Harmonic maps, Toda frames and extended Dynkin diagrams

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- Coxeter automorphism on $G^{\mathbb{C}}/T^{\mathbb{C}}$ and conditions for it to preserve a real form
- de Sitter spheres S^{2n}_1 and isotropic flag bundles
- Toda integrable system and relationship to cyclic primitive maps from a surface into G/T
- Solution in terms of ODEs (finite type)
- Applications to superconformal tori in S^{2n}_1
- Applications to Willmore tori in S^3.

Coxeter automorphism on $G^{\mathbb{C}}/T^{\mathbb{C}}$

Let $G^{\mathbb{C}}$ be a simple complex Lie group and $T^{\mathbb{C}}$ a Cartan subgroup.

The homogeneous space $G^{\mathbb{C}}/T^{\mathbb{C}}$ is naturally a k-symmetric space.

That is, we have an automorphism $\sigma : G^{\mathbb{C}} \to G^{\mathbb{C}}$ with $\sigma^k = 1$ and

$$(G^{\mathbb{C}}_{\sigma})_{\mathrm{id}} \subset T^{\mathbb{C}} \subset G^{\mathbb{C}}_{\sigma}.$$

Recall that a non-zero $\alpha \in (\mathfrak{t}^{\mathbb{C}})^*$ is a *root* with *root space* $\mathcal{G}^{\alpha} \subset \mathfrak{g}^{\mathbb{C}}$ if

$$[H, R_\alpha] = \alpha(H) R_\alpha \quad \forall H \in \mathfrak{t},\ R_\alpha \in \mathcal{G}^{\alpha}.$$

Choose a set of *simple roots*, that is roots $\{\alpha_1, \ldots, \alpha_N\}$ such that every root can be written uniquely as

$$\alpha = \sum_{j=1}^{N} m_j \alpha_j,$$

where all $m_j \in \mathbb{Z}^+$ or all $m_j \in \mathbb{Z}^-$.

The *height* of α is $h(\alpha) = \sum_{j=1}^{N} m_j$ and the root of minimal height is called the *lowest root*.

Let $\eta^1, \ldots, \eta^N \in \mathfrak{t}^{\mathbb{C}}$ be the dual basis to $\alpha_1, \ldots, \alpha_N$ and $\sigma : G^{\mathbb{C}} \to G^{\mathbb{C}}$ be conjugation by

$$\exp(\frac{2\pi i}{k} \sum_{j=1}^{N} \eta_j) \quad \text{(Coxeter automorphism).}$$

Then σ has order k, where $k-1$ is the maximal height of a root of $\mathfrak{g}^{\mathbb{C}}$.

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Let G be a real simple Lie group with Cartan subgroup T and assume that the Coxeter automorphism preserves the real form G.

I will describe class of harmonic maps from the surface into G/T which are given simply by solving ordinary differential equations and give a relationship between these maps and the Toda equations.

This will generalise work of Bolton, Pedit and Woodward for the case when G is compact.

Example: $SO(2n,1)$

Let $\mathbb{R}^{2n,1}$ denote $\mathbb{R}^{2n+1}$ with the Minkowski inner product

$$x^1y^1 + x^2y^2 + \cdots + x^{2n}y^{2n} - x^{2n+1}y^{2n+1}$$

Consider the de Sitter group $SO(2n,1)$ of orientation preserving isometries of $\mathbb{R}^{2n,1}$. Take as Cartan subgroup

$$T = \mathrm{diag}\left(1, SO(2), \ldots, SO(2), SO(1,1)\right).$$

Define $\tilde{a}_k \in \mathfrak{t}^*$, $k = 1, \ldots, n$ by

$$\tilde{a}_k \left(\mathrm{diag} \left\{ 0, \begin{pmatrix} 0 & a_1 \\ -a_1 & 0 \end{pmatrix}, \ldots \begin{pmatrix} 0 & a_n \\ a_n & 0 \end{pmatrix} \right\} \right) = a_k.$$

Take as simple roots of $\mathfrak{so}(2n, 1, \mathbb{C})$ the roots

$$\begin{aligned} \alpha_1 &= i\tilde{a}_1, \\ \alpha_k &= i\tilde{a}_k - i\tilde{a}_{k-1} \quad \text{for } 1 < k < n \quad \text{and} \\ \alpha_n &= \tilde{a}_n - i\tilde{a}_{n-1}. \end{aligned}$$

The lowest root is then $\alpha_0 = -\tilde{a}_n - i\tilde{a}_{n-1}$, which is of height $-2n + 1$.

