On the existence of pseudoharmonic maps from pseudohermitian manifolds into Riemannian manifolds with nonpositive sectional curvature

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1. Introduction

Let \((M^n, h_{\alpha\beta}), (N^m, g_{ij})\) be Riemannian manifolds with \(M\) closed and let \(f : M \to N\) be a smooth map. In the paper of Eells-Sampson ([6]), they considered the harmonic map heat flow on \(M \times [0, T)\):

\[
\begin{aligned}
\frac{\partial u^k}{\partial t} - \Delta u^k &= \tilde{\Gamma}^k_{ij} \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^j}{\partial x^\alpha}, & k = 1, \ldots, m \\
u(x, 0) &= f(x),
\end{aligned}
\]

(1.1)

where \(\Delta\) is the Laplace-Beltrami operator and \(\tilde{\Gamma}^k_{ij}\) are the Christoffel symbols of \(N\). They proved the following remarkable existence theorem of harmonic maps between Riemannian manifolds.

**Theorem ([6])**

Suppose that the sectional curvature \(\tilde{K}^N\) is nonpositive, then (1.1) admits a unique, smooth solution \(u \in C^\infty(M \times [0, \infty); N)\) which subconverges to a harmonic map \(u_\infty \in C^\infty(M; N)\) as \(t \to \infty\).
Let $M$ and $N$ be Riemannian manifolds with $M$ closed. Does there exist a harmonic map $u : M \to N$?

There always exists a harmonic $u : M \to N$ whenever the sectional curvature of $N$ is nonpositive.
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Let \((M^{2n+1}, J, \theta)\) be a closed pseudohermitian manifold, \((N^m, g)\) be a Riemannian manifold and we consider the pseudoharmonic map heat flow on \(M \times [0, T)\):

\[
\begin{aligned}
\frac{\partial \varphi^k}{\partial t} - \Delta_b \varphi^k &= 2\tilde{\Gamma}^k_{ij} \varphi^i_{\alpha} \varphi^j_{\bar{\alpha}}, \quad k = 1, \ldots, m \\
\varphi(x, 0) &= f(x), \quad f \in C^\infty(M; N).
\end{aligned}
\] (1.2)

Here \(\tilde{\Gamma}^k_{ij}\) are the Christoffel symbols of \(N\). We will follow the same method of Eells-Sampson to show the existence of global smooth solutions to the pseudoharmonic map heat flow (1.2).
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Remark

Geometrically, in the paper of E. Barletta, S. Dragomir and H. Urakawa ([1]), they discovered the following phenomenon: Let $K(M)$ be the canonical bundle over $(M^{2n+1}, J, \theta)$ and $C(M) = (K(M)\setminus \{0\})/\mathbb{R}_+$ be the associated Fefferman manifold which is the circle bundle over $M$. Let $\pi : C(M) \to M$ be the projection and $h$ be the associated Fefferman metric, a Lorentz metric on $C(M)$ (see [17] for details). Then $\varphi : (M^{2n+1}, J, \theta) \to (N, g)$ is pseudoharmonic if and only if its vertical lift $\varphi \circ \pi : (C(M), h) \to (N, g)$ is harmonic. From this point of view, pseudoharmonic maps on a pseudohermitian manifold is closely related to harmonic (wave) maps on the Minkowsky space ([22]).
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In fact, we have the following CR analogue of Eells-Sampson Theorem for the harmonic map heat flow.

**Theorem 1.1**

Let $(M^{2n+1}, J, \theta)$ be a closed pseudohermitian manifold, $(N^m, g)$ be a Riemannian manifold with nonpositive sectional curvature $\tilde{K}^N$. Assume that

$$[\Delta_b, T] = 0.$$ 

Then for any $f \in C^\infty(M; N)$, the pseudoharmonic map heat flow (1.2) admits a unique, smooth solution $\varphi \in C^\infty(M \times [0, \infty); N)$ which subconverges to a pseudoharmonic map $\varphi_\infty \in C^\infty(M; N)$ as $t \to \infty$. 
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Let \((M, \xi)\) be a \((2n + 1)\)-dimensional, orientable, contact manifold with contact structure \(\xi\), \(\dim_{\mathbb{R}} \xi = 2n\).

- A CR structure compatible with \(\xi\) is an endomorphism \(J : \xi \to \xi\) such that \(J^2 = -1\). We also assume that \(J\) satisfies the following integrability condition: If \(X\) and \(Y\) are in \(\xi\), then so are 
  \([JX, Y] + [X, JY]\) and 
  \(J([JX, Y] + [X, JY]) = [JX, JY] - [X, Y]\).

- A CR structure \(J\) can extend to \(\mathbb{C} \otimes \xi\) and decomposes \(\mathbb{C} \otimes \xi\) into the direct sum of \(T_{1,0}\) and \(T_{0,1}\) which are eigenspaces of \(J\) with respect to eigenvalues \(i\) and \(-i\), respectively.

- A pseudohermitian structure compatible with \(\xi\) is a CR structure \(J\) compatible with \(\xi\) together with a choice of contact form \(\theta\).
  Such a choice determines a unique real vector field \(T\) transverse to \(\xi\), which is called the characteristic vector field of \(\theta\), such that \(\theta(T) = 1\) and 
  \(\mathcal{L}_T \theta = 0\) or \(d\theta(T, \cdot) = 0\).
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- Let \( \{ T, Z_\alpha, Z_{\bar{\alpha}} \} \) be a frame of \( TM \otimes \mathbb{C} \), where \( Z_\alpha \) is any local frame of \( T_{1,0} \), \( Z_{\bar{\alpha}} = \overline{Z_\alpha} \in T_{0,1} \) and \( T \) is the characteristic vector field. Then \( \{ \theta, \theta^\alpha, \theta^{\bar{\alpha}} \} \), which is the coframe dual to \( \{ T, Z_\alpha, Z_{\bar{\alpha}} \} \), satisfies

\[
d\theta = \imath h_{\alpha\overline{\beta}} \theta^\alpha \wedge \overline{\theta^\beta},
\]

for some positive definite hermitian matrix of functions \( (h_{\alpha\overline{\beta}}) \). Actually, we can always choose \( Z_\alpha \) such that \( h_{\alpha\overline{\beta}} = \delta_{\alpha\beta} \).

