Special solutions to Lagrangian mean curvature flow

便箋標題 2010/4/12

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The 3rd OCAMI-TIMS Joint international workshop on Differential Geometry and Geometric Analysis

March 13-15, 2011

Osaka City University, Japan

- $$\left.\frac{dA_t}{dt}\right|_{t=0} = -\int_M \langle H, E\rangle \, dV$$

H mean curv. v.

$$E = \frac{dF_t(M)}{dt}$$

MCF. $$\frac{\partial F_t(x)}{\partial t} = H(F_t(x))$$

- Initial Lag L in KE

($\because$ $d\omega(H,\cdot) = Ric|_L$)

$= 0$

$\Rightarrow$ smooth sol of MCF is Lag

LMCF

- possible way to find Lag and minimal / sLag

short time existence

- Singularity may occur

To study singularities Huisken

- parabolic blow up $\Rightarrow$ solitons in $\mathbb{R}^n$

  $(y,t) \to (c_j y, c_j^2 t)$, $c_j \to \infty$

- self-similar sol'n $\leftrightarrow$ $H = \alpha F^\perp$

  $\leftrightarrow$ $F_t = \sqrt{2\alpha t}\, F$, a sol to MCF

  (self-shrinkers, self-expanders)

- translating sol'n $\leftrightarrow$ $H = (T)^\perp$ for a const vector T

  $\leftrightarrow$ $F_t = F + tT$, a sol to MCF

  (T : translating vector)

We construct many new such soliton solutions with different geometric properties (discussions)

√ 1) L — & M. T. Wang, JDG 83 (2009)

- Self-similar HS Lag, the same ansatz as SW cones
- Eternal Lag Brakke sols. no mass loss
- implications (i) distinguish (2,1) cone from others

  (ii) consider weak sols in different category

  $n=2$

  all.

  (iii) possible modles for surgeries

  pair of SW cones

2) L — & W —, Trans of AMS . 2010

generalize to hd. different toplogies.

3) D. Joyce, L — & M. T. Tsui, JDG, 2010

- any pair of Lag planes with a necessary condition $\Rightarrow \exists$ Lag self-Expanders asymptotic to the planes

  different modles for surgeries

- as close sLag as we like (Lag angle small)

- translating sols as close sLag as we like

  analog to cigar or Bryant sols in Ricci flow

- many families of other sol^{n}

  Non-HS eternal Lag Brakke flow, no mass loss

- self-similar sol^{n} $\Rightarrow$ translating sol^{n}

Weak solution ?

MCF codim 1 : Level set formulation Evans & Spruck

(Viscosity solution) Chen, Giga & Goto

Brakke flow (also high codim) Brakke

(varifold setting from GMT)

Elliptic regularization Ilmanen

Q: Weak sol for LMCF ?

Do not exist in general

Need further restrictions or modifications

- Lag minimizers with cone singularities. (not in the AG sense)

Minimize area Schoen - Wolfson

among Lag integral currents, oriented

$n=2$ in map setting $\Rightarrow$ regularity theory

- Schoen - Wolfson cone

assume $p > q$

$p, q \in \mathbb{N}$, $(p, q) = 1$, $0 \leq s < 2\pi$

$$\gamma_{pq}(s) = \left(\sqrt{\frac{q}{p+q}}\, e^{ips},\ i\sqrt{\frac{p}{p+q}}\, e^{-iqs} \right) \subset S^3$$

obstructions to the existence of sLag or mLag

The cone over $\gamma_{pq}(s)$ is Hamiltonian stationary Lag.
(HS)

C_{pq} stable iff $|p-q| = 1$

(no oriented Lag smoothing near the singularity due to Maslov index $p-q \neq 0$)

Q: Which cones can be realized as singular cone of Lag minimizers?

Conj: C_{21}

We show it is true infinitesmally via Brakke flow integral version for MCF:

if $\left(\frac{dF_t(x)}{dt}\right)^{\perp} = H(F_t(x))$ $\varphi \in C_0^1(\mathbb{R}^n)$, then

$$\frac{d}{dt}\int \varphi(F_t(x))\, dV_t = \int D\varphi \cdot H(F_t)\, dV_t - \int \varphi \langle H, H\rangle\, dV_t$$

$(\delta(V_t, \varphi)(h(V_t)))$

A family of varifolds V_t is said to form a solution of the Brakke motion

if $\bar{D}\|V_t\|(\phi) \le \delta(V_t,\phi)(h(V_t))$ for each $\phi \in C_0^1(\mathbb{R}^n)$ with $\phi \ge 0$,

"=" called without mass loss

where $\bar{D}\|V_t\|(\phi) = \overline{\lim}_{t_1 \to t} \frac{\|V_{t_1}\|(\phi) - \|V_t\|(\phi)}{t_1 - t}$.

