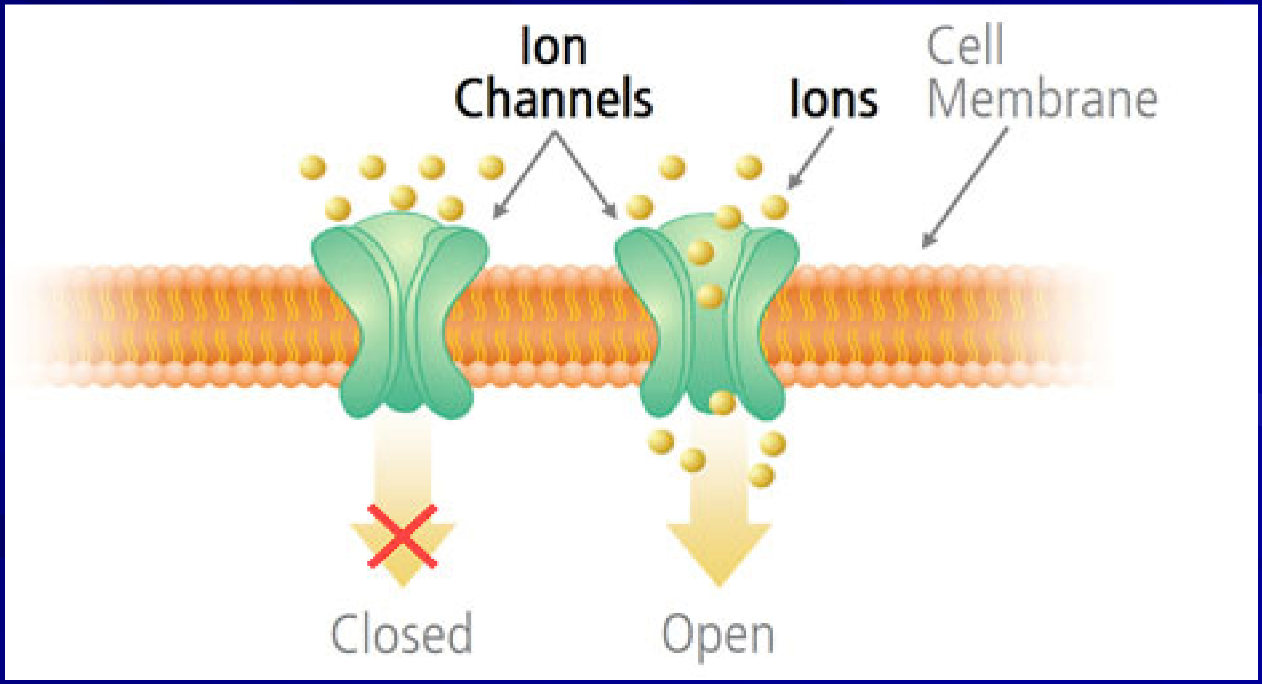
- Movement of salts and other electrolytes in the form of ions from place to place within living systems
- Ions may travel by themselves or as a group of two or more ions in the same or opposite directions
- The movement of ions across cell membranes through ion channels

Cell Membranes surround all biological cells.



Ion Channels of Membrane

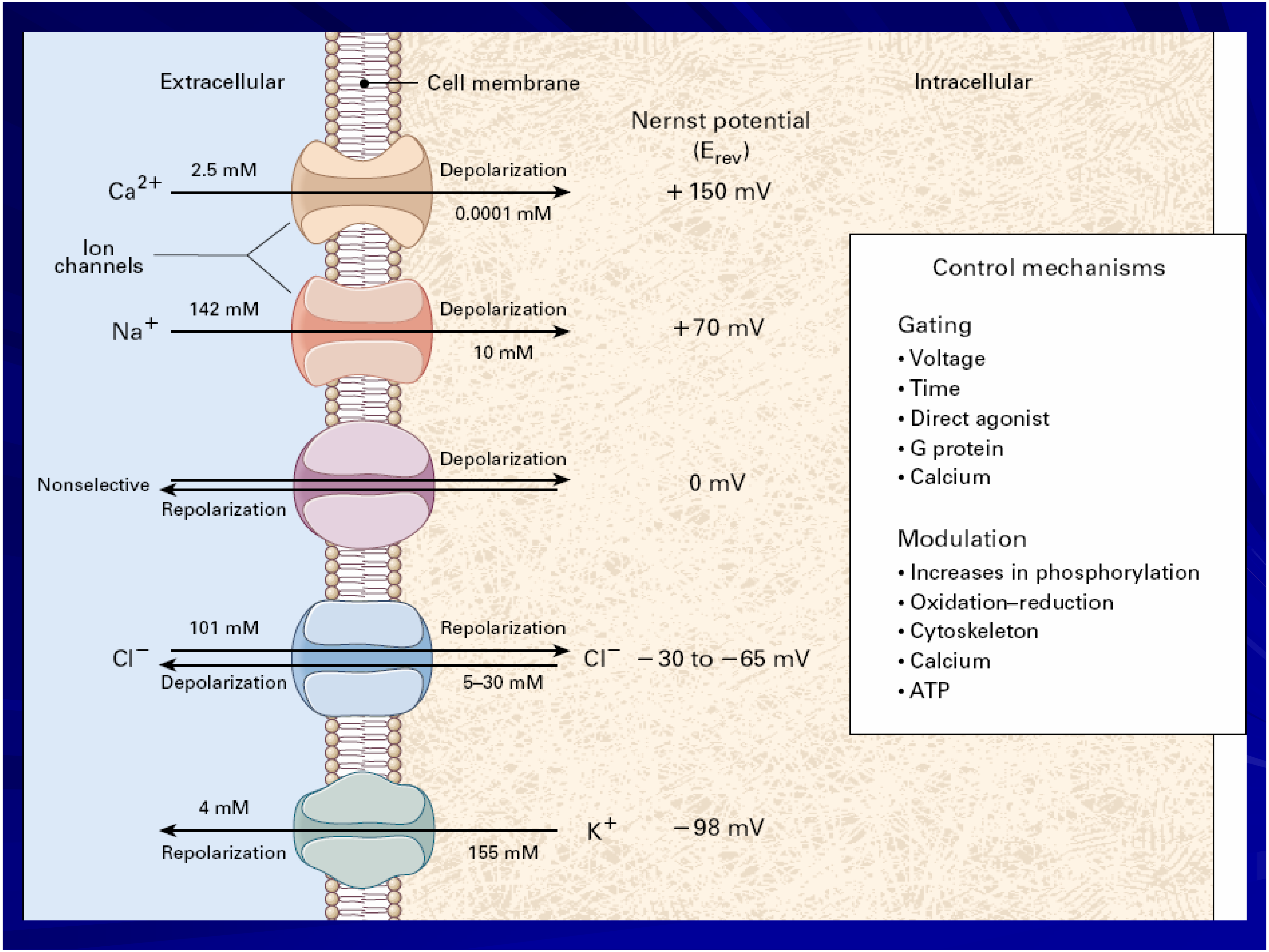
Ion channels are pores in cell membranes and the gatekeepers for cells to control the movement of anions (陰離子) and cations (陽離子) across cell membranes.