- The Levi form \( \langle \cdot, \cdot \rangle_{L_\theta} \) is the Hermitian form on \( T_{1,0} \) defined by

\[
\langle Z, W \rangle_{L_\theta} = -\imath d\theta(Z, \overline{W}).
\]

We can extend \( \langle \cdot, \cdot \rangle_{L_\theta} \) to \( T_{0,1} \) by defining \( \langle \overline{Z}, \overline{W} \rangle_{L_\theta} = \langle Z, W \rangle_{L_\theta} \) for all \( Z, W \in T_{1,0} \).
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The pseudohermitian connection of $(J, \theta)$ is the connection $\nabla$ on $TM \otimes \mathbb{C}$ given in terms of a local frame $Z_\alpha \in T_{1,0}$ by

$$\nabla Z_\alpha = \omega_\alpha^\beta \otimes Z_\beta, \quad \nabla Z_{\bar{\alpha}} = \omega_{\bar{\alpha}}^\bar{\beta} \otimes Z_{\bar{\beta}}, \quad \nabla T = 0,$$

where $\omega_\alpha^\beta$ are the 1-forms uniquely determined by the following equations:

$$d\theta^\beta = \theta^\alpha \wedge \omega_\alpha^\beta + \theta \wedge \tau^\beta,$$

$$0 = \tau_\alpha \wedge \theta^\alpha,$$

$$0 = \omega_\alpha^\beta + \omega_{\bar{\beta}}^\bar{\alpha},$$

We can write (by Cartan lemma) $\tau_\alpha = A_{\alpha \gamma} \theta^\gamma$ with $A_{\alpha \gamma} = A_{\gamma \alpha}$ the pseudohermitian torsion of $(M, J, \theta)$. 

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- Webster showed that the curvature forms

\[ \Pi^\beta_\alpha = \bar{\Pi}^\bar{\alpha}_\bar{\beta} = d\omega^\beta_\alpha - \omega^\gamma_\beta \wedge \omega^\alpha_\gamma \] satisfies the structure equation

\[ \Pi^\beta_\alpha = R^\beta_\alpha \rho \bar{\sigma} \theta^\rho \wedge \bar{\theta}^\bar{\sigma} + W^\beta_\alpha \rho \theta^\rho \wedge \theta - W^\alpha_\beta \bar{\rho} \theta^\bar{\rho} \wedge \theta + i \theta^\beta \wedge \tau^\alpha - i \tau^\beta \wedge \theta^\alpha, \]

where the coefficients satisfy

\[ R^\beta_\alpha \rho \bar{\sigma} = R^\alpha_\beta \rho \bar{\sigma} = R^\beta_\alpha \rho \bar{\sigma} = R^\alpha_\beta \rho \bar{\sigma} = R^\beta_\alpha \rho \bar{\sigma}, \quad \text{and} \quad W^\beta_\alpha \gamma = W^\alpha_\beta \gamma. \]

- We will denote components of covariant derivatives with indices preceded by comma; thus write \( A^\alpha_{\beta, \gamma} \). The indices \( \{0, \alpha, \bar{\alpha}\} \) indicate derivatives with respect to \( \{T, Z_\alpha, Z_{\bar{\alpha}}\} \). For derivatives of a scalar function, we will often omit the comma, for instance, \( u_\alpha = Z_\alpha u \), \( u_{\alpha \bar{\beta}} = Z_{\bar{\beta}} Z_\alpha u - \omega^\gamma_\alpha (Z_{\bar{\beta}}) Z_\gamma u \), \( u_0 = Tu \) for a smooth function \( u \).
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- For a real function $u$, the subgradient $\nabla_b u$ is defined by $\nabla_b u \in \xi$ and $\langle Z, \nabla_b u \rangle_{L_{\theta}} = du(Z)$ for all vector fields $Z$ tangent to contact plane. Locally, $\nabla_b u = \sum_{\alpha} u_{\bar{\alpha}} Z_{\alpha} + u_{\alpha} Z_{\bar{\alpha}}$.

- We can use the connection to define the subhessian as the complex linear map
  \[ (\nabla^H)^2 u : T_{1,0} \oplus T_{0,1} \rightarrow T_{1,0} \oplus T_{0,1} \]
  by
  \[ (\nabla^H)^2 u(Z) = \nabla_Z \nabla_b u. \]

  In particular,
  \[ |\nabla_b u|^2 = 2u_{\alpha} u_{\bar{\alpha}}, \quad |\nabla^2_b u|^2 = 2(u_{\alpha\beta} u_{\bar{\alpha}\bar{\beta}} + u_{\alpha\bar{\beta}} u_{\bar{\alpha}\beta}). \]
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The subLaplacian $\Delta_b$ of $(M, J, \theta)$ is locally given by

$$\Delta_b u = Tr \left( (\nabla^H)^2 u \right) = \sum_{\alpha} (u_{\alpha \bar{\alpha}} + u_{\bar{\alpha} \alpha}).$$

The pseudohermitian Ricci tensor and the torsion tensor on $T_{1,0}$ are defined by

$$Ric(X, Y) = R_{\alpha \bar{\beta}} X^\alpha Y^{\bar{\beta}},$$

and

$$Tor(X, Y) = i \sum_{\alpha, \beta} (A_{\bar{\alpha} \bar{\beta}} X^{\bar{\alpha}} Y^{\bar{\beta}} - A_{\alpha \beta} X^\alpha Y^\beta),$$

where $X = X^{\alpha} Z_\alpha$, $Y = Y^{\beta} Z_\beta$. 
Let \((M^{2n+1}, J, \theta)\) be a closed pseudohermitian \((2n + 1)\)-manifold, \((N^m, g)\) be a Riemannian \(m\)-manifold and let \(\varphi \in C^2(M; N)\). At each point \(p \in M\), we may take a local coordinate chart \(U_p \subset M\) of \(p\) and a local coordinate chart \(V_{\varphi(p)} \subset N\) of \(\varphi(p)\) such that \(\varphi(U_p) \subset V_{\varphi(p)}\). We define the energy density \(e(\varphi)\) of \(\varphi\) at the point \(x \in U_p\) by

\[
e(\varphi)(x) = \frac{1}{2} h^{\alpha\bar{\beta}}(x) g_{ij}(\varphi(x)) \varphi^i_x \varphi^j_{\bar{x}}.
\]

Here \(h^{\alpha\bar{\beta}}\) is the Levi metric on \((M^{2n+1}, J, \theta)\) and we may assume \(h^{\alpha\bar{\beta}} = \delta^{\alpha\bar{\beta}}\). It can be checked that the energy density is intrinsically defined, i.e., independent of the choice of local coordinates.
2. Preliminary lemmas

Now we define the energy $E(\varphi)$ of $\varphi$ by

$$E(\varphi) = \int_M e(\varphi)d\mu,$$

where $d\mu = \theta \wedge (d\theta)^n$. In the paper of E. Barletta, S. Dragomir and H. Urakawa ([1]), they introduced a notion of the pseudoharmonic map from a pseudohermitian $(2n + 1)$-manifold $(M^{2n+1}, J, \theta)$ into a Riemannian $m$-manifold $(N^m, g)$ as following:

**Definition 2.1**

A $C^2$-map $\varphi : (M^{2n+1}, J, \theta) \rightarrow (N^m, g)$ is said to be a pseudoharmonic map if it is a critical point of the energy functional $E$. 
2. Preliminary lemmas

Lemma 2.1 (CR Bochner formula)

Let \((M^{2n+1}, J, \theta)\) be a closed pseudohermitian manifold. For a real smooth function \(u\) on \((M, J, \theta)\),

\[
\frac{1}{2} \Delta_b |\nabla_b u|^2 = |\nabla^2_b u|^2 + \langle \nabla_b u, \nabla_b \Delta_b u \rangle_{L^0} + 2\langle J\nabla_b u, \nabla_b u_0 \rangle_{L^0} + [2\text{Ric} - (n - 2) \text{Tor}]((\nabla_b u)_c, (\nabla_b u)_c).
\]

(2.1)

Here \((\nabla_b u)_c = u_\alpha Z_\alpha\) is the corresponding complex \((1, 0)\)-vector field of \(\nabla_b u\).
2. Preliminary lemmas

The following lemma gives a necessary and sufficient condition for a map \( \varphi \in C^2(M; N) \) to be pseudoharmonic.