L-. & W.-

- $S(\mu,\theta) = (\cosh\mu\sqrt{q}\,e^{ip\theta},\ i\sinh\mu\sqrt{p}\,e^{-iq\theta})$ $\quad \mu \in \mathbb{R}$, $0 \le \theta < 2\pi$

  HSL. & $\vec{H}^{\perp} = -2CH$.

- $E(\mu,\theta) = (\sinh\mu\sqrt{q}\,e^{ip\theta},\ i\cosh\mu\sqrt{p}\,e^{-iq\theta})$

  HSL. & $\vec{H}^{\perp} = 2CH$ $\qquad C = \frac{pq}{2(p-q)} \quad (p>q)$

$$\Rightarrow\ S_t(\mu,\theta) = \sqrt{\frac{-t}{C}}\,(\cosh\mu\sqrt{q}\,e^{ip\theta},\ i\sinh\mu\sqrt{p}\,e^{-iq\theta}) \quad t<0$$

$$E_t(\mu,\theta) = \sqrt{\frac{t}{C}}\,(\sinh\mu\sqrt{q}\,e^{ip\theta},\ i\cosh\mu\sqrt{p}\,e^{-iq\theta}) \quad t>0$$

smooth sol for MCF.

- Castro & Lerma : HSL in $n=2$. must be locally as the examples we construct (proceeding of AMS, 2010)

Denote $x_1 = \sqrt{\frac{-t}{c}} \cosh \mu$, $x_2 = \sqrt{\frac{-t}{c}} \sinh \mu$

$$x_1^2 - x_2^2 = \frac{-t}{c} \Longrightarrow x_1^2 - x_2^2 = 0$$

$\mu > 0$ $\mu < 0$

as $t \to 0^-$, $S_t \to (+,+) \cup (+,-)$

$t \to 0^+$ $E_t \to (+,+) \cup (-,+)$

① modify S_t & E_t to match as $t \to 0^\pm$. for $\mu \in \mathbb{R}$

② Take $\mu \geq 0$

② Define $V_t = S_t^+$ for $t<0$ smooth

$C_{p,q}$ for $t=0$ singular at origin

E_t^+ for $t>0$ smooth

When $p>q>1$. V_t, $-\infty<t<\infty$, is an eternal sol for Brakke flow without mass loss

- $p-q>1$. $C_{p,q}$ unstable
- $C_{q+1,q}$. for $q>1$. can be resolved as above

area decreases

$\Rightarrow$ distinguish $C_{2,1}$ from others

Why can we smoothize $C_{p,q}$?

$\because$ create boundary

$t<0$ $\qquad$ $t=0$ $\qquad$ $t>0$ $\qquad$ $c = \frac{pq}{2(p-q)}$, $p>q$

$$\sqrt{\frac{-t}{c}}\sqrt{q}\,(e^{ips}, 0) \to 0 \to \sqrt{\frac{t}{c}}\sqrt{p}\,(0, ie^{-iqs}) \quad \text{boundary}$$

bound p copies hol. disks in z_1 plane

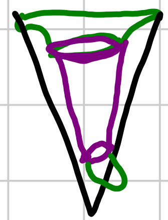
bound q copies anti-hol disk in z_2 plane

Key : no boundary contribution as varifolds if $p>1$, $q>1$

Q : Can we construct weak sol's for LMCF in general among this kind of varifolds ?

(no singular measure . ie no contribution from boundary)

- Our self-expanders + non-oriented filling will decrease area of C_{pq} $\forall p.q.$ for small t



Q: Can we construct weak sol/s for LMCF in general among $\mathbb{Z}_2$-currents ?

