Smith-Dold Branched Coverings and Cup-Length

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Plan

1. Definitions, examples and cohomology transfer.
2. A.Dold classification result, group action transfer.
3. The case of manifolds. "Wild" coverings and A.V.Chernavskii theorem.
4. Orientable manifolds case. I.Berstein-A.L.Edmonds inequality and Alexander theorem.
5. Main result.
6. Applications to nonorientable manifolds.
7. Our approach. Extension of V.M.Buchstaber-E.G.Rees theory to graded algebras.
8. Some remarks on branched coverings not of Smith-Dold type.

Definitions, examples and cohomology transfer.

Definition

Suppose X and Y are Hausdorff spaces. A continuous map $f : X \to Y$ is called an n-fold branched covering if it is

- open-closed and surjective
- finite-to-one
- $n := \max_{y \in Y} |f^{-1}(y)| < \infty$

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Remark

1. 1-fold branched covering is just a homeomorphism.
2. 2-fold branched covering is always equivalent to some projection map onto the quotient space under an involution $\pi : X \to X/\mathbb{Z}_2$.

Definitions, examples and cohomology transfer.

Some auxiliary definitions

Let X be a Hausdorff space.
Define $\exp_n(X) := \{A \subset X | 1 \leq |A| \leq n\}$ (with Vietoris topology)
Define $\mathrm{Sym}^n X := X^n / S_n$
Point of $\mathrm{Sym}^n X = [k_1 x_1, \ldots, k_s x_s] \in \mathrm{Sym}^n X$,
$k_i \in \mathbb{N}, k_1 + \ldots + k_s = n, \; x_i \in X, x_i \neq x_j, \forall i \neq j$

$< \cdot >: \mathrm{Sym}^n X \to \exp_n(X)$ — "forgetting multiplicities" map
$< [k_1 x_1, \ldots, k_s x_s] >= \{x_1, \ldots, x_s\} \in \exp_n(X)$

Definitions, examples and cohomology transfer.

Definition (L.Smith, 1983)

Suppose X and Y are Hausdorff spaces. A continuous map $f : X \to Y$ is called an n-fold Smith-Dold branched covering if there exists a continuous "n-inversion" map $g : Y \to \mathrm{Sym}^n X$ such that
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1. The map $g : Y \to \mathrm{Sym}^n X$ is often included into the structure of a branched covering.
2. n-fold S.-D. branched covering is always an m-fold branched covering (in the sense of the first definition) for some $m \leq n$.

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An example of a 5-fold S.-D. branched covering is on the board (Fig. 1).

Definitions, examples and cohomology transfer.

An example of a 3-fold branched covering which is not an n-fold branched covering of S.-D. type for any $n \in \mathbb{N}$. (Fig. 3)

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Question: Why S.-D. branched coverings are better than just simply defined branched coverings?

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Question: Why S.-D. branched coverings are better than just simply defined branched coverings?
Answer: For S.-D. branched coverings there exists a transfer in cohomology!

Definitions, examples and cohomology transfer.

Suppose X and Y are connected Hausdorff spaces, X is homotopy equivalent to a CW complex, and a pair of maps $f : X \to Y,\ g : Y \to \mathrm{Sym}^n X$ is an n-fold S.-D. branched covering.

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There exists a homology transfer
$\tau_S : H_*(Y; \mathbb{Z}) \to H_*(X; \mathbb{Z})$

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Tensoring by $\mathbb{Q}$ we obtain transfer $\tau_S : H_*(Y; \mathbb{Q}) \to H_*(X; \mathbb{Q})$

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In cohomology (by dualization) one obtains transfers
$\tau_S : H^*(X;\mathbb{Q}) \to H^*(Y;\mathbb{Q})$ and
$\tau_S : H^*(X;\mathbb{Z}_p) \to H^*(Y;\mathbb{Z}_p)\ \forall p,\ (p,n) = 1$

with the same property $\tau_S \circ f^* = n\mathrm{Id}_{H^*(Y)}$.

Definitions, examples and cohomology transfer.

Important consequence: For n-fold S.-D. branched covering $f : X \to Y$ the induced homomorphisms
$f^* : H^*(Y; \mathbb{Q}) \to H^*(X; \mathbb{Q})$ and
$f^* : H^*(Y; \mathbb{Z}_p) \to H^*(X; \mathbb{Z}_p)\ \forall p,\ (p, n) = 1$
are monomorphisms.