**Lemma 2.2**

Let \((M^{2n+1}, J, \theta)\) be a closed pseudohermitian manifold and \((N^m, g)\) be a Riemannian manifold. A \(C^2\)-map \( \varphi : (M, J, \theta) \to (N, g) \) is pseudoharmonic if and only if it satisfies the Euler-Lagrange equations

\[
\Delta_b \varphi^k + 2\bar{\Gamma}^k_{ij} \varphi^i_{\alpha} \varphi^j_{\overline{\alpha}} = 0, \quad k = 1, \cdots, m, \tag{2.2}
\]

where \(\bar{\Gamma}^k_{ij}\) are the Christoffel symbols of \((N^m, g)\).
3. On the pseudoharmonic map heat flow

The organization of this section is as follows.

- Applying Moser’s Harnack inequality to show that the energy density of the pseudoharmonic map heat flow (1.2) is uniformly bounded.
- Using the higher order regularity theory of Folland-Stein space $S^{k,p}$ to show the existence of global smooth solution to (1.2).

The main difficulty comes from the CR Bochner formula (2.1) with a mixed term $\langle J\nabla b u, \nabla b u_0 \rangle_{L^0}$ involving the covariant derivative of $u$ in the direction of $T$, which is hard to control. However, by adding an extra energy density

$$e_0(\varphi) := g_{ij} \varphi_0^i \varphi_0^j$$

and then estimate the total energy density

$$\hat{e}(\varphi) = 2e(\varphi) + e_0(\varphi),$$

we are able to overcome such a difficulty.
3. On the pseudoharmonic map heat flow

We first need the following lemma, which states that the energy $E(\varphi(t))$ for (1.2) is monotonically decreasing.

**Lemma 3.1**

For any $0 < T \leq \infty$, if $\varphi \in C^\infty(M \times [0, T); N)$ solves (1.2), then

$$E(\varphi(t)) + \int_0^t \int_M |\partial_s \varphi|^2 d\mu ds = E(f), \quad \forall \ t \in [0, T). \quad (3.1)$$
3. On the pseudoharmonic map heat flow

The following lemma gives the CR version of Bochner identity for the pseudoharmonic map heat flow (1.2).

**Lemma 3.2**

Let \((M^{2n+1}, J, \theta)\) be a closed pseudohermitian manifold, \((N^m, g)\) be a Riemannian manifold and let \(\phi \in C^\infty(M \times [0, T); N)\) be a solution to the pseudoharmonic map heat flow (1.2). If \([\Delta_b, T] = 0\), then there holds

\[
\left( \frac{\partial}{\partial t} - \Delta_b \right) \hat{e}(\phi) = - \sum_{k=1}^{m} \left[ |\nabla^2_b \phi^k|^2 + (2Ric - (n - 2) Tor)((\nabla_b \phi^k)_c, (\nabla_b \phi^k)_c) \right] \\
+ 2 \langle J\nabla_b \phi^k, \nabla_b \phi^k \rangle_{L_\theta} + 2 |\nabla_b \phi^k|^2 \\
+ \sum_{ij, k, \ell=1}^{m} \sum_{\alpha, \beta=1}^{n} \left[ 2\tilde{R}_{ijkl} \phi^i_\alpha \phi^j_\beta \phi^k_\alpha \phi^\ell_\beta + 2\tilde{R}_{ijkl} \phi^i_\alpha \phi^j_\beta \phi^k_\alpha \phi^\ell_\beta \right] \\
+ 4 \sum_{i, j, k, \ell=1}^{m} \sum_{\alpha=1}^{n} \tilde{R}_{ijkl} \phi^i_\alpha \phi^j_0 \phi^k_\alpha \phi^\ell_0.
\]
3. On the pseudoharmonic map heat flow

**Theorem 3.1**

Let $(M^{2n+1}, J, \theta)$ be a closed pseudohermitian manifold, $(N^n, g)$ be a Riemannian manifold with nonpositive sectional curvature $\tilde{K}^N$. Let $\varphi \in C^\infty(M \times [0, T); N)$ be a solution of (1.2). If

$$[\Delta_b, T] = 0,$$

then it holds

$$\left(\frac{\partial}{\partial t} - \Delta_b\right)\tilde{e}(\varphi) \leq C\tilde{e}(\varphi).$$

Here $C$ is a positive constant depends on the pseudohermitian Ricci tensor and torsion of $(M, J, \theta)$. 
3. On the pseudoharmonic map heat flow

Before we go further, let’s recall Moser’s Harnack inequality ([19]). Let

\[ \mathcal{L} = \frac{\partial}{\partial t} - \Delta_b \]

be the heat operator on \((M^{2n+1}, J, \theta)\). For \(z_0 = (x_0, t_0) \in M \times (0, \infty)\), let \(0 < \delta < \text{diam}(M)\), \(0 < \tau < t_0\) and let \(R(z_0, \delta, \tau)\) be the cylinder

\[ R(z_0, \delta, \tau) = \{(x, t) \in M \times [0, \infty) : |x - x_0| < \delta, \ t_0 - \tau < t < t_0\}. \]

**Lemma 3.3**

Let \(u\) be a positive smooth solution of

\[ \mathcal{L}u \leq 0 \]

in \(R(z_0, \delta, \tau)\). Then we have

\[ u(z_0) \leq C \int_{R(z_0, \delta, \tau)} u(x, t)d\mu dt, \]

where \(C > 0\) is a constant depends only on \(n, \delta\) and \(\tau\).
3. On the pseudoharmonic map heat flow

To prove our main theorem, we need one more lemma.

**Lemma 3.4**

Let \((M^{2n+1}, J, \theta)\) be a closed pseudohermitian manifold and \((N^m, g)\) be a Riemannian manifold with nonpositive sectional curvature \(\tilde{K}^N\). For any \(0 < T \leq \infty\), if \(\varphi \in C^\infty(M \times [0, T); N)\) solves (1.2), then

\[
\hat{E}(\varphi(t)) := E(\varphi(t)) + E^0(\varphi(t))
\]

is decreasing in \(t\). Here \(E^0(\varphi(t))\) is given by

\[
E^0(\varphi(t)) := \int_M e_0(\varphi) d\mu = \int_M g_{ij} \varphi_0^i \varphi_0^j d\mu.
\]

**Remark 3.1**

The energy \(E^0(\varphi(t))\) is decreasing in \(t\) under the flow only in case of \((N^m, g)\) with nonpositive sectional curvature \(\tilde{K}^N\).
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The proof of Theorem 1.1