Then writing η_j for the dual basis of $\mathfrak{t}^{\mathbb{C}}$, conjugation by

$$Q = \exp\Big(\frac{\pi i}{n}\sum_{j=1}^{n}\eta_j\Big) = \operatorname{diag}\left(1, R\left(\frac{\pi}{n}\right), R\left(\frac{2\pi}{n}\right), \ldots, R\left(\frac{r\pi}{n}\right), -I_2\right)$$

is an automorphism of order $2n$.

It is not hard to prove directly in this case that the real form $SO(2n, 1)$ is preserved by the Coxeter automorphism.

Let $\langle\cdot,\cdot\rangle$ denote the complex bilinear form

$$\langle z, w\rangle = z^1 w^1 + z^2 w^2 + \cdots + z^{2n} w^{2n} - z^{2n+1} w^{2n+1}$$

on $\mathbb{C}^{2n+1}$.

A subspace $V \subset \mathbb{C}^{2n+1}$ is isotropic if $\langle u, v\rangle = 0$ for all $u, v \in V$.

Geometrically,

$$SO(2n,1)/T = SO(2n,1)/(1 \times SO(2) \times \cdots \times SO(2) \times SO(1,1))$$

is the full isotropic flag bundle

$$\mathrm{Fl}(S_1^{2n}) = \{V_1 \subset V_2 \subset \cdots \subset V_{n-1} \subset T^{\mathbb{C}} S_1^{2n} \mid V_j \text{ is an isotropic sub-bundle of dimension } j\}$$

We now give conditions under which a choice of

- real form $\mathfrak{g}$ of a simple complex Lie algebra $\mathfrak{g}^{\mathbb{C}}$,
- Cartan subalgebra $\mathfrak{t}^{\mathbb{C}}$ and simple roots α_j

yield a Coxeter automorphism $\sigma = \mathrm{Ad}_{\exp(\frac{2\pi i}{k}\sum_{j=1}^{N}\eta_j)}$ which preserves the real Lie algebra $\mathfrak{g}$.

The condition for the Coxeter automorphism σ to preserve $\mathfrak{g}$ is that for the simple roots $\alpha_1, \dots, \alpha_N$ we have

$$\bar{\alpha}_j \in \{-\alpha_0, \dots, -\alpha_N\},$$

where $\bar{\alpha}(X) = \overline{\alpha(\bar{X})}$ and α_0 is the lowest root.

We will now use a Cartan involution to express this reality condition in terms of the extended Dynkin diagram.

A Cartan involution for $\mathfrak{g}$ is an involution Θ of $\mathfrak{g}^{\mathbb{C}}$ such that

$$\langle X, Y\rangle_\Theta = -\langle X, \Theta(Y)\rangle$$

is positive definite on $\mathfrak{g}$, where $\langle\cdot,\cdot\rangle$ denotes the Killing form. Alternatively, it is an involution for which

$$\mathfrak{k} \oplus i\mathfrak{m}$$

is compact, where

$$\begin{aligned}\mathfrak{k} &= +1\text{-eigenspace of } \Theta\\ \mathfrak{m} &= -1\text{-eigenspace of } \Theta.\end{aligned}$$

We may choose a Cartan involution which preserves the given Cartan subalgebra $\mathfrak{t}$

Proposition

Let $\mathfrak{g}$ be a real simple Lie algebra, $\mathfrak{t}$ a Cartan subalgebra and Θ be a Cartan involution preserving $\mathfrak{t}$. Choose simple roots $\alpha_1, \ldots, \alpha_N$ and let $\sigma = \mathrm{Ad}_{\exp(\frac{2\pi i}{k} \sum_{j=1}^{N} \eta_j)}$ be the corresponding Coxeter automorphism of $\mathfrak{g}^{\mathbb{C}}$. Then the following are equivalent:

1. *σ preserves the real form $\mathfrak{g}$,*
2. *σ commutes with Θ,*
3. *Θ defines an involution of the extended Dynkin diagram for $\mathfrak{g}^{\mathbb{C}}$ consisting of the usual Dynkin diagram augmented with the lowest root α_0.*