Information of ion channels

- Each channel can transport 1 million to 100 million ions per second (10^{-10} to 10^{-12} amperes).
- Close and open within a millisecond.
- Action potential: -70mV to 50mV

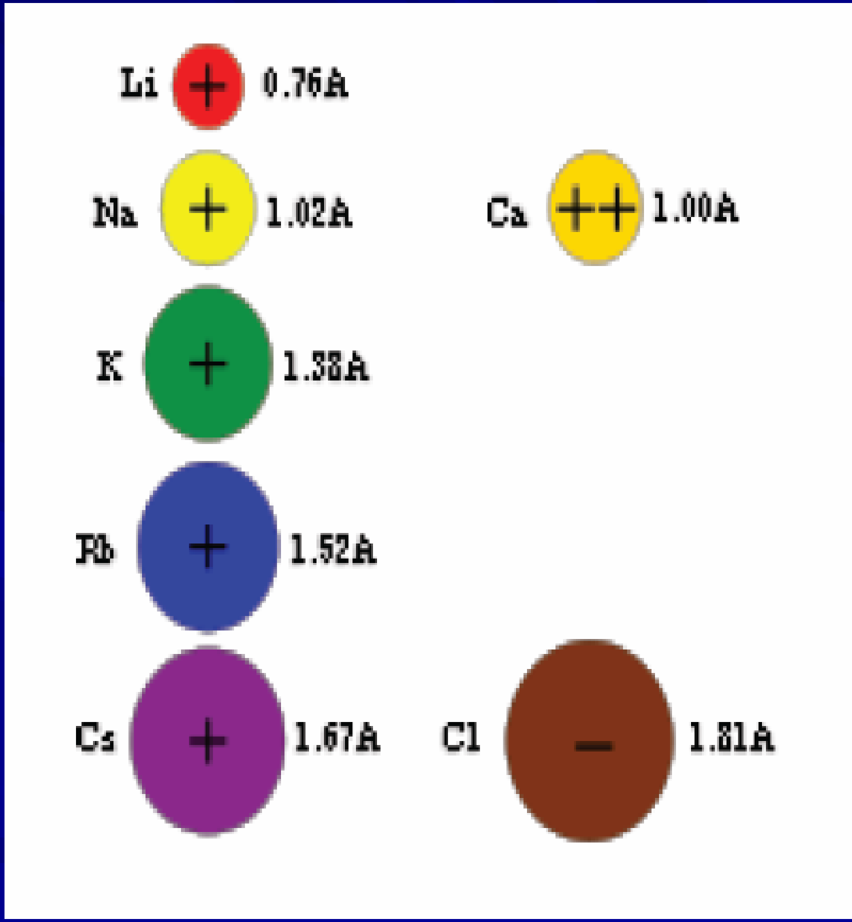
Continuous model is reasonable for the open channel



Extracellular
Cell membrane
Intracellular
Nernst potential
(E_{rev})
Ca^{2+}
2.5 mM
Depolarization
0.0001 mM
+ 150 mV
Ion channels
Na^{+}
142 mM
Depolarization
10 mM
+70 mV
Nonselective
Depolarization
Repolarization
0 mV
Cl^{-}
101 mM
Repolarization
Depolarization
5–30 mM
Cl^{-} −30 to −65 mV
4 mM
Repolarization
155 mM
K^{+}
−98 mV
Control mechanisms
Gating
• Voltage
• Time
• Direct agonist
• G protein
• Calcium
Modulation
• Increases in phosphorylation
• Oxidation–reduction
• Cytoskeleton
• Calcium
• ATP

Ion sizes and selectivity

- Despite the small differences in their radii, ions rarely go through the “wrong" channel.
- For example, sodium or calcium ions rarely pass through a potassium channel.



How to model the flow in ion channels ?