Definitions, examples and cohomology transfer.

There are 3 important for topology classes of maps, that are n-fold S.-D. branched coverings

1. (unbranched) n-fold coverings $f : X \to Y$.
2. projection maps $f : X \to X/G,\ G$ – a finite group, $|G| = n,\ X$ is a G-space.
3. usual branched coverings of manifolds $f : M^m \to N^m$ (smooth, PL or "wild").

A.Dold classification result, group action transfer.

Theorem (A.Dold, 1986)

(1) Let X be a Hausdorff G-space, G – a finite group, $H \subset G$ – a subgroup of index n, $[G : H] = n$. Then the natural projection map $\pi_{G,H} : X/H \to X/G$ is an n-fold S.-D. branched covering.
(2) For every n-fold S.-D. branched covering $f : X \to Y, \; g : Y \to \mathrm{Sym}^n X$ there exists a canonically obtained Hausdorff space W with the action of S_n such that $X = W/S_{n-1}, Y = W/S_n$ and $f = \pi_{S_n,S_{n-1}}$.

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From this statement one can obtain a transfer in a new way.

All spaces now are paracompact and locally contractible.
(One can simply consider arbitrary ENR spaces or arbitrary CW complexes).

A.Dold classification result, group action transfer.

Theorem (A)

Suppose X is a paracompact G-space, G — finite group, $|G| = n$, $\mathbb{K}$ — a field, $\mathrm{char}\mathbb{K} = 0$ or p, $(p, n) = 1$. Let $\pi : X \to X/G$ be a projection map. Then the induced homomorphism in Čech cohomology
$\pi^* : \check{H}^*(X/G; \mathbb{K}) \cong \check{H}^*(X; \mathbb{K})^G$
is an isomorphism onto the G-invariant cohomology.

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Theorem (B)

Let X be a locally contractible paracompact space. Then there is a canonical isomorphism of algebras
$H^*(X; \mathbb{K}) \cong \check{H}^*(X; \mathbb{K})$,
where $\mathbb{K} = \mathbb{Z}$ or is a field.

A.Dold classification result, group action transfer.

Suppose X and Y are locally contractible paracompact spaces. $f : X \to Y,\ g : Y \to \mathrm{Sym}^n X$ — an n-fold S.-D. branched covering.
There exists a Hausdorff space W (which by construction occurs to be paracompact) with the action of S_n such that

$$X = W/S_{n-1}, Y = W/S_n \text{ and } f = \pi_{S_n, S_{n-1}} : W/S_{n-1} \to W/S_n$$

Let $\mathbb{K} = \mathbb{Q}$ or $\mathbb{Z}_p,\ \forall p > n$ (we need (p,n!)=1)

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$f^* : H^*(Y; \mathbb{K}) \to H^*(X; \mathbb{K})$
$f^* = \pi^*_{S_n,S_{n-1}} : H^*(W/S_n; \mathbb{K}) \to H^*(W/S_{n-1}; \mathbb{K})$
$\pi^*_{S_n,S_{n-1}} : \check{H}^*(W/S_n; \mathbb{K}) \to \check{H}^*(W/S_{n-1}; \mathbb{K})$ — a monomorphism.

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$\pi^*_{S_n} : \check{H}^*(W/S_n;\mathbb{K}) \cong \check{H}^*(W;\mathbb{K})^{S_n}$ and
$\pi^*_{S_{n-1}} : \check{H}^*(W/S_{n-1};\mathbb{K}) \cong \check{H}^*(W;\mathbb{K})^{S_{n-1}}$ and
$\pi^*_{S_n} = \pi^*_{S_{n-1}} \circ \pi^*_{S_n,S_{n-1}} = \pi^*_{S_{n-1}} \circ f^*$

A.Dold classification result, group action transfer.

Denote $\check{H}^*(W;\mathbb{K}) = A^*$.

$H^*(X;\mathbb{K}) = (A^*)^{S_{n-1}}$ and $H^*(Y;\mathbb{K}) = (A^*)^{S_n}$ and

$f : H^*(Y;\mathbb{K}) \to H^*(X;\mathbb{K})$ is just $i : (A^*)^{S_n} \subset (A^*)^{S_{n-1}}$

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Generalize it a little:

Suppose A^* is a graded commutative algebra over a field $\mathbb{K}$ with the action of a finite group G, and $H \subset G$ is a subgroup of index n, $[G : H] = n$. $\mathrm{char}\mathbb{K} = 0$ or p, $(p, n) = 1$.