We first show that $|\nabla_b \varphi|$ is uniformly bounded. In fact, we will show that $\hat{e}(\varphi)$ is uniformly bounded. Let

$$F(x, t) := e^{-Ct} \hat{e}(\varphi(x, t)), \quad (x, t) \in M \times [0, T).$$

Here $C > 0$ is a constant satisfying

$$\left( \frac{\partial}{\partial t} - \Delta_b \right) \hat{e}(\varphi) \leq C \hat{e}(\varphi).$$

It is easy to check that

$$\left( \frac{\partial}{\partial t} - \Delta_b \right) F(x, t) = e^{-Ct} \left[ \left( \frac{\partial}{\partial t} - \Delta_b \right) \hat{e}(\varphi) - C \hat{e}(\varphi) \right] \leq 0.$$
3. On the pseudoharmonic map heat flow

Then for any $z_0 = (x_0, t_0) \in M \times [0, T)$ we have by Lemma 3.3, that

$$\hat{e}(\varphi(z_0)) \leq C_1 e^{-C t_0} \int_{R(z_0, \delta, 1)} F(x, s) d\mu ds$$

$$= C_1 \int_{t_0 - 1}^{t_0} \int_{B_\delta(x_0)} e^{-C(t_0 - s)} \hat{e}(\varphi(x, s)) d\mu ds$$

$$\leq C_1 \int_{t_0 - 1}^{t_0} \int_{B_\delta(x_0)} \hat{e}(\varphi(x, s)) d\mu ds$$

$$\leq C_1 \int_{t_0 - 1}^{t_0} \hat{E}(\varphi(s)) ds \leq C_1 \hat{E}(f),$$

since $\hat{E}(\varphi(t))$ is decreasing in $t$ by Lemma 3.4. This shows that $|\nabla_b \varphi|$ is uniformly bounded and we conclude that $\varphi \in C^\infty(M \times [0, \infty); N)$ by the higher order regularity theory of the Folland-Stein space (see [7] and [2]).
By a direct computation one has

\[(\frac{\partial}{\partial t} - \Delta_b)|\partial_t \varphi|^2\]

\[= -2 \sum_{k=1}^{m} \left| \nabla_b \left( \frac{\partial \varphi^k}{\partial t} \right) \right|^2 + 4 \sum_{i,j,k,\ell=1}^{m} \sum_{\alpha=1}^{n} \tilde{R}_{ijk\ell} \varphi_{\alpha}^i \frac{\partial \varphi^j}{\partial t} \varphi_{\alpha}^k \frac{\partial \varphi^\ell}{\partial t}\]

\[= -2 \sum_{k=1}^{m} \left| \nabla_b \left( \frac{\partial \varphi^k}{\partial t} \right) \right|^2

+ 4 \sum_{\alpha=1}^{n} \left[ \tilde{R}(\text{Re}(X_{\alpha}), W, \text{Re}(X_{\alpha}), W) + \tilde{R}(\text{Im}(X_{\alpha}), W, \text{Im}(X_{\alpha}), W) \right],\]

where \(X_{\alpha} = \varphi_{\alpha}^i \partial y_i\) and \(W = \frac{\partial \varphi^i}{\partial t} \partial y_i\) with \(\partial y_i = \partial / \partial y^i\) the local coordinates of \(N\). Thus, the sectional curvature \(\tilde{K}^N\) of \(N\) is nonpositive implies that

\[(\frac{\partial}{\partial t} - \Delta_b)|\partial_t \varphi|^2 \leq 0.\]
Thus, from Lemma 3.3 we have

$$\| \partial_t \varphi \|_{C^0(M \times [t-1,t])} \leq C \| \partial_t \varphi \|_{L^2(M \times [t-2,t])}.$$ 

Since from Lemma 3.1,

$$\int_0^t \int_M |\partial_s \varphi|^2 d\mu ds \leq E(f) < +\infty, \quad \forall \, t,$$

we see that

$$\lim_{t \to \infty} \int_{t-2}^t \int_M |\partial_s \varphi|^2 d\mu ds = 0$$

and so

$$\| \partial_t \varphi \|_{C^0(M \times [t-1,t])} \leq C \| \partial_t \varphi \|_{L^2(M \times [t-2,t])} \to 0$$

as $t \to \infty$. 

Therefore, we may choose a sequence \( \{t_\ell\}_\ell \) with \( t_\ell \uparrow \infty \) as \( \ell \to \infty \) such that \( \varphi_t(\cdot, t_\ell) \to 0 \) in \( C^0(M) \) and \( \varphi(\cdot, t_\ell) \to \varphi_\infty \) in \( C^2(M; N) \). Since

\[
\Delta_b(\varphi_\infty)^k + 2\tilde{\Gamma}^k_{ij}(\varphi_\infty)^i\alpha(\varphi_\infty)^j\alpha = \lim_{\ell \to \infty} \left[ \Delta_b \varphi^k(\cdot, t_\ell) + 2\tilde{\Gamma}^k_{ij}\varphi^i\alpha(\cdot, t_\ell)\varphi^j\alpha(\cdot, t_\ell) \right]
\]

\[
= \lim_{\ell \to \infty} \varphi_t(\cdot, t_\ell) = 0,
\]

\( \varphi_\infty \in C^2(M; N) \) satisfies the Euler-Lagrange equations and hence we conclude by Lemma 2.2, that \( \varphi_\infty \) is a pseudoharmonic map.
Open problems

1° The monotonicity inequality for the pseudoharmonic map heat flow.
2° Finite time blow-up of solutions.
Appendix I. The Folland-Stein space

We recall below what the Folland-Stein space $S^{k,p}$ is. Let $D$ denote a differential operator acting on functions. We say $D$ has weight $m$, denoted $w(D) = m$, if $m$ is the smallest integer such that $D$ can be locally expressed as a polynomial of degree $m$ in vector fields tangent to the contact bundle $\xi$. We define the Folland-Stein space $S^{k,p}$ of functions on $M$ by

$$S^{k,p} = \{ f \in L^p : Df \in L^p \text{ whenever } w(D) \leq k \}.$$ 

We define the $L^p$ norm of $\nabla_b f$, $\nabla_b^2 f$, ... to be $(\int |\nabla_b f|^p \theta \wedge d\theta)^{1/p}$, $(\int |\nabla_b^2 f|^p \theta \wedge d\theta)^{1/p}$, ..., respectively, as usual. So it is natural to define the $S^{k,p}$ norm of $f \in S^{k,p}$ as follows:

$$\|f\|_{S^{k,p}} \equiv \left( \sum_{0 \leq j \leq k} \|\nabla_b^j f\|^p_{L^p} \right)^{1/p}.$$ 

The function space $S^{k,p}$ with the above norm is a Banach space for $k \geq 0$, $1 < p < \infty$. There are also embedding theorems of Sobolev type. (\cite{2, 7, 8}).
Appendix II. The proofs

The proof of Lemma 2.1

First from [11], we have for a real smooth function \( u \)

\[
\frac{1}{2} \Delta_b |\nabla_b u|^2 = |\nabla_b^2 u|^2 + \langle \nabla_b u, \nabla_b \Delta_b u \rangle_{L_\theta} - 2i \sum_{\alpha=1}^{n} (u_\alpha u_{\bar{\alpha}0} - u_{\bar{\alpha}} u_\alpha 0) \tag{5.1}
\]

\[+ [2Ric - nTor] ((\nabla_b u)_C, (\nabla_b u)_C).\]