- Use EVA to find a PDE system which may describe the flow.
- Total energy consists of
- Hydrodynamics : incompressible Navier-Stokes equations
- Ion-exchange: PNP （Poisson-Nernst-Planck） systems
- Finite size effects give compressibility

Model for ion channels

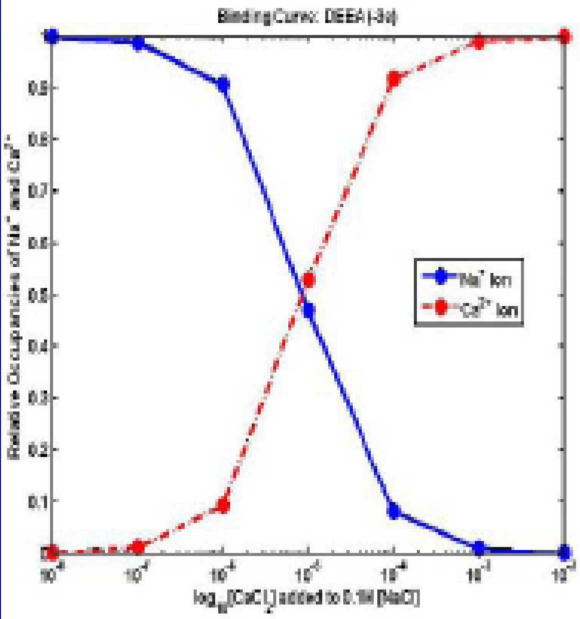
- A complicated PDE model (cf. Chun Liu et al, 2010) including the PNP system which is effective to simulate the ion selectivity of ion channels

▸ PNP equation:

$$
\begin{aligned}
\frac{\partial c_i}{\partial t} + u_f \cdot \nabla c_i &= \nabla \cdot \left[D_i \left(\nabla c_i + \frac{z_i e}{k_B T} c_i \nabla \phi \right) \right] \\
&\quad - \nabla \cdot \left[\frac{c_i}{k_B T} \int \frac{12 \varepsilon_{i,j} (a_i + a_j)^{12} (x - y)}{|x - y|^{14}} c_j(y)\, dy \right] \\
&\quad - \nabla \cdot \left[\int \frac{6 \varepsilon_{i,j} (a_i + a_j)^{12} (x - y)}{|x - y|^{14}} c_j(y)\, dy \right] \quad \text{for } i, j = n, p, \quad i \neq j \\
\nabla \cdot (\varepsilon \nabla \phi) &= -4\pi \left(\rho + \sum_{i=1}^{N} z_i e c_i \right).
\end{aligned}
$$

▸ Finite size (Equation of state, Lenard-Jones, DFT)

▸ Ion binding of ion channel



Two basic principles of IT

- Electro-neutrality (EN)
- The total amounts of the positive charge and the negative charge are the same
- Nonelectro-neutrality (NN)
- The total positive and negative charge densities are not equal to each other

Motivation

- For almost all biological systems, EN is presumed.
- NN is very rare but exists (cf. Hsu et al ’97, Lee et al ’97, Bazant et al ’05 and Riccardi et al ‘09)
- It is natural to believe that EN is quite stable even under the NN perturbation. Why?

Model of IT

- Electro-diffusion (Fick’s law)
- Electrophoresis (Kohlrausch’s laws)
- Electrostatic force (Poisson’s law)
- Nernst-Planck equations describe electro-diffusion and electrophoresis
- Poisson’s equation is used for the electrostatic force between ions

Poisson-Nernst-Planck (PNP) system for two ions

$$n_t = -\partial_x J_n, \qquad p_t = -\partial_x J_p, \tag{1.1}$$

$$J_n = -D_n\left(n_x - \frac{z_n e}{k_B T} n\phi_x\right), \quad J_p = -D_p\left(p_x + \frac{z_n e}{k_B T} p\phi_x\right), \tag{1.2}$$

$$\epsilon^2 \phi_{xx} = \rho + z_n e n - z_p e p, \qquad \text{for } x \in (-1,1),\ t > 0 \tag{1.3}$$

where ϕ is the electrostatic potential, n is the density of anions, p is the density of cations, ρ is the permanent(fixed) charge density in the domain, z_n, z_p are the

valence of ions, e is the unit charge, k_B is the Boltzmann constant, T is temperature, J_n, J_p are the ionic flux densities and D_n, D_p are their diffusion coefficients. The parameter $\epsilon > 0$ related to the ratio of the Debye length to a characteristic length scale can be assumed as a small parameter tending to zero. Such a hypothesis can

Energy (dissipation) law

As for Fokker-Planck equation, the energy law of PNP is given by

$$\frac{d}{dt}\int_{-1}^{1}\left(n\log n + p\log p + \epsilon^2\frac{|\nabla\phi|^2}{2}\right)dx = -\int_{-1}^{1}\left(n\left|\frac{\nabla n}{n} - \nabla\phi\right|^2 + p\left|\frac{\nabla p}{p} + \nabla\phi\right|^2\right)dx.$$

For simplicity, we consider monovalent ions, that is, $z_n = z_p = 1$ with $e/k_BT = 1$, $\rho = 0$, $D_n = D_p = 1$. Here we reuse the notation, ϵ again. Then the PNP system (1.1)-(1.3) becomes

$$n_t = -\partial_x J_n\,, \qquad p_t = -\partial_x J_p, \tag{1.4}$$

$$J_n = -(n_x - n\phi_x)\,, \quad J_p = -(p_x + p\phi_x), \tag{1.5}$$

$$\epsilon^2\phi_{xx} = n - p\,, \qquad \text{for } x \in (-1,1),\ t > 0. \tag{1.6}$$

