Killing fields, holonomy and the index of symmetry

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FaMAF, Universidad Nacional de Córdoba

Pacific Rim Geometry Conference, Osaka, 2011

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One has that M is symmetric if and only if $i_{\mathfrak{s}}(M) = n$

We are, of course, interested on non-symmetric spaces with positive index of symmetry. In this case one has that $i_{\mathfrak{s}}(M) \leq n - 2$, as we will see later (in other words the *co-index of symmetry* is at least 2).

We have only few general results.
On the other hand there are a lot of open questions and a large number of examples.

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We were able to classify the spaces with co-index of symmetry equals to 2, the smallest one, after proving a bound on the dimension n of the space, for a fixed positive co-index of symmetry k (for irreducible spaces, since the product by a symmetric space does not change the co-index of symmetry).

The concept of index of symmetry came out from the study of compact naturally reductive spaces such that the isotropy has non-trivial fixed vectors

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In some sense, our philosophy is in the direction of the concept of co-polarity by Claudio Gorodski, that measures how a representation, orbit like, differ from a symmetric (isotropy) representation (and also we try to classify those spaces when the defect is small).

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The index of symmetry.

Let M^n be a Riemannian manifold and denote by $\mathfrak{K}(M)$ the algebra of global Killing fields on M.
For $q \in M$, let us define the Cartan subspace $\mathfrak{p}^q$ at q, by

$$\mathfrak{p}^q := \{X \in \mathfrak{K}(M) : (\nabla X)_q = 0\}$$

The symmetric isotropy algebra at q is defined by

$$\mathfrak{k}^q := \{[X, Y] : X, Y \in \mathfrak{p}^q\}$$

Observe that $\mathfrak{k}^q$ is contained in the (full) isotropy subalgebra $\mathfrak{K}_q(M)$. In fact, if $X, Y \in \mathfrak{p}^q$, $[X, Y]_q = (\nabla_X Y)_q - (\nabla_Y X)_q = 0$. Moreover, since $\mathfrak{p}^q$ is left invariant by the isotropy at q,

$$\mathfrak{g}^q := \mathfrak{k}^q \oplus \mathfrak{p}^q$$

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The symmetric subspace at q, $\mathfrak{s}_q \subset T_q M$, is defined by

$$\mathfrak{s}_q := \{X.q : X \in \mathfrak{p}^q\} = \mathfrak{p}^q.q$$

The local version, involving local Killing fields, can be equivalently defined as follows

$$\mathfrak{s}_q^{loc} := \{v \in T_q M : \nabla_v^k R = 0, \ k = 0, \ldots, n + \frac{1}{2}n(n-1)\},$$

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For dealing with the distribution $q \mapsto \mathfrak{s}^q$ one needs to regard Killing fields as parallel sections of the so called canonical (vector) bundle over M,

$$TM \oplus \Lambda^2(TM) \simeq TM \oplus \mathfrak{so}(TM)$$

where the connection $\bar{\nabla}$ in $TM \oplus \mathfrak{so}(TM)$ is given by

$$\bar{\nabla}_Y(Z, B) = (\nabla_Y Z - BY, \nabla_Y B - R_{Y,Z})$$

The bijection is given by

$$Z \leftrightarrow (Z, \nabla Z)$$

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Lemma

Let $X, Y \in \mathfrak{p}^q$, regarded as Killing fields, and let Z be an arbitrary tangent field of M. Then

$$R_{X(q),Y(q)}Z(q) = -[[X,Y],Z](q)$$

Let $q \in M$ and assume that the index of symmetry at q is positive, i.e. $\dim \mathfrak{s}_q > 0$. Let us consider the Lie subalgebra $\mathfrak{g}^q$ of the full isometry algebra. One has that

$$\mathfrak{g}^q = \mathfrak{k}^q \oplus \mathfrak{p}^q$$

is an involutive Lie algebra. Let G^q be its associated Lie subgroup of $I(M)$. One has that the orbit $G^q.q$ is a global symmetric space, which is a totally geodesic immersed manifold of M.

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Let $X, Y \in \mathfrak{p}^q$, regarded as Killing fields, and let Z be an arbitrary tangent field of M. Then

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Proposition

If M is compact, then G^q acts almost effectively on the orbit $G^q.q$.

Identify $T_q(G^q.q) = \mathfrak{s}_q \simeq \mathfrak{p}^q$ and decompose

$$\mathfrak{p}^q = \mathfrak{p}_0 \oplus \mathfrak{p}_1 \oplus \ldots \oplus \mathfrak{p}_r$$

where $\mathfrak{p}_0$ corresponds to the Euclidean factor and $\mathfrak{p}_i$ corresponds to the irreducible factors, in the de Rham local decomposition of the orbit $G^q.q$ $(i = 1, \ldots, r)$.
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Facts: *assume that M^n compact.*

(a) *G_i^q is a compact Lie subgroup of $I(M)$, if $i \geq 1$.*

(b) *If $R_{u,v}|\mathfrak{s}_q = 0$, then $R_{u,v} = 0$, for any $u, v \in \mathfrak{s}_q$.*

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Theorem

Let M^n be a compact locally irreducible homogeneous Riemannian manifold, which is not locally symmetric, and let $k := n - i_{\mathfrak{s}}(M)$ be its co-index of symmetry. Then there is a subgroup of isometries $G \subset I(M)$, which acts transitively on M and such that $dim(G) \leq \frac{1}{2}k(k+1)$. Moreover, if the equality holds, then, up to a cover, $G = Spin(k+1)$ and G has non-trivial isotropy, if $k \geq 4$.