1

- $\exists$ eternal sol to Lag Brakke flow that $\boxed{\text{resolve}}$
  $C_{pq} \sqcup C'_{p,q}$ without mass loss

$t<0$. smooth HSL. & self-shrinkers

$t=0$ $C_{pq} \sqcup C'_{p,q}$ HSLC, single singularity at 0

$t>0$. smooth HSL & self-expanders

The smoothing works because we use

2 cones C_{pq} & C'_{pq}

Recall

$$S_t(\mu,\theta) = \sqrt{\frac{-t}{c}}\left(\cosh\mu\sqrt{q}\, e^{ip\theta},\ i\sinh\mu\sqrt{p}\, e^{-iq\theta}\right) \quad t<0$$

$$E_t(\mu,\theta) = \sqrt{\frac{t}{c}}\left(\sinh\mu\sqrt{q}\, e^{ip\theta},\ i\cosh\mu\sqrt{p}\, e^{-iq\theta}\right) \quad t>0$$

as $t \to 0^-$, $S_t \to \underset{\mu>0}{(+,+)} \cup \underset{\mu<0}{(+,-)}$

$t \to 0^+$ $E_t \to (+,+) \cup (-,+)$

does not match in general

<u>Case 1</u> p odd, q odd, $C_{+-} = C_{-+}$ $\quad \theta \to \theta + \pi$

match

V_t satisfy MCF. for $t<0$ & $t>0$

Only need to check at $t=0$

Proposition 3.1. *Suppose the varifold V_t, $a<t<b$ forms a smooth mean curvature flow in $\mathbb{R}^n$ except at $t=c\in(a,b)$ and $\|V_t\|$ converges in Radon measure to $\|V_c\|$ as $t\to c$. If $\lim_{t\to c^-}\frac{d}{dt}\|V_t\|(\phi)$ and $\lim_{t\to c^+}\frac{d}{dt}\|V_t\|(\phi)$ are both either finite or $-\infty$ and*

$$\lim_{t\to c^\pm}\frac{d}{dt}\|V_t\|(\phi)\le\delta(V_0,\phi)(h(V_0)) \tag{3.2}$$

for any $\phi\in C_0^1(\mathbb{R}^n)$ then V_t forms a solution of the Brakke motion.

We show (3.2) with "=", no mass loss

Case 2 : p odd. q even

Define $V_t = S_t$ for $t < 0$

$$V_0 = C_{++} \cup C_{+-}$$

$$V_t = e^{i\arg\mu} E_t\left(\mu,\ \theta + \frac{\arg\mu}{q}\right), \quad t > 0$$

Recall $E_t(\mu,\theta) = \sqrt{\frac{t}{c}}\left(\sinh\mu\sqrt{q}\, e^{ip\theta},\ i\cosh\mu\sqrt{p}\, e^{-iq\theta}\right)$

Case 3 : p even. q odd

Define $V_t = e^{i\arg\mu} S_t\left(\mu,\ \theta + \frac{\arg\mu}{p}\right)$ $t < 0$

$V_0 = C_{++} \cup C_{-+}$, $V_t = E_t$ $t > 0$

Theorem 1.2. *For any Schoen-Wolfson cone $C_{p,q}$ with $p > q > 1$, there exists a Hamiltonian stationary self-similar eternal Brakke motion V_t such that V_t approaches $C_{p,q}$ as $t \to 0$ from either direction.*

Pf: take $\mu \geq 0$ only.

Prove no boundary contribution

i.e $\int \operatorname{div} W = -\int W \cdot \vec{h}$

take $t > 0$ for example

boundary curve $\{\sqrt{\frac{t}{c}}(0, i\sqrt{p}\, e^{-iq\theta})\}$

Recall $E_t(\mu, \theta) = \sqrt{\frac{t}{c}}\,(\sinh \mu \sqrt{q}\, e^{ip\theta}, i \cosh \mu \sqrt{p}\, e^{-iq\theta})$

$$\frac{\partial E_t}{\partial \mu} = \sqrt{\frac{t}{c}}\left(\cosh\mu \sqrt{q}\, e^{ip\theta},\ i \sinh\mu \sqrt{p}\, e^{-iq\theta}\right)$$

$\therefore$ unit normal $(e^{ip\theta}, 0)$

The boundary contribution is

$$\int_0^{2\pi} W\cdot(e^{ip\theta},0)\sqrt{\frac{tp}{c}}\,d\theta$$

$$= \sqrt{\frac{tp}{c}}\int_0^{\frac{2\pi}{q}} W\cdot(e^{ip\theta},0)\left(\underbrace{1+e^{i\frac{2\pi p}{q}}+\cdots+e^{i\frac{2\pi p(q-1)}{q}}}_{=\,0 \text{ if } q>1}\right)d\theta = 0.$$