Next from Lemma 2.2 of [3], one has that

\[i \sum_{\alpha} (u_\alpha u_{\bar{\alpha}0} - u_{\bar{\alpha}} u_\alpha 0) = i \sum_{\alpha} (u_\alpha u_0{\bar{\alpha}} - u_{\bar{\alpha}} u_0 \alpha) - Tor ((\nabla_b u)_C, (\nabla_b u)_C)\]

\[= -\langle J\nabla_b u, \nabla_b u_0 \rangle_{L_\theta} - Tor ((\nabla_b u)_C, (\nabla_b u)_C). \tag{5.2}\]

Then Lemma 2.1 follows from (5.1) and (5.2).
The proof of Lemma 2.2

Let $\varphi_t, -\varepsilon < t < \varepsilon$, be a smooth variation of $\varphi$ so that

$$
\varphi_0 = \varphi, \quad \text{and} \quad \frac{d\varphi_t}{dt} \bigg|_{t=0} = V \in \Gamma(\varphi^{-1}TN).
$$

$\varphi_t$ may be viewed as a map from $(-\varepsilon, \varepsilon) \times M$ into $N$. By direct computations one has

$$
\frac{d}{dt} E(\varphi_t) = \frac{1}{2} \frac{d}{dt} \int_M g_{ij}(\varphi_t) \varphi_t^i \varphi_t^j d\mu
$$

$$
= -\frac{1}{2} \int_M g_{k\ell} \left[ \Delta_b \varphi_t^k + 2\tilde{\Gamma}_{ij}^k \varphi_t^i \varphi_t^{j\bar{\alpha}} \right] \frac{d\varphi_t^\ell}{dt} d\mu
$$

$$
= -\frac{1}{2} \int_M \left< \frac{d\varphi_t}{dt}, \tau(\varphi_t) \right>_N d\mu.
$$
Thus, the first variational formula is given by

\[
\frac{d}{dt} E(\varphi_t) \bigg|_{t=0} = -\frac{1}{2} \int_M \langle V, \tau(\varphi) \rangle_N d\mu,
\]

where \( \tau(\varphi) \) is so called the tension field of \( \varphi \), which is defined by

\[
\tau(\varphi) = \sum_{k=1}^{m} \left[ \Delta_b \varphi^k + 2\tilde{\Gamma}_{ij}^k \varphi^i \varphi^j \alpha \right] \frac{\partial}{\partial y_k}.
\]

Therefore \( \varphi \in C^2(M; N) \) is a critical point of the energy functional \( E \) if and only if its tension field \( \tau(\varphi) \) vanishes identically. That is, \( \varphi \) is pseudoharmonic if and only if it satisfies the Euler-Lagrange equations (2.2).

\[ \square \]
Appendix II. The proofs

The proof of Lemma 3.1

By integration by parts, one has

\[ \frac{d}{ds} E(\varphi(s)) = \int_M \frac{\partial}{\partial s} \left( g_{ij} \varphi_\beta^i \varphi_\beta^j \right) d\mu \]

\[ = \int_M g_{ij,k} \varphi_\beta^i \varphi_\beta^j \frac{\partial \varphi^k}{\partial s} d\mu - \int_M g_{ij} \frac{\partial \varphi^i}{\partial s} \varphi_\beta^j d\mu - \int_M g_{ij,k} \varphi_\beta^i \varphi_\beta^j \frac{\partial \varphi^k}{\partial s} d\mu \]

\[ - \int_M g_{ij} \frac{\partial \varphi^i}{\partial s} \varphi_\beta^j d\mu - \int_M g_{ij,k} \varphi_\beta^i \varphi_\beta^j \frac{\partial \varphi^k}{\partial s} d\mu \]

\[ = -\int_M g_{ij} \frac{\partial \varphi^i}{\partial s} \Delta_b \varphi^j d\mu - \int_M \left( g_{kj,i} + g_{ik,j} - g_{ij,k} \right) \varphi_\beta^i \varphi_\beta^j \frac{\partial \varphi^k}{\partial s} d\mu \]

\[ = -\int_M g_{ij} \frac{\partial \varphi^i}{\partial s} \left[ \Delta_b \varphi^j + 2\tilde{\Gamma}_{pq}^j \varphi_\beta^p \varphi_\beta^q \right] d\mu = -\int_M g_{ij} \frac{\partial \varphi^i}{\partial s} \frac{\partial \varphi^j}{\partial s} d\mu. \]

That is,

\[ \frac{d}{ds} E(\varphi(s)) = -\int_M |\partial_s \varphi|^2 d\mu. \]

Integrating the above equality over \([0, t]\) gives (3.1).
The proof of Lemma 3.4

From (5.6) and (5.8) one has

\[
\frac{d}{dt} E^0(\varphi(t)) = \int_M \frac{\partial}{\partial t} (e_0(\varphi)) \, d\mu \\
= \int_M \left[ 4\tilde{R}^0 - 2 \sum_{k=1}^m |\nabla b \varphi_0^k|^2 \right] \, d\mu \leq 0,
\]

provided the sectional curvature $\tilde{K}^N$ of $N$ is nonpositive. This says that $E^0(\varphi(t))$ is decreasing in $t$. Since $E(\varphi(t))$ is also decreasing in $t$ (see Lemma 3.1), we then conclude that $\hat{E}(\varphi(t))$ is decreasing in $t$. \qed
The proof of Lemma 3.2

Recall that \( \hat{e}(\varphi) = 2e(\varphi) + e_0(\varphi) \) with

\[
e_0(\varphi) := g_{ij}\varphi_0^i\varphi_0^j.
\]

Since \( e(\varphi) \) and \( \hat{e}(\varphi) \) are independent of the choice of local coordinates, for each point \((p, t)\) one may choose a normal coordinate chart \( U \) at \((p, t)\) and a normal coordinate chart \( V \) at \( \varphi(p, t) \) such that \( \varphi(U) \subset V \) and then fulfill the following computations at the point \((p, t)\).
Appendix II. The proofs

(a) We first compute \((\frac{\partial}{\partial t} - \Delta_b)(2e(\varphi))\).