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$(A^*)^G \subset (A^*)^H$ — an inclusion of algebras.
$G = \{g_1 H\} \sqcup \ldots \sqcup \{g_n H\}$ — left cosets.
$g_i : A^* \to A^*$ — automorphisms.

$\tau_G = g_1 + g_2 + \ldots + g_n : (A^*)^H \to A^*$ (a sum of n $\mathbb{K}$-linear homomorphisms)
$\mathrm{Im}\tau_G = (A^*)^G$.

A.Dold classification result, group action transfer.

Consequence:

$\tau_G : (A^*)^H \to (A^*)^G$ is a $(A^*)^G$-linear transfer
$\tau_G(a) = na \; \forall a \in (A^*)^G$ (the expected property)

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$f^* : H^*(Y;\mathbb{K}) \to H^*(X;\mathbb{K})$ is a monomorphism, and
$\tau_G : H^*(X;\mathbb{K}) \to H^*(Y;\mathbb{K})$ is a $H^*(Y;\mathbb{K})$-linear transfer with the expected property

$\tau_G \circ f^* = n\mathrm{Id}_{H^*(Y;\mathbb{K})}$

The case of manifolds. "Wild" coverings and A.V.Chernavskii theorem.

Suppose X and Y are connected PL (TOP) manifolds of equal dimension.

Definition (Classical, PL case)

A continuous map $f : M^m \to N^m$ is a branched covering if it is

- open-closed and PL ($\Rightarrow$ it is discrete)
- finite-to-one

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- finite-to-one

The case of manifolds. "Wild" coverings and A.V.Chernavskii theorem.

Theorem (A.V.Chernavskii,1964)

Suppose $f : M^m \to N^m$ *is purely continuous branched covering of connected TOP (PL) manifolds of dimension* $m \geq 3$. *Then the following hols:*
(1) $n := \max_{y \in N^m} |f^{-1}(y)| < \infty$. *The set* $U = \{y \in N^m \mid |f^{-1}(y)| = n\}$ *is an open dense domain in* N^m*.*

(2) Define the branch set
$B_f = \{x \in M^m \mid f \text{ is not a local homeomorphism at } x\} \subset M^m$.
($B_f \subset M^m$ *is closed and also* $f(B_f) \subset N^m$ *is closed).*
Then $\dim B_f \leq m - 2$.

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($B_f \subset M^m$ is closed and also $f(B_f) \subset N^m$ is closed).
Then $\dim B_f \leq m - 2$.

At the late 70-s there was constructed examples of coverings with $\dim B_f = m - 4$ for all $m \geq 5$. Purely continuous coverings may be very wild.

The case of manifolds. "Wild" coverings and A.V.Chernavskii theorem.

Theorem (I.Berstein-A.L.Edmonds,1978)

For every n-fold branched covering of connected TOP manifolds $f : M^m \to N^m$ there exists a locally compact separable metric space W with the action of some finite group G provided with a subgroup $H \subset G$ of index n such that

$$M^m = W/H, \; N^m = W/G \text{ and } f = \pi_{G,H}.$$

Orientable manifolds case. I.Berstein-A.L.Edmonds inequality and Alexander theorem.

Definition

Let X be a topological space and R — commutative ring with identity element. The the cup-length $L_R(X)$ over R is the maximal number k such that there exists homogeneous elements $a_1, \dots, a_k \in H^{*\geq 1}(X; R)$ of positive degrees with nonzero product $a_1 a_2 \dots a_k \neq 0$.

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Theorem (Classical)

For an arbitrary connected ANR space X and the arbitrary ring R the following double inequality holds:

$$L_R(X) \leq \mathrm{Cat}(X) \leq \mathrm{dim} X.$$

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$\mathrm{Cat}(X)$ is Lusternik-Shnirelmann category of X

$\mathrm{Cat}(X)$ is the minimal $k \geq 0$ such that there exists a closed cover $X = \bigcup_{s=o}^{k} X_s$ with the property that all inclusion maps $i_s : X_s \subset X, 0 \leq s \leq k$, are nullhomotopic.