Known results for PNP

$$u_t = \nabla\cdot(\nabla u + u\nabla\phi)\,,\qquad v_t = \nabla\cdot(\nabla v - v\nabla\phi)\,,\qquad \Delta\phi = v - u\,,$$

$$n_t = -\partial_x J_n\,,\qquad p_t = -\partial_x J_p\,,$$
$$J_n = -(n_x - n\phi_x)\,,\quad J_p = -(p_x + p\phi_x)\,,$$
$$\epsilon^2\phi_{xx} = n - p\,,\qquad \text{for } x\in(-1,1),\ t>0.$$

- No small parameter ϵ
- Existence, uniqueness and long time (i.e. time goes to infinity) asymptotic behaviors (Arnold et al,'99 and Biler et al, '00)
- However, in general, bio-systems can not have such a long life
- Nothing to do with NN and EN

The small parameter

- $\epsilon = \left(\epsilon_0 U_T / (d^2 e S)\right)^{1/2} > 0,$
- ϵ_0 is the dielectric constant of the electrolyte
- U_T is the thermal voltage
- d is the length of the domain
- S is the appropriate concentration scale

Problems and results

- The equilibrium (steady state) of the PNP system using a new Poisson-Boltzmann type of equations (with Chiun-Chang Lee 2010)
- Linear stability of the equilibrium with respect to the PNP system
- We show that near the equilibrium, NN may evolve into EN in an extremely short time

Model steady state PNP

- Conventional way: Poisson-Boltzmann Eqn (PB)
- New way: a new Poisson-Boltzmann type (PB_n) equation

$$n_t = -\partial_x J_n\,, \qquad p_t = -\partial_x J_p, \tag{1.4}$$

$$J_n = -(n_x - n\phi_x)\,, \quad J_p = -(p_x + p\phi_x), \tag{1.5}$$

$$\epsilon^2 \phi_{xx} = n - p\,, \qquad \text{for } x \in (-1,1),\ t > 0. \tag{1.6}$$

- PB: solve J_n=J_p=0 directly
- PB_n: conservation law of total charges

Conservation law of total charges

- no-flux boundary conditions

$$J_n = J_p = 0 \quad \text{for } x = \pm 1,\ t > 0. \quad (1.7)$$

$$\frac{d}{dt}\int_{-1}^{1} n dx = -\int_{-1}^{1} \partial_x J_n dx = -J_n\Big|_{x=-1}^{x=1} = 0,$$
$$\frac{d}{dt}\int_{-1}^{1} p dx = -\int_{-1}^{1} \partial_x J_p dx = -J_p\Big|_{x=-1}^{x=1} = 0, \quad \text{for } t > 0.$$

Consequently, we have

$$\int_{-1}^{1} n\, dx = \alpha, \quad \int_{-1}^{1} p\, dx = \beta, \quad \text{for } t > 0 \qquad (1.8)$$

where α and β are positive constants only determined by the initial conditions.

Steady state PNP

$$\partial_x(n_x - n\phi_x) = 0, \quad \text{for} \quad x \in (-1,1), \tag{1.10}$$

$$\partial_x(p_x + p\phi_x) = 0, \quad \text{for} \quad x \in (-1,1), \tag{1.11}$$

$$\epsilon^2\phi_{xx} + p - n = 0, \quad \text{for} \quad x \in (-1,1) \tag{1.12}$$

no-flux boundary conditions:

$$(n_x - n\phi_x)(\pm 1) = 0, \tag{1.13}$$

$$(p_x + p\phi_x)(\pm 1) = 0 \tag{1.14}$$

- (1.8) gives $$\int_{-1}^{1} n dx = \alpha, \quad \int_{-1}^{1} p dx = \beta \tag{1.15}$$

$$n = n(x) = \tilde{\alpha} e^{\phi(x)}, \quad p = p(x) = \tilde{\beta} e^{-\phi(x)}, \quad \tilde{\alpha} = \frac{\alpha}{\int_{-1}^{1} e^{\phi} dx}, \quad \tilde{\beta} = \frac{\beta}{\int_{-1}^{1} e^{-\phi} dx}$$

PB_n

$$\epsilon^2 \phi_{xx} = \frac{\alpha\, e^{\phi}}{\int_{-1}^{1} e^{\phi} dx} - \frac{\beta\, e^{-\phi}}{\int_{-1}^{1} e^{-\phi} dx} \quad \text{for } x \in (-1,1). \tag{1.18}$$

- Differential and integral equations with nonlocal terms
- Nice variational structure

$$E_\epsilon[u] = \frac{\epsilon^2}{2}\int_{-1}^{1} |u'|^2\, dx + \alpha \log\left(\int_{-1}^{1} e^{u}\, dx\right) + \beta \log\left(\int_{-1}^{1} e^{-u}\, dx\right),$$

- Asymptotic behaviors for EN and NN

Ω: bounded smooth domain

PB_n equation:

$$-\epsilon^2 \Delta\phi(x) = -\sum_{k=1}^{N_1} \frac{a_k \alpha_k e^{a_k \phi(x)}}{\int_\Omega e^{a_k \phi(y)} dy} + \sum_{l=1}^{N_2} \frac{b_l \beta_l e^{-b_l \phi(x)}}{\int_\Omega e^{-b_l \phi(y)} dy} \quad in \ \ \Omega$$

