On G_2-holonomy metrics based on $S^3 \times S^3$

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Theorem (Berger, 1955)

Let M^n be simply connected irreducible Riemannian manifold which is not isometric to symmetric space. Then one of the following cases takes place.

1) $Hol(M) = SO(n)$,
2) $n = 2m$, for $m \geq 2$ и $Hol(M) = U(m) \subset SO(2m)$,
3) $n = 2m$, for $m \geq 2$ и $Hol(M) = SU(m) \subset SO(2m)$,
4) $n = 4m$, for $m \geq 2$ и $Hol(M) = Sp(m) \subset SO(4m)$,
5) $n = 4m$, for $m \geq 2$ и $Hol(M) = Sp(m)Sp(1) \subset SO(4m)$,
6) $n = 7$ and $Hol(M) = G_2 \subset SO(7)$,
7) $n = 8$ and $Hol(M) = Spin(7) \subset SO(8)$.

- Compact examples of Riemannian manifolds with holonomy group G_2: Joyce, Kovalev.
- Noncompact examples: many authors.

Representation of Lie group G_2:

$$G_2 = Aut(\mathbb{C}a).$$

The following approach is more convenient for explicit computations:

- In the Euclidean space $\mathbb{R}^7$ consider orthonormal co-frame $e^1, \dots, e^7$ and 3-forms

$$\Phi_0 = e^{123} + e^{147} + e^{165} + e^{246} + e^{257} + e^{354} + e^{367},$$

$$\star\Phi_0 = e^{4567} + e^{2356} + e^{2374} + e^{1357} + e^{1346} + e^{1276} + e^{1245},$$

where $e^{i\dots k} = e^i \wedge \dots \wedge e^k$. Then

$$G_2 = \{A \in GL(7) | A^*\Phi_0 = \Phi_0, A^*(\star\Phi_0) = \star\Phi_0\}.$$

Definition

Riemannian manifold (M, g) has G_2-structure, if it is orientable and there exist global 3-form Φ on M such that for each point $p \in M$ one can find preserving orientation linear map $\phi : T_pM \to \mathbb{R}^7$ with property $\phi^*\Phi_0 = \Phi|_p$.

If additionally form Φ on M is closed and coclosed then it is parallel (this is result of Gray) and M has G_2-holonomy.

Let $G = SU(2)$ with bi-invariant metric. Consider three Killing vector fields

$$\xi^1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \; \xi^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \; \xi^3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

$$[\xi^i, \xi^{i+1}] = 2\xi^{i+2}.$$

Let η_1, η_2, η_3 be dual co-frame,

$$d\eta_i = -2\eta_{i+1} \wedge \eta_{i+2}.$$

On the space $M = G \times G$ we have 6 pairwise orthonormal 1-forms η_i, $\tilde{\eta}_i$, $i = 1, 2, 3$. Consider cone $\bar{M} = \mathbb{R}_+ \times M$ over M with Riemannian metric

$$d\bar{s}^2 = dt^2 + \sum_{i=1}^{3} A_i(t)^2 \left(\eta_i + \tilde{\eta}_i\right)^2 + \sum_{i=1}^{3} B_i(t)^2 \left(\eta_i - \tilde{\eta}_i\right)^2,$$

where $A_i(t)$ и $B_i(t)$ be positive functions which control deformation of standard cone metric over M.

Consider co-frame

$$\begin{aligned} e^1 &= A_1\,(\eta_1 + \tilde{\eta_1})\,, \quad e^4 = B_1\,(\eta_1 - \tilde{\eta_1})\,, \\ e^2 &= A_2\,(\eta_2 + \tilde{\eta_2})\,, \quad e^5 = B_2\,(\eta_2 - \tilde{\eta_2})\,, \\ e^3 &= A_3\,(\eta_3 + \tilde{\eta_3})\,, \quad e^6 = B_3\,(\eta_3 - \tilde{\eta_3})\,, \\ e^7 &= dt \end{aligned}$$

and define forms Φ and $\star\Phi$ as above. So the G_2-holonomy (sufficient) condition for $\bar{M}$ has the form of the next equations:

$$d\Psi = 0, d * \Psi = 0.$$

To simplify further computations we consider particular case $A_1 = A_2$, $B_1 = B_2$.