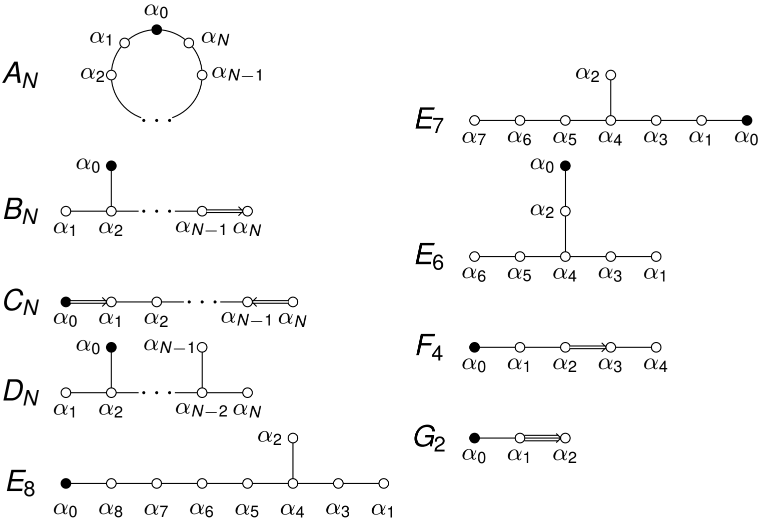
For a Θ-stable Cartan subalgebra $\mathfrak{t}$,

$\mathfrak{t}$ is maximally compact	$\Leftrightarrow$	Θ defines a permutation of the Dynkin diagram for $\mathfrak{g}^{\mathbb{C}}$

and so when $\mathfrak{t}$ is maximally compact (e.g. $\mathfrak{g}$ is compact), the real form $\mathfrak{g}$ is preserved by any Coxeter automorphism defined by simple roots for $\mathfrak{t}$.

The more interesting case is when we have an involution of the extended Dynkin diagram which does not restrict to an involution of the Dynkin diagram (i.e. $\mathfrak{t}$ is not maximally compact).

Call these *non-trivial* involutions.



There are nontrivial involutions for all root systems except E_8, F_4 and G_2.

Theorem

Every involution of the extended Dynkin diagram for a simple complex Lie algebra $\mathfrak{g}^{\mathbb{C}}$ is induced by a Cartan involution of a real form of $\mathfrak{g}^{\mathbb{C}}$.

More precisely, let $\mathfrak{g}^{\mathbb{C}}$ be a simple complex Lie algebra with Cartan subalgebra $\mathfrak{t}^{\mathbb{C}}$ and choose simple roots $\alpha_1, \ldots, \alpha_N$ for the root system $\Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$. Given an involution π of the extended Dynkin diagram for Δ, there exists a real form $\mathfrak{g}$ of $\mathfrak{g}^{\mathbb{C}}$ and a Cartan involution Θ of $\mathfrak{g}$ preserving $\mathfrak{t} = \mathfrak{g} \cap \mathfrak{t}^{\mathbb{C}}$ such that Θ induces π and $\mathfrak{t}$ is a real form of $\mathfrak{t}^{\mathbb{C}}$. The Coxeter automorphism σ determined by $\alpha_1, \ldots, \alpha_N$ preserves the real form $\mathfrak{g}$.

Primitive Maps and Loop Groups

The Coxeter automorphism $\sigma : \mathfrak{g} \to \mathfrak{g}$ of order k induces a $\mathbb{Z}_k$-grading

$$\mathfrak{g}^{\mathbb{C}} = \bigoplus_{j=0}^{k-1} \mathfrak{g}_j^\sigma, \quad [\mathfrak{g}_j^\sigma, \mathfrak{g}_l^\sigma] \subset \mathfrak{g}_{j+l}^\sigma,$$

where $\mathfrak{g}_j^\sigma$ denotes the $e^{j\frac{2\pi i}{k}}$-eigenspace of σ.

We have the reductive splitting

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

with

$$\mathfrak{p}^{\mathbb{C}} = \bigoplus_{j=1}^{k-1} \mathfrak{g}_j^\sigma, \qquad \mathfrak{k}^{\mathbb{C}} = \mathfrak{g}_0^\sigma,$$

and if φ is a $\mathfrak{g}$-valued form we may decompose it as $\varphi = \varphi_{\mathfrak{k}} + \varphi_{\mathfrak{p}}$.