PB equation:

$$-\epsilon^2 \Delta\phi(x) = -\sum_{k=1}^{N_1} A_k e^{a_k \phi(x)} + \sum_{l=1}^{N_2} B_l e^{-b_l \phi(x)} \quad in \ \ \Omega$$

Boundary Conditions

No-flux boundary condition: $J_i(\partial\Omega, t) \cdot \vec{\nu} = 0, \quad t > 0$

conservation of ionic charge: $\frac{d}{dt}\int_\Omega c_i dx = -\int_\Omega \nabla \cdot J_i dx = 0$

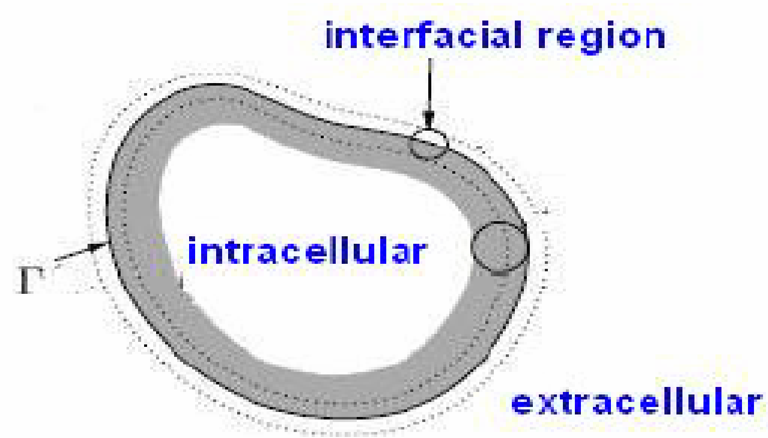
Interfacial boundary condition (of electrostatic potential ϕ)

$$\phi + \eta \frac{\partial \phi}{\partial \nu}\Big|_\Gamma \approx \phi_S + \eta \frac{\partial \phi_S}{\partial \nu} = \phi_{extra}$$

$\eta = \epsilon_S / C_S$: Stern layer thickness

C_S: capacitance of the Stern layer

ϵ_S: effective permittivity of the Stern layer



Existence

Energy:

$$E[\phi] = \int_\Omega \frac{\epsilon^2}{2}|\nabla\phi|^2 + \sum_k \alpha_k \log \int_\Omega e^{a_k\phi} + \sum_l b_l \log \int_\Omega e^{-b_l\phi}$$
$$+ \frac{\epsilon^2}{2\eta}\int_{\partial\Omega}(\phi-\phi_0)^2 dS, \quad \phi \in H^1(\Omega)$$

$*$ Friedrichs' inequality : $\int_\Omega |u|^2 \le C(\int_\Omega |\nabla u|^2 + \int_{\partial\Omega} u^2 dS)$

Direct Method $\Rightarrow$ Weak Solution ϕ^*

$\Rightarrow$ $\phi^* \in L^\infty(\Omega)$ + Elliptic Regularity $\Rightarrow$ Classical Solution

Uniqueness

$$\epsilon^2 \Delta\phi(x) = \sum_{k=1}^{N_1} \frac{a_k \alpha_k e^{a_k \phi(x)}}{\int_\Omega e^{a_k \phi(y)} dy} - \sum_{l=1}^{N_2} \frac{b_l \beta_l e^{-b_l \phi(x)}}{\int_\Omega e^{-b_l \phi(y)} dy}$$

$$\phi + \eta_\epsilon \frac{\partial \phi}{\partial \nu}\bigg|_{\partial\Omega} = \phi_0$$

* Subtracting PB_n for ϕ_2 from that for ϕ_1 and multiplaying by $\phi_1 - \phi_2$
* $A_i(x) = a_k \phi_i(x) - \log \int_\Omega e^{a_k \phi_i}$

$$a_k \alpha_k \int_\Omega (e^{A_1(x)} - e^{A_2(x)})(\phi_1(x) - \phi_2(x))$$

$$= \alpha_k \int_\Omega (e^{A_1(x)} - e^{A_2(x)}) \left(A_1(x) - A_2(x) + \log \frac{\int_\Omega e^{\phi_1}}{\int_\Omega e^{\phi_2}} \right)$$

$$\geq \alpha_k \int_\Omega (e^{A_1(x)} - e^{A_2(x)}) \log \frac{\int_\Omega e^{\phi_1}}{\int_\Omega e^{\phi_2}} = 0$$

Main Result: Electroneutrality

Theorem: Assume $\sum_{k=1}^{N_1} a_k \alpha_k = \sum_{l=1}^{N_2} b_l \beta_l$ and $\phi_0(1) = -\phi_0(-1) > 0$

then

$$\lim_{\epsilon \downarrow 0} \phi(\pm 1) = \pm t \text{ and } \lim_{\epsilon \downarrow 0} \phi(x) = c \text{ for } x \in (-1, 1)$$

(i) If $\lim_{\epsilon \downarrow 0} \frac{\eta_\epsilon}{\epsilon} = \infty$, then $c = t = 0$ and $\lim_{\epsilon \downarrow 0} \eta_\epsilon \phi'(\pm 1) = \phi_0(1)$.