Previous equations are equivalent to the next system of ODE:

$$
\begin{aligned}
\frac{dA_1}{dt} &= \frac{1}{4}\left(\frac{B_1^2 - A_1^2 + B_3^2}{B_1 B_3} - \frac{A_3}{A_1}\right)\\
\frac{dB_1}{dt} &= \frac{1}{4}\left(\frac{A_1^2 - B_1^2 + B_3^2}{A_1 B_3} + \frac{A_3}{B_1}\right)\\
\frac{dA_3}{dt} &= \frac{1}{4}\left(\frac{A_3^2}{A_1^2} - \frac{A_3^2}{B_1^2}\right)\\
\frac{dB_3}{dt} &= \frac{1}{2}\left(\frac{B_1^2 + A_1^2 - B_3^2}{B_1 A_1}\right)
\end{aligned}
$$

Two types of regularity conditions for solutions of the system:

- Type I:

$$\begin{aligned} A_1(0) = A_3(0) = 0, \\ B_1(0) = B_3(0) \neq 0, \\ \frac{dA_1}{dt}(0) = \frac{dA_3}{dt}(0) = \frac{1}{4}, \\ \frac{dB_1}{dt}(0) = \frac{dB_3}{dt}(0) = 0. \end{aligned}$$

  In this case $\bar{M}$ is diffeomorphic to $S^3 \times \mathbb{R}^4$.

- Type II:

$$
\begin{aligned}
&B_3(0) = 0,\\
&A_1(0) = B_1(0) \neq 0,\\
&A_3(0) \neq 0,\\
&\frac{dB_3}{dt}(0) = 1,\\
&\frac{dA_1}{dt}(0) = -\frac{dB_1}{dt}(0)\\
&\frac{dA_3}{dt}(0) = 0.
\end{aligned}
$$

In this case $\bar{M}$ is diffeomorphic to $S^3 \times H$, where $H \to S^2$ is complex line bundle over S^2 with $c_1(H) = 4$.

Explicit example of solution of type I
(Brandhuber-Gomis-Gubser-Gukov, 2011):

$$d\bar{s}^2 = \frac{\left(r-\frac{3}{2}\right)\left(r+\frac{3}{2}\right)}{\left(r-\frac{9}{2}\right)\left(r+\frac{9}{2}\right)} dr^2 +$$

$$\frac{1}{12}\left(r-\frac{9}{2}\right)\left(r+\frac{3}{2}\right)\left((\eta_1+\tilde{\eta_1})^2+(\eta_2+\tilde{\eta_2})^2\right)+$$

$$\frac{\left(r-\frac{9}{2}\right)\left(r+\frac{9}{2}\right)}{\left(r-\frac{3}{2}\right)\left(r+\frac{3}{2}\right)}(\eta_3+\tilde{\eta_3})^2+$$

$$\frac{1}{12}\left(r+\frac{9}{2}\right)\left(r-\frac{3}{2}\right)\left((\eta_1-\tilde{\eta_1})^2+(\eta_2-\tilde{\eta_2})^2\right)+\frac{r^2}{9}(\eta_3-\tilde{\eta_3})^2 .$$

Definition

Solution corresponded to deformation functions $A_i(t)$, $B_i(t)$ is called ALC (asymptotically locally conical), if there exists linear functions $\tilde{A}_i$, $\tilde{B}_i$ such that

$$\frac{A_i - \tilde{A}_i}{\tilde{A}_i} \to 0, \frac{B_i - \tilde{B}_i}{\tilde{B}_i} \to 0, \text{ as } t \to \infty.$$

Theorem (B.-B.)

There exist one-parameter family of (pairwise non-homothetic) complete Riemannian G_2-holonomy ALC metrics of type II. This family is controlled by parameter

$$\tau = \frac{\frac{dA_1}{dt}(0)}{A_1(0)}$$

and asymptotically metrics locally look like $S^1 \times C(S^2 \times H)$

In $\mathbb{R}^4$ let $R(t) = (A_1(t), A_2(t), A_3(t), B(t))^T$ and $V : \mathbb{R}^4 \to \mathbb{R}^4$ be the right hand of our ODE system. Then system looks like

$$\frac{dR}{dt} = V(R).$$

As is invariant with respect to homothety we put $R(t) = f(t)S(t)$, where

$$|S(t)| = 1, f(t) = |R(t)|,$$
$$S(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t), \alpha_4(t)).$$

Our system splits to radial and tangential parts:

$$\frac{dS}{du} = V(S) - \langle V(S), S\rangle S = W(S),$$

$$\frac{1}{f}\frac{df}{du} = \langle V(S), S\rangle,$$

$$dt = fdu.$$

The first equation is the autonomous system on the sphere S^3 and the other equations can be solved by ordinary integration if solution of the first equation is known.