A smooth map f of a surface into a symmetric space $(G/K, \sigma)$ is harmonic if and only if for a smooth lift $F : U \to G$ of $f : U \to G/K$, the form $\varphi = F^{-1}dF$ has the property that for each $\lambda \in S^1$

$$\varphi_\lambda = \lambda \varphi'_{\mathfrak{p}} + \varphi_{\mathfrak{k}} + \lambda^{-1} \varphi''_{\mathfrak{p}}$$

satisfies the Maurer-Cartan equation

$$d\varphi_\lambda + \frac{1}{2}[\varphi_\lambda \wedge \varphi_\lambda] = 0.$$

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When $k > 2$, the condition that

$$\varphi_\lambda = \lambda \varphi'_{\mathfrak{p}} + \varphi_{\mathfrak{k}} + \lambda^{-1} \varphi''_{\mathfrak{p}}$$

satisfies the Maurer-Cartan equation

$$d\varphi_\lambda + \frac{1}{2}[\varphi_\lambda \wedge \varphi_\lambda] = 0.$$

characterises (not merely harmonic but) primitive maps ψ of a surface into the k-symmetric space G/K.

ψ is primitive if for a smooth lift $F : U \to G$ of $\psi : U \to G/K$, $\varphi' = F^{-1}\partial F$ takes values in $\mathfrak{g}_0^\sigma \oplus \mathfrak{g}_1^\sigma$. We say that F is a primitive frame.

Primitive maps ψ are in particular harmonic.

For studying maps into G/T it is helpful to consider the twisted loop group

$$\Omega^\sigma G = \{\gamma : S^1 \to G : \gamma(e^{\frac{2\pi i}{k}}\lambda)\} = \sigma(\gamma(\lambda))\}$$

and corresponding twisted loop algebra $\Omega^\sigma \mathfrak{g}$. The (possibly doubly infinite) Laurent expansion

$$\xi(\lambda) = \sum_j \xi_j \lambda^j, \quad \xi_j \in \mathfrak{g}_j^\sigma \subset \mathfrak{g}^{\mathbb{C}}, \quad \Phi_{-j} = \bar{\Phi}_j$$

allows us to filtrate $\Omega^\sigma \mathfrak{g}^{\mathbb{C}}$ by finite-dimensional subspaces

$$\Omega_d^\sigma = \{\xi \in \Omega\mathfrak{g} \mid \xi_j = 0 \text{ whenever } |j| > d\}.$$

Suppose $\xi : \mathbb{R}^2 \to \Omega_d^\sigma$ satisfies the Lax equation

$$\frac{\partial \xi}{\partial z} = [\xi, \lambda \xi_d + \frac{1}{2}\xi_{d-1}].$$

Then

$$\varphi_\lambda(z) = \Big(\lambda \xi_d(z) + \frac{1}{2}\xi_{d-1}(z)\Big)dz + \Big(\lambda^{-1}\xi_{-d}(z) + \frac{1}{2}\overline{\xi_{d-1}(z)}\Big)d\bar{z}$$

satisfies the Maurer-Cartan equation and so defines a primitive map $f : \mathbb{R}^2 \to G/T$.

Maps f obtained in this simple way are said to be of finite type.

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The equation

$$\frac{1}{2}(X(\xi) - iY(\xi)) = \left(\lambda\xi_d + \frac{1}{2}\xi_{d-1}\right)$$

defines vector fields X, Y on Ω_d^σ.

Assume the vector fields X, Y are complete (e.g. G is compact). The vector fields X, Y commute and so define an action

$$(x, y) \cdot \xi(\lambda) = X^x \circ Y^y(\xi(\lambda))$$

of $\mathbb{R}^2$ on Ω_d. Define $\xi(z, \lambda) := (x, y) \cdot \xi^0(\lambda)$ for any initial $\xi^0(\lambda) \in \Omega_d$, where $z = x + iy$. Then

$$\varphi_\lambda(z) = \left(\lambda\xi_d(z) + \frac{1}{2}\xi_{d-1}(z)\right)dz + \left(\lambda^{-1}\xi_{-d}(z) + \frac{1}{2}\overline{\xi_{d-1}(z)}\right)d\bar{z}$$

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For the Coxeter automorphism on G/T, $\mathfrak{g}_0^\sigma = \mathfrak{t}$ and $\mathfrak{g}_1^\sigma$ is the sum of the simple and lowest root spaces.

We say that a primitive map ψ / frame F is in addition *cyclic* if the image of $F^{-1}\partial F$ contains a cyclic element.