(ii) If $\lim_{\epsilon \downarrow 0} \frac{\eta_\epsilon}{\epsilon} = \gamma < \infty$, then $|c| < t \leq \phi_0(1)$

$$\begin{cases} \phi_0(1) - t = \gamma (f(t-c) - f(0))^{1/2}, \\ f(t-c) = f(-t-c) \end{cases}$$

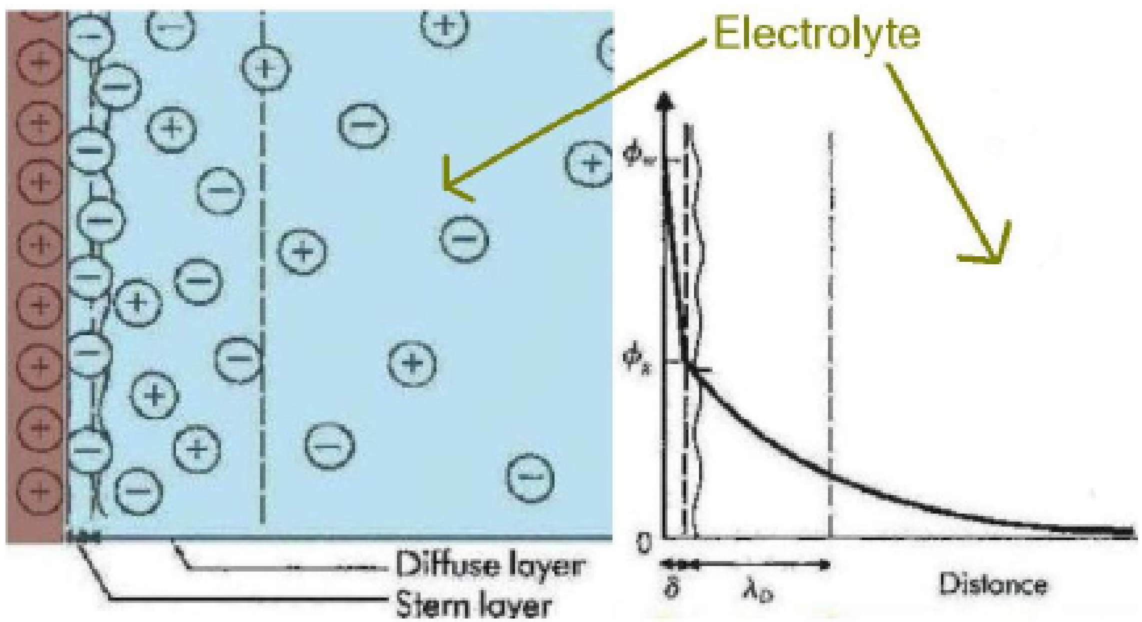
and $\lim_{\epsilon \downarrow 0} \epsilon \phi'(\pm 1) = (f(t-c) - f(0))^{1/2}$

where $f(s) = \sum_{k=1}^{N_1} \alpha_k e^{a_k s} + \sum_{l=1}^{N_2} \beta_l e^{-b_l s}$

Debye screening Length

Debye screening length $$\lambda_D := \epsilon \left(\sum_{i=1}^{N} \frac{z_i^2 e^2 c_i^\infty}{k_B T} \right)^{-1/2}$$

$$\frac{\eta_\epsilon}{\epsilon} \sim \frac{\textit{Stern layer} \quad \delta}{\textit{Debye length} \quad \lambda_D}$$



Main Result: Non-electroneutrality

Theorem: Assume that $\sum_{k=1}^{N_1} a_k \alpha_k < \sum_{k=1}^{N_2} b_l \beta_1$.

Then for all $x \in K \Subset (-1, 1)$

$$\phi(x) - \phi(\pm 1) = \frac{2}{k_\epsilon} \log \frac{1}{\epsilon} + O(1) \quad as\, 0 < \epsilon \ll 1$$

where $b_1 \leq k_\epsilon \leq b_{N_2}$

Main Result: Non-electroneutrality

Assume that $N_1 = N_2 = 1$ and $a_1\alpha_1 < b_1\beta_1$

(i) If $\lim_{\epsilon\downarrow 0} \frac{\eta_\epsilon}{\epsilon^2} = 0$, then $\lim_{\epsilon\downarrow 0} \phi(1) = \phi_0(1)$ and $\lim_{\epsilon\downarrow 0} \phi(-1) = \phi_0(-1)$ and

$$\lim_{\epsilon\downarrow 0} \epsilon^2\phi'(1) = \frac{e^{b_1\phi_0(-1)/2}}{e^{b_1\phi_0(1)/2} + e^{b_1\phi_0(-1)/2}} (a_1\alpha_1 - b_1\beta_1)\,,$$

$$\lim_{\epsilon\downarrow 0} \epsilon^2\phi'(-1) = -\frac{e^{b_1\phi_0(1)/2}}{e^{b_1\phi_0(1)/2} + e^{b_1\phi_0(-1)/2}} (a_1\alpha_1 - b_1\beta_1)\,.$$

(ii) If $\lim_{\epsilon\downarrow 0} \frac{\eta_\epsilon}{\epsilon^2} = \infty$, then $\lim_{\epsilon\downarrow 0}(\phi(-1) - \phi(1)) = 0$ and

$$\lim_{\epsilon\downarrow 0} \epsilon^2\phi'(-1) = -\lim_{\epsilon\downarrow 0} \epsilon^2\phi'(1) = -\frac{1}{2} (a_1\alpha_1 - b_1\beta_1)\,.$$