L. & Wang (Tran. of AMS. 2010)

generalize to higher dims $L^n \subset \mathbb{C}^n$

- HSL cones $\approx S^1 \times S^{k-1} \times S^{n-k-1} \times \mathbb{R}^+$
  a generalization of Schoen-Wolfson cones to hd
- cpt HSL self-shrinkers $\approx S^1 \times S^{n-1}$
- HSL self-shrinkers $\approx S^1 \times S^{k-1} \times \mathbb{R}^{n-k}$ slag
- HSL self-expanders $\approx S^1 \times \mathbb{R}^k \times S^{n-k-1}$

glue to eternal sols of Lag Brakke flow, no mass loss

L–, W–. $n \geq 2$

const

$$\left\{ \left(x_1 e^{i\lambda_1 s}, \cdots, x_n e^{i\lambda_n s} \right) : \sum_{j=1}^{n} \lambda_j x_j^2 = C, \quad x_j \in \mathbb{R} \right\}$$

$0 \leq s < 2\pi$

$\longrightarrow 0 \leq s < \pi$, conditions for orientability & embeddness

- If $\sum \lambda_j = 0$. slag (explore the family, Top changes?)
- If $\sum \lambda_j \neq 0$. HSL. & self-similar

$$F^{\perp} = \frac{-C}{\sum_{j=1}^{n} \lambda_j} H.$$

$C = 0 \longrightarrow$ HSL cone

Define $V_t = \{ (x_1 e^{i\lambda_1 s}, \dots, x_n e^{i\lambda_n s}) \quad 0 \le s < \pi \}$

$$\sum_{j=1}^{n} \lambda_j x_j^2 = -2t \sum_{j=1}^{n} \lambda_j$$

λ_j integer. $\sum \lambda_j > 0$, $\lambda_j > 0$ $1 \dots k$, $\lambda_j < 0$. $k+1, \dots n$

<u>Thm</u> V_t : eternal sol to Lag Brakke flow. no mass loss

Topology changes

$$S^1 \times S^{k-1} \times \mathbb{R}^{n-k} \longrightarrow S^1 \times S^{k-1} \times S^{n-k-1} \times \mathbb{R}^+ \longrightarrow S^1 \times \mathbb{R}^k \times S^{n-k-1}$$

Joyce, L. & Tsui (JDG. 2010)

① Lag self-expanders $L \approx S^{n-1} \times \mathbb{R}$, asymptotic to a pair of Lag planes $P_1 \cup -P_2$

$P_1 = \{ (t_1 e^{i\bar{\varphi}_1}, \cdots t_n e^{i\bar{\varphi}_n}) : t_j \in \mathbb{R} \}$ $0 < \bar{\varphi}_j$, $\sum_{j=1}^{n} \bar{\varphi}_j < \pi/2$

$P_2 = \{ (t_1 e^{-i\bar{\varphi}_1}, \cdots t_n e^{-i\bar{\varphi}_n}, t_j \in \mathbb{R} \}$

Lag angle θ of $L \in (\frac{\pi}{2} - \varepsilon, \frac{\pi}{2} + \varepsilon)$ if $\sum \bar{\varphi}_j = \pi/2 - \varepsilon$

n-parameter families (up to scaling and $U(n)$ action)

- have explicit formula
- converge to slag as $\sum_{j=1}^{n} \overline{\varphi}_j \to \frac{\pi}{2}$. or one of shortest radii $\to 0$ or $\alpha \to 0$

$H = \alpha F^{\perp}$. $\alpha > 0$

- Maslov class 0
- optimal. ∵ $P_1 \cup -P_2$ vol minimizing iff $\sum \overline{\varphi}_j \geq \pi/2$

assume. $0 \leq \overline{\varphi}_j \leq \frac{\pi}{2}$ Angle criterion: Nance, Lawlor

$\sum \overline{\varphi}_j = \frac{\pi}{2}$. Lawlor's slag (c.f Anciaux)

- From these examples, can construct Lag translating solns $\approx \mathbb{R}^n$, with arbitrarily small Θ

a surprise to people's expection. (almost calibrated)

similar to cigar / Brayn soliton in Ricci flow

Q: Can these examples occur as blow-ups for MCF?

(Neves & Tian)

If yes, $\Rightarrow$ no regularity in general

② cpt Lag self-shrinkers $\approx S^{n-1} \times S^1$, HS or non-HS

③ self-shrinkers $\approx S^{k-1} \times \mathbb{R}^{n-k} \times S^1$ HS or non-HS

④ self-expanders $\approx S^{k-1} \times \mathbb{R}^{n-k} \times S^1$ HS or non-HS

- ②.③.④ a dense set in n-parameter families respectively (up to scaling and $U(n)$ action). Maslov class $\neq 0$
- Examples in ③ & ④ (different k) that are asymptotic to the same cone, can glue with the cone to yield eternal sols to Lag Brakke flow without mass loss

  $\alpha \to 0$ SLags by Joyce
- From ②.③.④ & others in the n-parameter families, we can also construct corresponding translating sols.