For example:

(1) $X = T^m$. $L_{\mathbb{Q}}(T^m) = m = \dim T^m$ So, $\mathrm{Cat}(T^m) = m$

(2) $X = \mathbb{R}P^m$. $L_{\mathbb{Z}_2}(\mathbb{R}P^m) = m = \dim \mathbb{R}P^m$ So, $\mathrm{Cat}(\mathbb{R}P^m) = m$

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(3) $\mathrm{Cat}(\mathbb{C}P^m) = m$ and $\mathrm{Cat}(\mathbb{H}P^m) = m$

Orientable manifolds case. I.Berstein-A.L.Edmonds inequality and Alexander theorem.

Let us consider the case of branched coverings of closed connected orientable manifolds.
Using the existence of group action transfer, I.Berstein and A.L.Edmonds obtained the following crucial result.

Theorem (I.Berstein-A.L.Edmonds,1978)

Suppose $f : M^m \to N^m$ is an n-fold branched covering of closed connected orientable manifolds. Then the following inequality holds:
$nL_{\mathbb{Q}}(N^m) \geq L_{\mathbb{Q}}(M^m)$.

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Let us rewrite: $L_{\mathbb{Q}}(N^m) \geq \frac{L_{\mathbb{Q}}(M^m)}{n}$.

The rational cup-length of the base has the lower bound in terms of the cup-length of the covering space!

Orientable manifolds case. I.Berstein-A.L.Edmonds inequality and Alexander theorem.

Let us rewrite once more: $n \geq \frac{L_{\mathbb{Q}}(M^m)}{L_{\mathbb{Q}}(N^m)}$.
The degree of the branched covering has a lower bound in terms of cohomology rings of the base and the covering space!

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Special case: branched coverings over the sphere.

Theorem (Alexander,1920)

For every closed connected orientable PL manifold M^m there exists a PL branched covering $f : M^m \to S^m$ over the m-sphere.

The Alexander construction is represented on the board ($m = 2$).

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The degree of a branched covering in Alexander constructon is $n = \frac{(m+1)!}{2}\mathrm{Number}[m\text{-simplexes}] >> m$.

Orientable manifolds case. I.Berstein-A.L.Edmonds inequality and Alexander theorem.

Consider B.-E. inequality in this case:

for arbitrary $f : M^m \to S^m$ one has $n \geq L_{\mathbb{Q}}(M^m)$.

The maximal value of the cup-length here is m (e.g. for T^m), so $n = m$ is not forbidden anyway.

Orientable manifolds case. I.Berstein-A.L.Edmonds inequality and Alexander theorem.

Problem: Does the branched covering over the m-sphere with $n = m$ always exist for every PL orientable closed connected M^m.

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History:

- $m = 2$ hyperelliptic surfaces give the positive answer: $n = 2$.
- $m = 3$ (1974, H.M. Hilden, U.Hirsh and J.M.Montesinos – independently) Every M^3 is a 3-fold branched cover of S^3 and the branch set $f(B_f) \subset S^3$ is a knot.
- $m = 4$ (1995, R.Piegallini) Every PL M^4 is a 4-fold branched cover of S^4 and the branch set $f(B_f) \subset S^4$ is a trasversally immersed PL surface (with double points).
- $m \geq 5$ Open Problem!

Main result.

Theorem (G.,2011)

Suppose X and Y are locally contractible paracompact spaces and $f : X \to Y$ is an n-fold S.-D. branched covering. Then the following general inequality holds:

$nL_{\mathbb{Q}}(Y) + n - 1 \geq L_{\mathbb{Q}}(X)$ *and*

$nL_{\mathbb{Z}_p}(Y) + n - 1 \geq L_{\mathbb{Z}_p}(X) \quad \forall p > n.$

Both inequalities are sharp for $n = 2$.

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Both inequalities are sharp for $n = 2$.

Theorem (G.,2011)

Suppose $f : X^m \to Y^m$ is an n-fold S.-D. branched covering, where Y^m is a closed connected orientable manifold and X^m is an ENR space (pseudomanifold). Then the B.-E. inequality holds true:

$$nL_{\mathbb{Q}}(Y^m) \geq L_{\mathbb{Q}}(X^m) \text{ and}$$

$$nL_{\mathbb{Z}_p}(Y^m) \geq L_{\mathbb{Z}_p}(X^m) \quad \forall p > n.$$

Main result.

Method of the proof (briefly):

A detailed examination of algebraic structure of the group action transfer and its properties.

Applications to nonorientable manifolds.