Main Result: Non-electroneutrality

(iii) If $\lim_{\epsilon\downarrow 0}\frac{\eta_\epsilon}{\epsilon^2}=\gamma$, $0<\gamma<\infty$, then $\lim_{\epsilon\downarrow 0}\phi(1)=\phi_1^*$ and $\lim_{\epsilon\downarrow 0}\phi(-1)=\phi_2^*$

$$\begin{cases}\phi_1^*+\phi_2^*=\phi_0(1)+\phi_0(-1)+\gamma\left(b_1\beta_1-a_1\alpha_1\right),\\ (\phi_0(1)-\phi_1^*)e^{b_1\phi_1^*/2}+(\phi_2^*-\phi_0(-1))e^{b_1\phi_2^*/2}=0,\\ \phi_1^*>\phi_0(1),\quad \phi_2^*>\phi_0(-1),\end{cases}$$

$$\lim_{\epsilon\downarrow 0}\epsilon^2\phi'(1)=\frac{\phi_0(1)-\phi_1^*}{\gamma},\quad \lim_{\epsilon\downarrow 0}\epsilon^2\phi'(-1)=-\frac{\phi_0(-1)-\phi_2^*}{\gamma}.$$

Idea 1: Pohozaev's identity

1. *Pohozaev's identity*

$$\sum_{k=1}^{N_1} \frac{\alpha_k \left(e^{a_k\phi(1)} + e^{a_k\phi(-1)}\right)}{\int_{-1}^{1} e^{a_k\phi(y)}\,dy} + \sum_{l=1}^{N_2} \frac{\beta_l \left(e^{-b_l\phi(1)} + e^{-b_l\phi(-1)}\right)}{\int_{-1}^{1} e^{-b_l\phi(y)}\,dy}$$
$$+\frac{\epsilon^2}{2}\int_{-1}^{1} \phi'^2(x)dx = \frac{\epsilon^2}{2}\left(\phi'^2(1) + \phi'^2(-1)\right) + f(0)$$

2. For any $x^* \in (-1, 1)$,

$$\sum_{k=1}^{N_1} \frac{\alpha_k e^{a_k\phi(x^*)}}{\int_{-1}^{1} e^{a_k\phi(y)}\,dy} + \sum_{l=1}^{N_2} \frac{\beta_l e^{-b_l\phi(x^*)}}{\int_{-1}^{1} e^{-b_l\phi(y)}\,dy}$$
$$+\frac{\epsilon^2}{4}\left(\int_{-1}^{1} \phi'^2(x)dx - 2\phi'^2(x^*)\right) = \frac{1}{2}f(0)$$

Idea 2: Inverse Hölder's type inequality

1. $\exists 1 \leq \bar{k} \leq N_1$ and $1 \leq \bar{l} \leq N_2$ s.t.

$$\sup_{\epsilon>0} \left(\int_{-1}^{1} e^{a_{\bar{k}}\phi} \right)^{1/a_{\bar{k}}} \left(\int_{-1}^{1} e^{-b_{\bar{l}}\phi} \right)^{1/b_{\bar{l}}} < \infty.$$

2. $k = 1, ..., N_1$

$$\sup_{\epsilon>0} \left(\int_{-1}^{1} e^{a_k\phi} \right)^{1/a_k} \left(\int_{-1}^{1} e^{-b_{N_2}\phi} \right)^{1/b_1} e^{\frac{b_{N_2}-b_1}{b_1}\phi(1)} < \infty$$

- $\min\{\phi_0(1), \phi_0(-1)\} \leq \phi(x) \leq M^*$, where

$$M^* = \max_{\substack{1\leq k\leq N_1 \\ 1\leq l\leq N_2}} \left\{ \frac{1}{a_k + b_l} \log \frac{N_2 b_l \beta_l \int_{-1}^{1} e^{a_k\phi}}{N_1 a_k \alpha_k \int_{-1}^{1} e^{-b_l\phi}}, \phi_0(1), \phi_0(-1) \right\}$$

Asymptotic behavior of boundary layer

Asymptotic Behaviors: $(N_1 = 1,\ N_2 = 2,\ a_1 = b_1 = 1,\ b_2 = 2)$

$$\phi(x) \sim c + \ln\left\{1 + B^{+}_{i,\epsilon}\operatorname{csch}^2\left[\frac{C^{+}_{i,\epsilon}}{\epsilon}(1-x) + \ln D^{+}_{i,\epsilon}\right]\right\},\ x \in (y^{+}_{\epsilon}, 1)$$

$$\phi(x) \sim c + \ln\left\{1 - B^{-}_{i,\epsilon}\operatorname{sech}^2\left[\frac{C^{-}_{i,\epsilon}}{\epsilon}(1+x) + \ln D^{-}_{i,\epsilon}\right]\right\},\ x \in (-1, y^{-}_{\epsilon})$$

where $-1 < y^{-}_{\epsilon} < y^{+}_{\epsilon} < 1$ satisfy $\lim_{\epsilon\downarrow 0}\phi(y^{\pm}_{\epsilon}) = c$, and