An element of $\mathfrak{g}_0^{\sigma_r} \oplus \mathfrak{g}_1^{\sigma_r}$ is cyclic if its projection to each of the root spaces $\mathcal{G}^{\alpha_1}, \ldots, \mathcal{G}^{\alpha_n}, \mathcal{G}^{\alpha_0}$ is non-zero.

I will now describe the relationship between cyclic primitive maps into G/T and the Toda equations.

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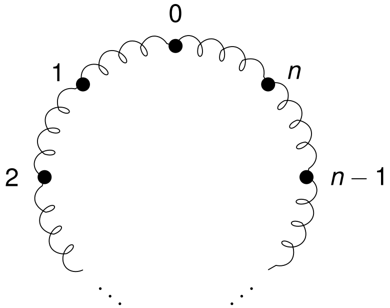
Toda equation

The classical 1-dimensional affine Toda integrable system describes the motion of finitely many particles of equal mass arranged in a circle, joined by “exponential springs".



$$m\frac{d^2 x_j}{dt^2} = e^{(x_{j-1} - x_j)} - e^{(x_j - x_{j+1})}.$$

We may generalise this to any simple Lie algebra as

$$2\frac{d^2\Omega}{dt^2} = \sum_{j=0}^{n} m_j e^{2\alpha_j(\Omega)}[R_{\alpha_j}, R_{-\alpha_j}]$$

or for a 2-dimensional domain

$$2\Omega_{z\bar{z}} = \sum_{j=0}^{N} m_j e^{2\alpha_j(\Omega)}[R_{\alpha_j}, R_{-\alpha_j}] \quad (1)$$

where $\Omega : \mathbb{C} \to i\mathfrak{t}$ is a smooth map, $m_j \in \mathbb{R}^+$ satisfies $m_{\pi(j)} = \overline{m_j}$ and R_{α_j} are root vectors satisfying $\overline{R_{\alpha_j}} = R_{-\alpha_{\pi(j)}}$.

To recover the classical Toda equation:

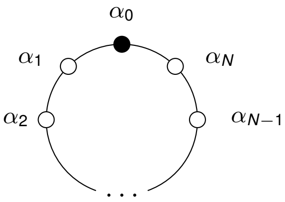
1. Take the standard simple roots for $\mathfrak{su}(n+1)$.
2. Set $m_0 = 1$ and let

$$\alpha_0 = -\sum_{j=1}^{N} m_j \alpha_j$$

be the expression for the lowest root α_0.

3. Choose root vectors R_{α_j} so that $[R_{\alpha_j}, R_{-\alpha_j}]$ is the dual of α_j with respect to the Killing form.

Notice that the extended Dynkin diagram for $\mathfrak{su}(n+1)$ looks like



Given a cyclic element $W = \sum_{j=0}^{N} r_j R_{\alpha_j}$ of $\mathfrak{g}_1^\sigma$, we say that a lift $F : \mathbb{C} \to G$ of $\psi : \mathbb{C} \to G/T$ is a *Toda frame* with respect to W if there exists a smooth map $\Omega : \mathbb{C} \to i\mathfrak{t}$ such that

$$F^{-1} F_z = \Omega_z + \mathrm{Ad}_{\exp \Omega} W.$$

We call Ω an *affine Toda field* with respect to W.

Lemma

The affine Toda field equation (1) *is the integrability condition for the existence of a Toda frame with respect to W.*

Here $W = \sum_{j=0}^{N} r_j R_{\alpha_j}$ is a cyclic element of $\mathfrak{g}_1^\sigma$ such that $m_{\pi(j)} = \overline{m_j}$ and $\overline{R_{\alpha_j}} = R_{-\alpha_{\pi(j)}}$ and we take $m_j = r_j \overline{r_j}$ for $j = 0, \ldots, N$.

Toda frame and cyclic primitive

Theorem

A map $\psi : \mathbb{C} \to G/T$ possesses a Toda frame if and only if it has a cyclic primitive frame F for which $c_0 \prod_{j=1}^{N} c_j^{m_j}$ is constant, where

$$F^{-1} F_z|_{\mathfrak{g}_1^\sigma} = \sum_{j=0}^{N} c_j R_{\alpha_j}.$$

The Toda frame is then cyclic primitive with respect to any $W = \sum_{j=0}^{N} r_j R_{\alpha_j}$ for which

$$r_0 \prod_{j=1}^{N} r_j^{m_j} = c_0 \prod_{j=1}^{N} c_j^{m_j}.$$

Finite-type

Theorem

Let G be a simple real Lie group, T a Cartan subgroup and assume that the Coxeter automorphism preserves G. Suppose $\psi : \mathbb{C}/\Lambda \to G/T$ has a Toda frame $F : \mathbb{C}/\Lambda \to G$. Then ψ is of finite type.