\[
\frac{\partial}{\partial t} (2e(\varphi)) = \frac{\partial}{\partial t} \left( \sum_{i,j = 1}^{m} \sum_{\beta = 1}^{n} g_{ij} \varphi_{\beta}^i \varphi_{\bar{\beta}}^j \right)
\]

\[
= \sum_{k=1}^{m} \sum_{\beta = 1}^{n} \left\{ \varphi_{\beta}^k \left( \frac{\partial \varphi_{\beta}^k}{\partial t} \right)_{\bar{\beta}} + \varphi_{\bar{\beta}}^k \left( \frac{\partial \varphi_{\beta}^k}{\partial t} \right)_{\bar{\beta}} \right\}
\]

\[
= \sum_{k=1}^{m} \sum_{\alpha, \beta = 1}^{n} \left[ \varphi_{\beta}^k \left( \Delta_b \varphi_{\beta}^k + 2 \tilde{\Gamma}_{ij}^k \varphi_{\alpha}^i \varphi_{\bar{\alpha}}^j \right)_{\bar{\beta}} + \varphi_{\bar{\beta}}^k \left( \Delta_b \varphi_{\beta}^k + 2 \tilde{\Gamma}_{ij}^k \varphi_{\alpha}^i \varphi_{\bar{\alpha}}^j \right)_{\bar{\beta}} \right]
\]

\[
= \sum_{k=1}^{m} \left\langle \nabla_b \varphi_{\beta}^k, \nabla_b \Delta_b \varphi_{\beta}^k \right\rangle_{L_0}
\]

\[
+ 2 \sum_{i,j,k,\ell = 1}^{m} \sum_{\alpha, \beta = 1}^{n} \left[ \tilde{\Gamma}_{ij}^k \varphi_{\alpha}^i \varphi_{\alpha}^{j} \varphi_{\beta}^k \varphi_{\bar{\beta}}^\ell + \tilde{\Gamma}_{ij}^k \varphi_{\alpha}^i \varphi_{\alpha}^{j} \varphi_{\bar{\beta}}^k \varphi_{\beta}^\ell \right].
\]
Appendix II. The proofs

From the CR Bochner formula (Lemma 2.1), one has

\[ \Delta_b(2e(\varphi)) = \Delta_b\left( \sum_{i,j=1}^{m} g_{ij} \varphi^i_{\alpha} \varphi^i_{\bar{\alpha}} \right) \]

\[ = \frac{1}{2} \sum_{k=1}^{m} \Delta_b |\nabla_b \varphi^k|^2 + \sum_{i,j=1}^{m} \varphi^i_{\alpha} \varphi^i_{\bar{\alpha}} \Delta_b(g_{ij}) \]

\[ = \sum_{k=1}^{m} \left[ |\nabla^2_b \varphi^k|^2 + \langle \nabla_b \varphi^k, \nabla_b \Delta_b \varphi^k \rangle_{L_\theta} + 2 \langle J\nabla_b \varphi^k, \nabla_b \varphi^k_0 \rangle_{L_\theta} \right. \]

\[ + \left. (2Ric - (n - 2) Tor) ((\nabla_b \varphi^k)_C, (\nabla_b \varphi^k)_C) \right] \]

\[ + \sum_{i,j=1}^{m} \sum_{\alpha=1}^{n} \varphi^i_{\alpha} \varphi^i_{\bar{\alpha}} \Delta_b(g_{ij}). \]
Appendix II. The proofs

Thus

\[
\left( \frac{\partial}{\partial t} - \Delta_b \right) (2e(\varphi)) = - \sum_{k=1}^{m} \left[ |\nabla_b \varphi^k|^2 + 2 \langle J \nabla_b \varphi^k, \nabla_b \varphi_0^k \rangle_{L^\theta} \right.

+ (2Ric - (n - 2) Tor) \left( (\nabla_b \varphi^k)_C, (\nabla_b \varphi^k)_C \right) \\

+ 2 \sum_{i,j,k,\ell=1}^{m} \sum_{\alpha,\beta=1}^{n} \left[ \tilde{\Gamma}^{k}_{ij,\ell} \varphi^i_\alpha \varphi^j_\alpha \varphi^k_\beta \varphi^\ell_\beta + \tilde{\Gamma}^{k}_{ij,\ell} \varphi^i_\alpha \varphi^j_\alpha \varphi^k_\beta \varphi^\ell_\beta \right] \\

- \sum_{i,j=1}^{m} \sum_{\alpha=1}^{n} \varphi^i_\alpha \varphi^j_\alpha \Delta_b(g_{ij}).
\]

(5.3)
Furthermore at the point \((p, t)\),

\[
2 \sum_{i,j,k,\ell=1}^{m} \sum_{\alpha,\beta=1}^{n} \left[ \tilde{\Gamma}^{k}_{ij,\ell} \varphi_{\alpha} \varphi_{\alpha} \varphi_{\beta} \varphi_{\beta} + \tilde{\Gamma}^{k}_{ij,\ell} \varphi_{\alpha} \varphi_{\alpha} \varphi_{\beta} \varphi_{\beta} \right] - \sum_{i,j=1}^{m} \sum_{\alpha=1}^{n} \varphi_{\alpha} \varphi_{\alpha} \Delta b(g_{ij})
\]

\[
= \sum_{i,j,k,\ell=1}^{m} \sum_{\alpha,\beta=1}^{n} \left[ 2\tilde{\Gamma}^{k}_{ij,\ell} \varphi_{\alpha} \varphi_{\alpha} \varphi_{\beta} \varphi_{\beta} + 2\tilde{\Gamma}^{k}_{ij,\ell} \varphi_{\alpha} \varphi_{\alpha} \varphi_{\beta} \varphi_{\beta} - 2g_{ij,\ell} \varphi_{\alpha} \varphi_{\alpha} \varphi_{\beta} \varphi_{\beta} \right]
\]

\[
= \sum_{i,j,k,\ell=1}^{m} \sum_{\alpha,\beta=1}^{n} \left[ (g_{kj,\ell} + g_{ik,\ell} - g_{ij,\ell}) \varphi_{\alpha} \varphi_{\alpha} \varphi_{\beta} \varphi_{\beta} \right]
\]

\[
- (g_{ik,\ell} + g_{kj,\ell} + g_{ij,\ell}) \varphi_{\alpha} \varphi_{\alpha} \varphi_{\beta} \varphi_{\beta} - 2g_{ij,\ell} \varphi_{\alpha} \varphi_{\alpha} \varphi_{\beta} \varphi_{\beta} \right]
\]

\[
= \sum_{i,j,k,\ell=1}^{m} \sum_{\alpha,\beta=1}^{n} g_{ij,\ell} \left[ \varphi_{\alpha} \varphi_{\beta} \varphi_{\alpha} \varphi_{\beta} + \varphi_{\alpha} \varphi_{\beta} \varphi_{\alpha} \varphi_{\beta} \right]
\]

\[
+ \varphi_{\alpha} \varphi_{\beta} \varphi_{\alpha} \varphi_{\beta} + \varphi_{\alpha} \varphi_{\beta} \varphi_{\alpha} \varphi_{\beta} - 4\varphi_{\alpha} \varphi_{\alpha} \varphi_{\beta} \varphi_{\beta} \right].
\]
Appendix II. The proofs