Corollary

Let M^n, $n \geq 3$, be a compact locally irreducible Riemannian manifold with co-index of symmetry equals to 2. Then $n = 3$. Moreover, if M is simply connected then $M = Spin(3) \simeq S^3$ with a left invariant metric that belongs to one of two families g_s^1 , g_t^2 described in the next.

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Examples.

- Left invariant metrics in Spin(3).

Since $Ad(\mathrm{Spin}(3)) = \mathrm{SO}(\mathfrak{so}(3)) \simeq \mathrm{SO}(3)$, with respect to the bi-invariant metric of curvature 1.

any left invariant metric, modulo isometries and rescaling, is determined by a triple of positive numbers

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The metrics with co-index of symmetry 2 are given exactly by the two families of metrics g^1_s and g^2_t associated to the triples

$$(1, s, 1-s), \qquad 0 < s < \frac{1}{2}$$

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The isometry group for the first family is Spin(3) and for the second family is Spin(3) $\times S^1$ (and the tranvections do not lie in Spin(3)), if $t \neq \frac{1}{2}$.

Observe that (Spin(3), g^2_t) is a Berger sphere. Or equivalently, up to a cover, it is the unit tangent bundle over the 2-sphere of constant curvature different from 1 (in which case the metric would be bi-invariant and the space symmetric).

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The distribution of symmetry $\mathfrak{s}$, of the unit tangent bundle M^{2n-1} of the sphere S^n_2 of curvature 2, coincides with the vertical distribution ν. In particular, $i_{\mathfrak{s}} = n - 1$, where $i_{\mathfrak{s}} = \dim(\mathfrak{s})$ is the index of symmetry (or equivalently, the co-index of symmetry is equals to n).

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- Naturally reductive spaces whose isotropy has fixed vectors

Let $M = G/H$ be a homogeneous compact Riemannian manifold with a G-invariant metric $\langle \ , \ \rangle$.

The space M is said to be *naturally reductive* if there exists a reductive decomposition

$$\mathcal{G} = \mathfrak{h} \oplus \mathfrak{m},$$

where $\mathcal{G} = \mathrm{Lie}(G)$, $\mathfrak{h} = \mathrm{Lie}(H)$, $\mathrm{Ad}(H)\mathfrak{m} \subset \mathfrak{m}$, such that the geodesics by $p = [e]$ are given by

$$\gamma_{X.p} = \mathrm{Exp}(tX).p$$

for al $X \in \mathfrak{m}$. In other words, the Riemannian geodesics coincide with the ∇^c-geodesics, where ∇^c is the canonical connection, which is a metric connection, of M associated to the reductive decomposition. This is in fact equivalent to the property that $[X, \cdot\,]_{\mathfrak{m}} : \mathfrak{m} \to \mathfrak{m}$ is skew-symmetric, for all $X \in \mathfrak{m}$ ($\mathfrak{m} \simeq T_pM$).

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The Levi-Civita connection is given by

$$\nabla_v \tilde{w} = \tfrac{1}{2}[\tilde{v}, \tilde{w}]_p,$$

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The tensor D is totally skew, i.e. $\langle D_v w, z\rangle$ is a 3-form.

Let M be a compact locally irreducible (non-symmetric) naturally reductive space. Let now, keeping the previous notation,

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Such a field is parallel with respect to the canonical connection. In fact, any G-invariant tensor is ∇^c-parallel. Then, for any $v \in \mathfrak{m} \simeq T_pM$, $w \in \mathfrak{m}_0$,

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Remark. There are no more new Killing fields in M, since the canonical connection is unique (unless M is round sphere, or a Lie group, with a bi-invariant metric). This is by making use of the so-called *skew-torsion holonomy theorem* (O.- Reggiani)

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Examples associated to any canonical connection on a compact Lie group.

Given any (non-symmetric) element of the one-parameter family of reductive decomposition of $G \times G/\mathrm{diag}(G \times G) \simeq G$, then there is a left invariant metric on $G \times G$ such that:

- $G \times G$ is an irreducible Riemannian manifold.
- The projection map into the symmetric quotient $\pi : G \times G \to G \times G/\mathrm{diag}(G \times G) \simeq G$ is a Riemannian submersion whose horizontal subspace correspond to the reductive decomposition.
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– Are the leaves of the distribution of symmetry compact (or equivalently, is the flat factor compact?).

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– Find new examples.

– Classify the case of co-index of symmetry equals to 3 (in which case de dimension is at most 6).

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