Let us consider branched coverings $f : M^m \to N^m$ of closed connected manifolds with a nonorientable base.

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Example:

$f : S^{2m} \to \mathbb{R}P^{2m}, \quad n = 2$

B.-E. inequality is not correct: $2 \ngeq \frac{L_{\mathbb{Q}}(S^{2m})}{L_{\mathbb{Q}}(\mathbb{R}P^{2m})} = \frac{1}{0}$.

General inequality holds: $2 \geq \frac{1+1}{0+1} = 2$.

Applications to nonorientable manifolds.

R.Piergallini theorem implies the existense of a 4-fold branched covering

$f_1 : T^4 \to S^4, \quad n_1 = 4$

$f_2 : S^4 \to \mathbb{R}P^4, \quad n_2 = 2$

By composition: $f = f_2 \circ f_1 : T^4 \to \mathbb{R}P^4, \quad n = n_1 n_2 = 8.$

Problem: What is the minimal degree (number of sheets) n of a branched covering $f : T^4 \to \mathbb{R}P^4$?

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General inequality: $n \geq \frac{L_{\mathbb{Q}}(T^4)+1}{L_{\mathbb{Q}}(\mathbb{R}P^4)+1} = \frac{4+1}{0+1} = 5. \quad \Rightarrow$

$5 \leq n \leq 8.$

Applications to nonorientable manifolds.

What can still give the B.-E. inequality?

Case 1. Suppose $f : T^4 \to \mathbb{R}P^4$ can be lifted to $\widehat{f} : T^4 \to S^4$, $f = \pi \circ \widehat{f}, \pi : S^4 \to \mathbb{R}P^4$ — 2-sheet covering. Then $n = \mathrm{deg} f = 2\mathrm{deg}\widehat{f} \geq 8$. (A stronger estimate)

Case 1 occures iff $f_* : \pi_1(T^4) \to \pi_1(\mathbb{R}P^4)$ is zero.

Case 2. Suppose $f_* : \pi_1(T^4) \to \pi_1(\mathbb{R}P^4)$ is an epimorphism. By pullback one obtains $n \geq 4$. (See the whiteboard)

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Remark

For odd n there is a transfer in $\mathbb{Z}_2$ cohomology. So the induced homomorphism $f^*: H^*(\mathbb{R}P^4; \mathbb{Z}_2) \to H^*(T^4; \mathbb{Z}_2)$ must be a monomorphism. But $H^*(\mathbb{R}P^4; \mathbb{Z}_2) = \mathbb{Z}_2[u]/(u^5 = 0)$ and $H^*(T^4; \mathbb{Z}_2) = \mathbb{Z}_2[v_1, \ldots, v_4]/(v_i^2 = 0)$. So one has a contradiction: $f^*(u^2) = (f^*u)^2 = 0$.

Applications to nonorientable manifolds.

Conclusion:

General inequality gives the following possible values: $n = 6, 8$.

B.-E. inequality gives only $n = 4, 6, 8$.

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Conjecture 1. The minimal degree of a branched covering $f : T^4 \to \mathbb{R}P^4$ is 8.

Applications to nonorientable manifolds.

Generalization: Consider branched coverings over $\mathbb{Q}$-acyclic manifolds.

$$N^{2m_k} \xrightarrow{\mathbb{R}P^{2s_{k-1}}} N^{2m_{k-1}} \to \ldots \xrightarrow{\mathbb{R}P^{2s_1}} N^{2m_1} = \mathbb{R}P^{2m_1}$$

Suppose M^{2m} is orientable, $L_{\mathbb{Q}}(M^{2m}) = 2m$ and N^{2m} is $\mathbb{Q}$-acyclic. Then for an n-fold branched covering $f : M^{2m} \to N^{2m}$ general inequality gives the estimate $n \geq 2m + 1.$

B.-E. inequality gives only $n \geq 2m$ if f induces an epimorphism of fundamental groups.

Applications to nonorientable manifolds.

Conjecture 2: Every nonorientable PL closed connected manifold M^4 is a branched cover over $\mathbb{R}P^4$. ($n \leq 8$?)

Remark

- (I.Berstein-A.L.Edmonds,1979) $M^3_{nonori} \to S^1 \times \mathbb{R}P^2$ with $n \leq 6$.
- For $\{M^5_{nonori}\}$ there is no terminal oblect!
  $S^1 \times \mathbb{R}P^4$ and $S^3 \times \mathbb{R}P^2$

Our approach. Extension of V.M.Buchstaber-E.G.Rees theory to graded algebras.