Type II metrics correspond to solutions of autonomous system with initial point $S_0 = (\lambda, \lambda, \mu, 0)$, where $2\lambda^2 + \mu^2 = 1$.

Lemma

For every S_0 as above there exist (at least locally) solution $S(u)$ with $S(0) = S_0$.

The following proposition shows the role of stationary points:

Lemma

Stationary solutions of autonomous system on S^3 correspond to ALC metrics on $\bar{M}$

The system on S^3 has following stationary solutions (zeros of vector field W on S^3):

$$\pm\left(\frac{\sqrt{3}}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, \frac{\sqrt{3}}{2\sqrt{2}}\right), \ \pm\left(\frac{1}{2\sqrt{2}}, \frac{\sqrt{3}}{2\sqrt{2}}, -\frac{1}{2\sqrt{2}}, \frac{\sqrt{3}}{2\sqrt{2}}\right),$$

$$\pm\left(\frac{\sqrt{3}}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, -\frac{1}{2\sqrt{2}}, -\frac{\sqrt{3}}{2\sqrt{2}}\right), \ \pm\left(\frac{1}{2\sqrt{2}}, \frac{\sqrt{3}}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, -\frac{\sqrt{3}}{2\sqrt{2}}\right),$$

$$\pm\left(-\frac{\sqrt{3}}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, -\frac{1}{2\sqrt{2}}, \frac{\sqrt{3}}{2\sqrt{2}}\right), \ \pm\left(\frac{1}{2\sqrt{2}}, -\frac{\sqrt{3}}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, \frac{\sqrt{3}}{2\sqrt{2}}\right),$$

$$\pm\left(-\frac{\sqrt{3}}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, -\frac{\sqrt{3}}{2\sqrt{2}}\right), \ \pm\left(\frac{1}{2\sqrt{2}}, -\frac{\sqrt{3}}{2\sqrt{2}}, -\frac{1}{2\sqrt{2}}, -\frac{\sqrt{3}}{2\sqrt{2}}\right),$$

$$\pm\left(\frac{\sqrt{3}}{\sqrt{10}}, \frac{\sqrt{3}}{\sqrt{10}}, 0, \frac{\sqrt{2}}{\sqrt{5}}\right), \ \pm\left(\frac{\sqrt{3}}{\sqrt{10}}, \frac{\sqrt{3}}{\sqrt{10}}, 0, -\frac{\sqrt{2}}{\sqrt{5}}\right),$$

$$\pm\left(\frac{\sqrt{3}}{\sqrt{10}}, -\frac{\sqrt{3}}{\sqrt{10}}, 0, \frac{\sqrt{2}}{\sqrt{5}}\right), \ \pm\left(\frac{\sqrt{3}}{\sqrt{10}}, -\frac{\sqrt{3}}{\sqrt{10}}, 0, -\frac{\sqrt{2}}{\sqrt{5}}\right).$$

Consider domain $\Pi \subset S^3$:

$$\Pi = \{(\alpha_1, \alpha_3, \beta_1, \beta_3) | \beta_1 \geq \alpha_1 \geq 0, \alpha_3 \geq 0, \beta_3 \geq 0\}$$

Lemma

Function

$$F = \ln \frac{\beta_3 \left(\beta_1^2 - \alpha_1^2\right)}{\alpha_1 \beta_1 \alpha_3}$$

increases in the domain Π along trajectories of autonomous system

Theorem

For every initial point S_0 there exist unique solution $S(u)$ which converges at infinity to stationary point $S_\infty = (\frac{\sqrt{3}}{\sqrt{10}}, \frac{\sqrt{3}}{\sqrt{10}}, 0, \frac{\sqrt{2}}{\sqrt{5}})$.

THANK YOU!