Harmonic maps into S_1^{2n}

The *isotropy order* of a harmonic map f of a surface into S_1^{2n} is the maximal integer $r \geq 0$ such that the derivatives $\partial_z F, \partial_z^2 F, \ldots, \partial_z^r f$ span an isotropic subspace at each point.

If f has the maximal isotropy order $r = n$ we say it is isotropic.

Isotropic surfaces in S_1^{2n} include all harmonic maps of S^2, and can be expressed holomorphically in terms of a Weierstrass-type representation (Bryant 84, Ejiri 88)

Harmonic maps $f : M^2 \to S_1^{2n}$ with the penultimate isotropy order $r = n - 1$ are said to be superconformal.

Applying Gram-Schmidt, we define the harmonic sequence $\{f_0, f_1, \ldots, f_r\}$ of a non-constant harmonic map $f : M^2 \to S_1^{2n}$ by

$$f_0 = f, \qquad f_{j+1} = \partial_z f_j - \frac{\langle \partial_z f_j, \overline{f_j} \rangle}{\|f_j\|^2} f_j \text{ wherever } \|f_j\|^2 \neq 0$$

and extend by continuity wherever $f_j = 0$. Then

$$\begin{aligned} \partial_{\bar z} f_{j+1} &= -\frac{\|f_{j+1}\|^2}{\|f_j\|^2} f_j \quad \text{for } 0 \leq j \leq r \\ \langle f_j, \overline{f_k} \rangle &= 0 \quad \text{unless } j = k \end{aligned}$$

and the zeros of the f_j are isolated whenever f_j does not vanish identically (Hulett 05).

Applying Gram-Schmidt, we define the harmonic sequence $\{f_0, f_1, \ldots, f_r\}$ of a non-constant harmonic map $f : M^2 \to S^{2n}_1$ by

$$f_0 = f, \qquad f_{j+1} = \partial_z f_j - \frac{\langle \partial_z f_j, \overline{f_j} \rangle}{\|f_j\|^2} f_j \text{ wherever } \|f_j\|^2 \neq 0$$

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Theorem

A harmonic map $f : \mathbb{C} \to S^{2n}_1$ has a cyclic primitive lift $\psi : \mathbb{C} \to Fl(S^{2n}_1)$ if and only if it is superconformal and the entries $\{f_1, \ldots, f_{n-1}\}$ of its harmonic sequence are defined everywhere.

We have for each $1 \leq j \leq r$

$$f_j = 2^{j-1} c_1 \ldots c_j F(e_{2j} + ie_{2j+1}) \text{ for each } 1 \leq j \leq n-1$$

where the c_j are root vector coefficients with respect to particular choices of the root vectors appearing in $\mathfrak{g}_1^{\sigma_r}$.

Corollary

Let $f : \mathbb{C}/\Lambda \to S^{2n}_1$ be a superconformal harmonic map with globally defined harmonic sequence $\{f_1, \ldots, f_n\}$. Then f has a lift $\psi : \mathbb{C}/\Lambda \to SO(2n, 1)/T$ of finite type.

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An immersed surface $\phi : M^2 \to \mathbb{R}^3$ is *Willmore* if it is critical for the Willmore functional

$$\mathcal{W} = \int_{M^2} H^2 \, dA,$$

where H denotes the mean curvature of ϕ and dA the area form.

Due to Gauss-Bonnet, it is equivalent to seek critical surfaces for

$$\int_{M^2} (H^2 - K) \, dA, = \int_{M^2} (k_2 - k_1)^2 \, dA$$

where K is the Gauss curvature and k_1, k_2 are the principal curvatures.

This latter functional is clearly conformally invariant and so we instead consider immersions into S^3.