Recall $\tau = \tau_G : g_1 + \ldots + g_n : (A^*)^H \to (A^*)^G$

A^* is a commutative graded algebra with an action of a finite group G, $H \subset G$ is a subgroup of index n, $G = \{g_1 H\} \sqcup \ldots \sqcup \{g_n H\}$

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Generalize Suppose A^* and B^* are commutative graded $\mathbb{K}$-algebras, $\mathrm{char}\mathbb{K} = 0$ or $p, p > n$.

$f_1, f_2, \ldots, f_n : A^* \to B^*$ — algebra homomorphisms.
$f = f_1 + \ldots + f_n : A^* \to B^*$ — a $\mathbb{K}$-linear map.

A map $f : A^* \to B^*$ is not an algebra homomorphism, as $f(ab) \neq f(a)f(b)$ for many $a, b \in A^*$.

Our approach: There is still some weak multiplicativity property for such maps f.

Our approach. Extension of V.M.Buchstaber-E.G.Rees theory to graded algebras.

$n = 2$ The "weak multiplicativity" property:

$$f(abc) = \frac{1}{2}(-f(a)f(b)f(c) + f(a)f(bc) + f(ab)f(c) + (-1)^{|a||b|}f(b)f(ac)) \quad \forall a, b, c \in A^*.$$

Our approach. Extension of V.M.Buchstaber-E.G.Rees theory to graded algebras.

Definition

Suppose $f : A^* \to B^*$ is a $\mathbb{K}$-linear map. Define by induction polylinear symmetric maps $\Phi_m(f) : (A^*)^{\times m} \to B^*$:

$\Phi_1(f)(a_1) = f(a_1)$

$\Phi_2(f)(a_1, a_2) = f(a_1)f(a_2) - f(a_1 a_2)$

Graded Frobenius Recursion:

$$\Phi_{m+1}(f)(a_1, a_2, \ldots, a_{m+1}) = f(a_1)\Phi_m(f)(a_2, \ldots, a_{m+1}) -$$
$$- \Phi_m(f)(a_1 a_2, a_3 \ldots, a_{m+1}) -$$
$$(-1)^{|a_1||a_2|}\Phi_m(f)(a_2, a_1 a_3, \ldots, a_{m+1}) -$$
$$(-1)^{|a_1||a_2|+|a_1||a_3|}\Phi_m(f)(a_2, a_3, a_1 a_4, \ldots, a_{m+1}) -$$
$$\ldots - (-1)^{|a_1||a_2|+\ldots+|a_1||a_m|}\Phi_m(f)(a_2, \ldots, a_m, a_1 a_{m+1}).$$

Our approach. Extension of V.M.Buchstaber-E.G.Rees theory to graded algebras.

Definition

A $\mathbb{K}$-linear map $f : A^* \to B^*$ of graded commutative $\mathbb{K}$-algebras with 1 is called a graded Frobenius n-homomorphism if the following holds:

1. $f(1_{A^*}) = n1_{B^*}$
2. $\Phi_{n+1}(f)(a_1, \ldots, a_{n+1}) = 0 \;\; \forall a_1, \ldots, a_{n+1} \in A^*$

Original definition for ungraded algebras was introduces by V.M.Buchstaber and E.G.Rees in 1996.

Remark

- 1-homomorphisms are just algebra homomorphisms.
- The second axiom implies $f(a_1 a_2 \ldots a_{n+1})$ is a polynomial of $f(a_{i_1} a_{i_2} \ldots a_{i_s})$ for $1 \leq s \leq n$ and $1 \leq i_1 < i_2 < \ldots < i_s \leq n+1$.

Our approach. Extension of V.M.Buchstaber-E.G.Rees theory to graded algebras.

Theorem (G., 2011)

Suppose $f : A^* \to B^*$ is an n-homomorphism and $g : A^* \to B^*$ is an m-homomorphism. Then the sum $f + g : A^* \to B^*$ is an $(n+m)$-homomorphism.

Corollary

The sum $f = f_1 + \ldots + f_n : A^* \to B^*$ is an n-homomorphism for arbitrary algebra homomorphisms $f_1, \ldots, f_n : A^* \to B^*$.

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Thank you for your attention!