On the other hand, we have

\[
\sum_{i,j,k,\ell=1}^{m} \sum_{\alpha,\beta=1}^{n} \left[ 2\tilde{R}_{ijk\ell} \phi^i_{\alpha} \phi^j_{\beta} \phi^k_{\alpha} \phi^\ell_{\beta} + 2\tilde{R}_{ijk\ell} \phi^i_{\alpha} \phi^j_{\beta} \phi^k_{\alpha} \phi^\ell_{\beta} \right] = \sum_{i,j,k,\ell=1}^{m} \sum_{\alpha,\beta=1}^{n} \left[ 2(\tilde{\Gamma}_{j\ell,k} - \tilde{\Gamma}_{jk,\ell}) \phi^i_{\alpha} \phi^j_{\beta} \phi^k_{\alpha} \phi^\ell_{\beta} + 2(\tilde{\Gamma}_{j\ell,k} - \tilde{\Gamma}_{jk,\ell}) \phi^i_{\alpha} \phi^j_{\beta} \phi^k_{\alpha} \phi^\ell_{\beta} \right]
\]

\[
= \sum_{i,j,k,\ell=1}^{m} \sum_{\alpha,\beta=1}^{n} \left[ (g_{i\ell,jk} + g_{ji,\ell k} - g_{j\ell,ik}) - (g_{ik,\ell j} + g_{ji,\ell k} - g_{jk,i\ell}) \right] \phi^i_{\alpha} \phi^j_{\beta} \phi^k_{\alpha} \phi^\ell_{\beta} + \phi^i_{\alpha} \phi^j_{\beta} \phi^k_{\alpha} \phi^\ell_{\beta}
\]

\[
+ \sum_{i,j,k,\ell=1}^{m} \sum_{\alpha,\beta=1}^{n} \left[ (g_{i\ell,jk} + g_{ji,\ell k} - g_{j\ell,ik}) - (g_{ik,\ell j} + g_{ji,\ell k} - g_{jk,i\ell}) \right] \phi^i_{\alpha} \phi^j_{\beta} \phi^k_{\alpha} \phi^\ell_{\beta}
\]

\[
= \sum_{i,j,k,\ell=1}^{m} \sum_{\alpha,\beta=1}^{n} g_{ij,\ell k} \left[ \phi^i_{\alpha} \phi^j_{\beta} \phi^k_{\alpha} \phi^\ell_{\beta} + \phi^i_{\alpha} \phi^j_{\beta} \phi^k_{\alpha} \phi^\ell_{\beta} \right]
\]

\[
+ \phi^i_{\alpha} \phi^j_{\beta} \phi^k_{\alpha} \phi^\ell_{\beta} + \phi^i_{\alpha} \phi^j_{\beta} \phi^k_{\alpha} \phi^\ell_{\beta} - 4\phi^i_{\alpha} \phi^j_{\beta} \phi^k_{\alpha} \phi^\ell_{\beta} \right].
\]
Appendix II. The proofs

Therefore, equation (5.3) gives

\[
\left( \frac{\partial}{\partial t} - \Delta_b \right) (2e(\varphi)) = - \sum_{k=1}^{m} \left[ |\nabla^2_b \varphi^k|^2 + 2 \langle J \nabla_b \varphi^k, \nabla_b \varphi_0^k \rangle_{L^\theta} \right.
\]

\[
+ \left. (2 \text{Ric} - (n - 2) \text{Tor}) \left( (\nabla_b \varphi^k)_C, (\nabla_b \varphi^k)_C \right) \right]
\]

\[
+ \sum_{i,j,k,\ell=1}^{m} \sum_{\alpha,\beta=1}^{n} \left[ 2 \tilde{R}_{ijkl} \varphi^{i}_\alpha \varphi^{j}_\beta \varphi^{k}_{\bar{\alpha}} \varphi^{\ell}_{\bar{\beta}} + 2 \tilde{R}_{ijkl} \varphi^{i}_\alpha \varphi^{j}_\beta \varphi^{k}_{\bar{\alpha}} \varphi^{\ell}_{\bar{\beta}} \right].
\]
Appendix II. The proofs

(b) We next compute \((\frac{\partial}{\partial t} - \Delta_b)e_0(\varphi)\). Again, we have at the point \((p, t)\), that

\[
\frac{\partial}{\partial t}(e_0(\varphi)) = \frac{\partial}{\partial t}\left(\sum_{i,j=1}^{m} g_{ij}\varphi_i^0\varphi_j^0\right) = 2 \sum_{k=1}^{m} \varphi_k^0\left(\frac{\partial\varphi_k^0}{\partial t}\right)_0. 
\]

We also compute

\[
\Delta_b(e_0(\varphi)) = \Delta_b\left(\sum_{i,j=1}^{m} g_{ij}\varphi_i^0\varphi_j^0\right) = \sum_{k=1}^{m} \left[2\varphi_k^0\Delta_b\varphi_k^0 + 2|\nabla_b\varphi_k^0|^2\right] + \sum_{i,j=1}^{m} \varphi_i^0\varphi_j^0\Delta_b(g_{ij}).
\]

Ting-Hui Chang (Jointed with Prof. Shu-ChOn the existence of pseudoharmonic maps fi}
Thus, under the assumption that $[\Delta_b, T] = 0$ we have

\[
\left( \frac{\partial}{\partial t} - \Delta_b \right) (e_0(\phi)) = \sum_{k=1}^{m} \left[ \left( \frac{\partial}{\partial t} - \Delta_b \right) \phi^k \right)_0 - 2|\nabla_b \phi_0^k|^2 - \sum_{i,j=1}^{m} \phi_0^i \phi_0^j \Delta_b (g_{ij}) - 2 \sum_{k=1}^{m} |\nabla_b \phi_0^k|^2. \tag{5.5}
\]
Appendix II. The proofs

As what we computed in part (a), it is easy to see that

\[
4 \sum_{i,j,k,\ell=1}^{m} \sum_{\alpha=1}^{n} \tilde{\Gamma}_{ij}^{k} \varphi_{\alpha}^{i} \varphi_{\partial}^{j} \varphi_{0}^{k} \varphi_{0}^{\ell} - \sum_{i,j=1}^{m} \varphi_{0}^{i} \varphi_{0}^{j} \Delta_{b}(g_{ij})
\]

\[= 4 \sum_{i,j,k,\ell=1}^{m} \sum_{\alpha=1}^{n} \tilde{R}_{ijk\ell} \varphi_{\alpha}^{i} \varphi_{0}^{j} \varphi_{\partial}^{k} \varphi_{0}^{\ell} \]

and so

\[
\left(\frac{\partial}{\partial t} - \Delta_{b}\right)(e_{0}(\varphi)) = 4 \sum_{i,j,k,\ell=1}^{m} \sum_{\alpha=1}^{n} \tilde{R}_{ijk\ell} \varphi_{\alpha}^{i} \varphi_{0}^{j} \varphi_{\partial}^{k} \varphi_{0}^{\ell} - 2 \sum_{k=1}^{m} |\nabla b \varphi_{0}^{k}|^{2}. \tag{5.6}
\]