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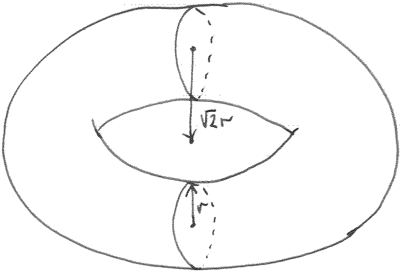
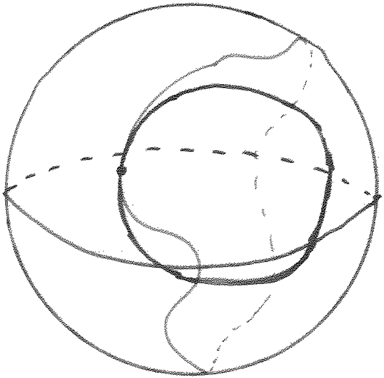
The Willmore conjecture proposes that $\mathcal{W}(\mathbb{C}/\Lambda) \geq 2\pi^2$ for any immersed torus with equality if and only if the torus is conformally equivalent to



The conformal Gauss map of an immersion $\phi : M^2 \to S^3$ associates to each point on the surface M^2 its central sphere, that is the oriented 2-sphere in S^3 with the same normal vector and mean curvature.



A 2-sphere in S^3 is the intersection of S^3 and a hyperplane in $\mathbb{R}^4$;

$$S^3 \cap \{x_1, x_2, x_3, x_4 : a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 - b = 0\}.$$

For this hyperplane to intersect with S^3 at more than one point requires $a_1^2 + a_2^2 + a_3^2 + a_4^2 - b^2 > 0$ and hence we can scale (a_1, a_2, a_3, a_4, b) so that $a_1^2 + a_2^2 + a_3^2 + a_4^2 - b^2 = 1$.

De Sitter space S_1^{2n} is the unit sphere in $\mathbb{R}^{2n+1}$ with respect to the Minkowski metric

$$x^1y^1 + x^2y^2 + \cdots + x^{2n}y^{2n} - x^{2n+1}y^{2n+1}.$$

Thus each 2-sphere in S^3 can be identified with two antipodal points $\pm(a_1, a_2, a_3, a_4, b) \in S_1^4$.

Choosing an orientation for the 2-sphere gives a well-defined element of S_1^4.

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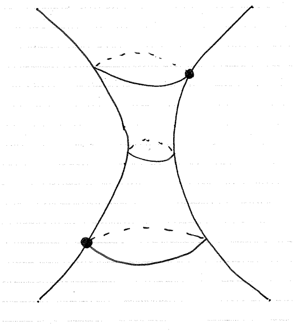
Hence we see that the space of oriented 2-spheres in S^3 is naturally identified with S^4_1.



The conformal Gauss map $f : M^2 \to S^4_1$ is given explicitly by

$$f(z) = H(z) \cdot \Phi(z) + N(z)$$

where $\Phi(z) = (\phi(z), 1)$, $N = (n, 0)$.

The conformal Gauss map f is weakly conformal and an immersion away from the umbilic points of ϕ.

The area form on M^2 induced by f is given by $(H^2 - K)dA$

Thus $\phi : M^2 \to S^3$ is a Willmore immersion without umbilic points if and only if $f : M^2 \to S^4_1$ is a minimal immersion, or equivalently is conformal and harmonic.

A minimal immersion $f : M^2 \to S^4_1$ can only have isotropy order $r = 1$ (superconformal) or $r = 2$ (isotropic).

Recall that the second fundamental form of f is $II(X, Y) = (\nabla_X Y)^\perp$, where $\perp$ denotes projection to the orthogonal complement of TM^2 in TS^4_1.

The curvature ellipse of f at $p \in M^2$ is the image of the unit circle in T_pM^2 under the second fundamental form.

It is a circle precisely when $\langle f_{zz}(p), f_{zz}(p)\rangle = 0$. This quantity is holomorphic, hence constant when M^2 is compact.

The curvature ellipse of f is thus a circle precisely when f is isotropic. All isotropic f have been constructed by Bryant using holomorphic data.

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We have seen that the first ellipse of curvature being a non-circular ellipse corresponds to f being superconformal.

For superconformal $f : M^2 \to S^4_1$ the cyclic primitive frame F constructed previously consists of

$$F = (f, f_x, f_y, v, w)$$

where the last two columns of F are determined by the principal directions of the curvature ellipse.

Corollary

A Willmore immersion $\phi : T^2 \to S^3$ without umbilic points may be constructed either

1. *from holomorphic Weierstrass data*
2. *by integrating a pair of commuting vector fields on a finite-dimensional space*