Ansatz 3.1. Fix $n \geqslant 1$. Consider n-submanifolds L in $\mathbb{C}^n$ of the form:

$$L = \left\{ \left(x_1 w_1(s), \ldots, x_n w_n(s) \right) : s \in I,\, x_1, \ldots, x_n \in \mathbb{R},\, \sum_{j=1}^n \lambda_j x_j^2 = C \right\}, \quad (2)$$

λ_j, C const $\in \mathbb{R} \setminus \{0\}$

$w_j : I \longrightarrow \mathbb{C} \setminus \{0\}$ smooth fun.

If (*)
$$\begin{cases} \dfrac{\mathrm{d}w_j}{\mathrm{d}s} = \lambda_j e^{i\theta(s)} \overline{w_1 \cdots w_{j-1} w_{j+1} \cdots w_n}, & j = 1, \ldots, n, \\ \dfrac{\mathrm{d}\theta}{\mathrm{d}s} = \alpha \operatorname{Im}(e^{-i\theta(s)} w_1 \cdots w_n), & \end{cases}$$

$\alpha F^{\perp} = CH$.

$\Longrightarrow$ L Lag, selfsimilar. Lag angle θ
$(\alpha \neq 0)$

generalization of ....

- $\alpha = 0 \rightsquigarrow$ sLag

① all $\lambda_j > 0 \rightsquigarrow$ expanders with small θ

$\alpha > 0$ asymptotic to 2 planes

② all $\lambda_j > 0$ $\rightsquigarrow$ cpt shrinkers $S^{n-1} \times S^1$

$\alpha < 0$

③ some $\lambda_j > 0$ $\rightsquigarrow$ eternal sol to Lag Brakke flow

some $\lambda_j < 0$

② & ③ need to find periodic soln of $(*)$

Lag Translating sol

Ansatz:

$$L = \{(x_1 w_1(s), \ldots, x_{n-1} w_{n-1}(s), -\tfrac{1}{2}\textstyle\sum_{j=1}^{n-1} \lambda_j x_j^2 + \beta(s)) : x_1, \ldots, x_{n-1} \in \mathbb{R},\ s \in I\}$$

$$(*) \quad \begin{cases} \dfrac{\mathrm{d}w_j}{\mathrm{d}s} = \lambda_j e^{i\theta(s)} \overline{w_1 \cdots w_{j-1} w_{j+1} \cdots w_{n-1}}, & j = 1, \ldots, n-1, \\ \dfrac{\mathrm{d}\theta}{\mathrm{d}s} = \alpha \operatorname{Im}(e^{-i\theta} w_1 \cdots w_{n-1}), \\ \dfrac{\mathrm{d}\beta}{\mathrm{d}s} = e^{i\theta(s)} \overline{w_1 \cdots w_{n-1}}, \end{cases}$$

$w_j : I \longmapsto \mathbb{C} \setminus \{0\}$

$\beta : I \longmapsto \mathbb{C}$

$\lambda_j > 0 \rightsquigarrow$ Lag Translating with small θ

Self-similar $\longrightarrow$ translating

Let F_i be a seq of self-similar sols

Satisfying $\quad F_i^{\perp} = R_i H_i \quad (\text{or } F_i^{\perp} = -R_i H_i)$

$R_i \to \infty$

Define $\quad L_i = F_i - (0, \cdots 0, R_i)$

$(H_{L_i} = H_{F_i} = H_i)$

$$\therefore \quad \big(L_i + (0, \cdots 0, R_i)\big)^{\perp} = R_i H_{L_i}$$

$$\Rightarrow \quad \left(\frac{L_i}{R_i} + (0, \cdots 0, 1)\right)^{\perp} = H_{L_i}$$

If $L_i \to L_\infty$ with $H_{L_i} \to H_{L_\infty}$ as $i \to \infty$

then $H_{L_\infty} = (0, \cdots, 0, 1)^\perp \in \mathbb{C}^n$

i.e. L_∞ is a translating sol

- We can arrange to make it hold in our cases. $\Rightarrow$ get translating solns (*)

$n=1 \Rightarrow$ the limit of Abresch & Langer's self-shrinkers is Grim reapers

also for self-expanders