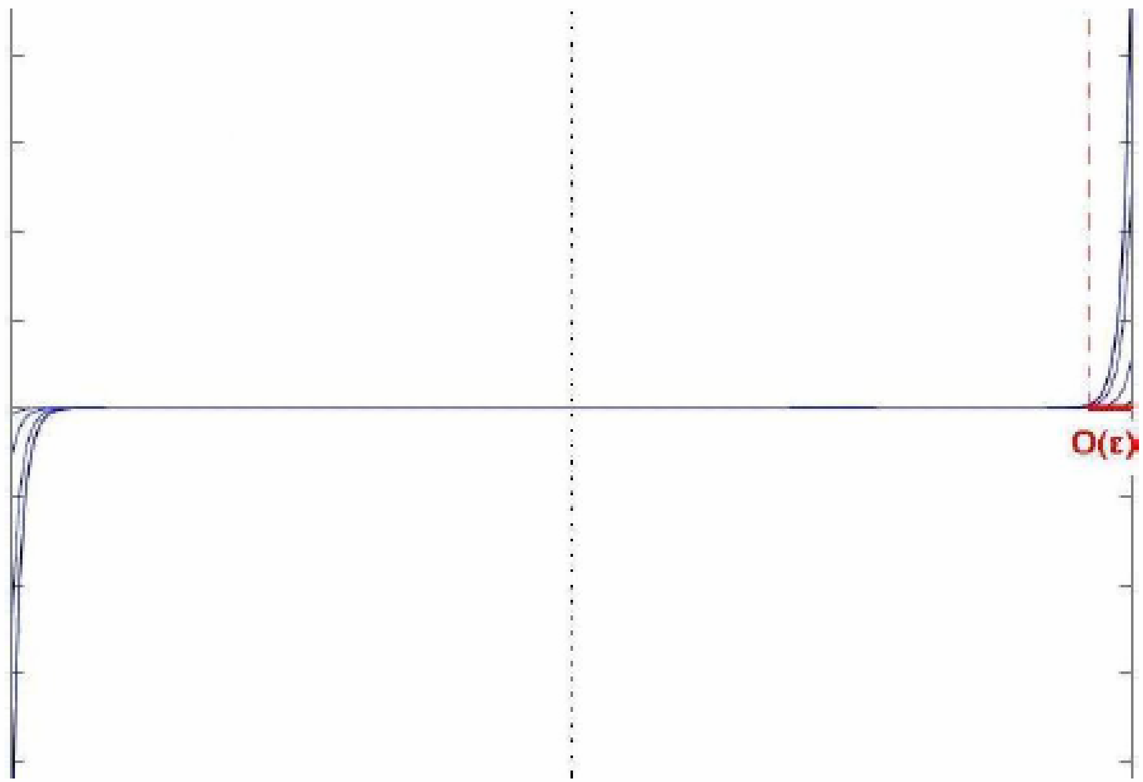
$$B^{\pm}_{i,\epsilon} \to 1 + \frac{\beta_2}{\alpha_1}, \quad C^{\pm}_{i,\epsilon} \to \sqrt{\alpha_1 + \beta_2},$$

$$D^{\pm}_{i,\epsilon} \to \frac{\sqrt{\alpha_1 e^{\pm t - c} + \beta_2} + \sqrt{\alpha_1 + \beta_2}}{\pm\sqrt{\alpha_1 e^{\pm t - c} + \beta_2} \mp \sqrt{\alpha_1 + \beta_2}}$$ as ϵ goes to zero.

Remark

$$\phi(1-\epsilon y) - c \xrightarrow{\epsilon\downarrow 0} 4\left(1+\frac{\beta_2}{\alpha_1}\right)\frac{\sqrt{\alpha_1 e^{t-c}+\beta_2}-\sqrt{\alpha_1+\beta_2}}{\sqrt{\alpha_1 e^{t-c}+\beta_2}+\sqrt{\alpha_1+\beta_2}}\, e^{\sqrt{\alpha_1+\beta_2}\,y}$$

uniformly in $K \subset\subset (0,\infty)$



Linear stability of PNP

- Small perturbations
$$\begin{cases} n = n^0 + \tilde{n}, \\ p = p^0 + \tilde{p}, \\ \phi = \psi + \tilde{\psi}, \end{cases}$$
$$n^0 = \frac{e^{\psi}}{\int_{-1}^{1} e^{\psi} dx}, \quad p^0 = \frac{e^{-\psi}}{\int_{-1}^{1} e^{-\psi} dx}.$$
$$\begin{cases} \varepsilon \psi_{xx} = \frac{e^{\psi}}{\int_{-1}^{1} e^{\psi} dx} - \frac{e^{-\psi}}{\int_{-1}^{1} e^{-\psi} dx} \quad \text{in} \quad (-1,1), \\ \psi(\pm 1) \pm \eta_{\varepsilon} \psi_x(\pm 1) = \phi_0(\pm 1), \end{cases}$$
- To observe EN and NN, we set
$$\tilde{\delta} = \tilde{n} - \tilde{p}, \quad \tilde{\eta} = \tilde{n} + \tilde{p},$$

Linearized problem and result

- Then the linearized problem becomes

$$\begin{cases} \tilde{\delta}_t = \tilde{\delta}_{xx} - (\eta^0 \tilde{\psi}_x)_x - (\tilde{\eta}\psi_x)_x, \\ \tilde{\eta}_t = \tilde{\eta}_{xx} - (\delta^0 \tilde{\psi}_x)_x - (\tilde{\delta}\psi_x)_x, \\ \varepsilon \tilde{\psi}_{xx} = \tilde{\delta}, \end{cases}$$

- We prove that

$\tilde{\delta}$ may tend to zero weakly in an extremely short time as the small parameter ϵ goes to zero

$\tilde{\eta}$ is governed by the standard heat equation

Main difficulty

- Due to the existence of boundary layer, spectrum analysis becomes very difficult to get the positive lower bound.
- We use the energy method to get the weak convergence
- From the experimental data

- We may believe that the weak convergence is reasonable

Main ideas for the proof

- Method I: Projection (Galerkin) method with a specific orthonormal basis
- Estimate the infinite dimensional system of ordinary differential equations
- Method II: Find the energy law of the linearized problem (the idea may come from Method I)
- Derive the associated estimates from the energy law

Summary

- Asymptotic behaviors of 1D PB_n
- Steady state solutions with EN have linear stability
- NN perturbation may tend to EN in an extremely short time