Therefore, Lemma 3.2 follows from (5.4) and (5.6).
Appendix II. The proofs

The proof of Theorem 3.1

By Lemma 3.2 and using the Cauchy inequality, one has

\[
\left( \frac{\partial}{\partial t} - \Delta_b \right) \widehat{e}(\varphi) \\
\leq - \sum_{k=1}^{m} \left[ |\nabla_b^2 \varphi_k|^2 + (2\text{Ric} - (n - 2)\text{Tor}) \left( (\nabla_b \varphi^k)_c, (\nabla_b \varphi^k)_c \right) \right] \\
+ \sum_{k=1}^{m} \left[ (\varepsilon - 2) |\nabla_b \varphi_0^k|^2 + \frac{1}{\varepsilon} |\nabla_b \varphi^k|^2 \right] + 2\widehat{R} + 4\widehat{R}^0,
\]

for some positive constant \( \varepsilon \). Here

\[
\widehat{R} = \sum_{ij, k, \ell=1}^{m} \sum_{\alpha, \beta=1}^{n} \left[ \widehat{R}_{ijkl} \varphi_i^\alpha \varphi_j^\beta \varphi^k_{\alpha} \varphi^\ell_{\beta} + \widehat{R}_{ijkl} \varphi_i^\alpha \varphi_j^\beta \varphi^k_{\alpha} \varphi^\ell_{\beta} \right]
\]

and

\[
\widehat{R}^0 = \sum_{i, j, k, \ell=1}^{m} \sum_{\alpha=1}^{n} \widehat{R}_{ijkl} \varphi_i^\alpha \varphi_0^j \varphi^k_{\alpha} \varphi_0^\ell.
\]
Appendix II. The proofs

Let \( \partial y_i = \partial / \partial y^i \) be the local coordinate of \( N \) and
\[
\tilde{R}(U, V, W, Z) := \langle \tilde{R}(W, Z) V, U \rangle.
\]
Thus
\[
\tilde{R}_{ijk\ell} \varphi^i_\alpha \varphi^j_\beta \varphi^k_\alpha \varphi^\ell_\beta
= \langle \tilde{R}(\varphi^k_\alpha \partial y_k, \varphi^\ell_\beta \partial y_\ell) \varphi^j_\beta \partial y_j, \varphi^i_\alpha \partial y_i \rangle = \tilde{R}(X_\alpha, Y_\beta, \overline{X_\alpha}, \overline{Y_\beta})
\]
\[
= \tilde{R}(\text{Re}(X_\alpha), \text{Re}(Y_\beta), \text{Re}(X_\alpha), \text{Re}(Y_\beta)) + \tilde{R}(\text{Re}(X_\alpha), \text{Im}(Y_\beta), \text{Re}(X_\alpha), \text{Im}(Y_\beta))
+ \tilde{R}(\text{Im}(X_\alpha), \text{Re}(Y_\beta), \text{Im}(X_\alpha), \text{Re}(Y_\beta)) + \tilde{R}(\text{Im}(X_\alpha), \text{Im}(Y_\beta), \text{Im}(X_\alpha), \text{Im}(Y_\beta))
- 2 \tilde{R}(\text{Re}(X_\alpha), \text{Re}(Y_\beta), \text{Im}(X_\alpha), \text{Im}(Y_\beta)) + 2 \tilde{R}(\text{Re}(X_\alpha), \text{Im}(Y_\beta), \text{Im}(X_\alpha), \text{Re}(Y_\beta)),
\]
where \( X_\alpha = \varphi^i_\alpha \partial y_i, \ Y_\beta = \varphi^j_\beta \partial y_j. \)
Similarly, we have

\[ \tilde{R}_{ijk\ell} \varphi^i_\alpha \varphi^j_\beta \varphi^k_\alpha \varphi^\ell_\beta = \langle \tilde{R}(\varphi^k_\alpha \partial y_k, \varphi^\ell_\beta \partial y_\ell) \varphi^j_\beta \partial y_j, \varphi^i_\alpha \partial y_i \rangle = \tilde{R}(X_\alpha, Y_\beta, X_\alpha, Y_\beta) \]

\[ = \tilde{R}(\text{Re}(X_\alpha), \text{Re}(Y_\beta), \text{Re}(X_\alpha), \text{Re}(Y_\beta)) + \tilde{R}(\text{Re}(X_\alpha), \text{Im}(Y_\beta), \text{Re}(X_\alpha), \text{Im}(Y_\beta)) + \tilde{R}(\text{Im}(X_\alpha), \text{Re}(Y_\beta), \text{Im}(X_\alpha), \text{Re}(Y_\beta)) + \tilde{R}(\text{Im}(X_\alpha), \text{Im}(Y_\beta), \text{Im}(X_\alpha), \text{Im}(Y_\beta)) + 2\tilde{R}(\text{Re}(X_\alpha), \text{Re}(Y_\beta), \text{Im}(X_\alpha), \text{Im}(Y_\beta)) - 2\tilde{R}(\text{Re}(X_\alpha), \text{Im}(Y_\beta), \text{Im}(X_\alpha), \text{Re}(Y_\beta)) \]
and so
\[
\tilde{R} = \tilde{R}_{ijkl} \varphi_{\alpha}^i \varphi_{\beta}^j \varphi_{\alpha}^k \varphi_{\beta}^\ell + \tilde{R}_{ijkl} \varphi_{\alpha}^i \varphi_{\beta}^j \varphi_{\alpha}^k \varphi_{\beta}^\ell
\]
\[
= 2\tilde{R}(\text{Re}(X_{\alpha}), \text{Re}(Y_{\beta}), \text{Re}(X_{\alpha}), \text{Re}(Y_{\beta}))
\]
\[
+ 2\tilde{R}(\text{Re}(X_{\alpha}), \text{Im}(Y_{\beta}), \text{Re}(X_{\alpha}), \text{Im}(Y_{\beta}))
\]
\[
+ 2\tilde{R}(\text{Im}(X_{\alpha}), \text{Re}(Y_{\beta}), \text{Im}(X_{\alpha}), \text{Re}(Y_{\beta}))
\]
\[
+ 2\tilde{R}(\text{Im}(X_{\alpha}), \text{Im}(Y_{\beta}), \text{Im}(X_{\alpha}), \text{Im}(Y_{\beta})).
\]

Furthermore, similar computations show that
\[
\tilde{R}^0 = \tilde{R}(\text{Re}(X_{\alpha}), Z, \text{Re}(X_{\alpha}), Z) + \tilde{R}(\text{Im}(X_{\alpha}), Z, \text{Im}(X_{\alpha}), Z),
\]
where \(X_{\alpha} = \varphi_{\alpha}^i \partial y_i\) and \(Z = \varphi_{0}^i \partial y_i\). Therefore, if the sectional curvature \(\tilde{K}^N\) of \(N\) is nonpositive, we see that
\[
\tilde{R} \leq 0 \quad \text{and} \quad \tilde{R}^0 \leq 0.
\]
Appendix II. The proofs

Now by taking $\varepsilon = 2$ in (5.7) we obtain

\[
\left( \frac{\partial}{\partial t} - \Delta_b \right) \hat{e}(\varphi) 
\leq \sum_{k=1}^{m} \left[ (2Ric - (n - 2) Tor) \left( (\nabla_b \varphi^k)_C, (\nabla_b \varphi^k)_C \right) + \frac{1}{2} |\nabla_b \varphi^k|^2 \right]
\leq C \sum_{k=1}^{m} |\nabla_b \varphi^k|^2 \leq C \hat{e}(\varphi).
\]

Here $C$ is a positive constant depends on the pseudohermitian Ricci tensor and torsion of $(M, J, \theta)$.

\[\blacksquare\]


