Integral transforms and the twistor theory for indefinite metrics

Fuminori NAKATA

Tokyo University of Science

Dec. 5, 2011, The 10th Pacific Rim Geometry Conference

Introduction

Twistor correspondence is a correspondence between

- complex manifolds with a family of $\mathbb{CP}^1$ or holomorphic disks, and
- manifolds equipped with a certain integrable structure.

	self-dual conformal 4-mfd	Einstein-Weyl 3-mfd
complex	Penrose (1976)	Hitchin (1982)
Riemannian	Atiyah-Hitchin-Singer (1978)	Hitchin (1982) Pedersen-Tod (1993)
indefinite	LeBrun-Mason (2007)	LeBrun-Mason (2009)

Results in hyperbolic PDE and integral transforms are obtained in the way of constructing explicit examples of twistor correspondences.

Introduction

Twistor correspondence is a correspondence between

- complex manifolds with a family of $\mathbb{CP}^1$ or holomorphic disks, and
- manifolds equipped with a certain integrable structure.

	self-dual conformal 4-mfd	Einstein-Weyl 3-mfd
complex	Penrose (1976)	Hitchin (1982)
Riemannian	Atiyah-Hitchin-Singer (1978)	Hitchin (1982) Pedersen-Tod (1993)
indefinite	LeBrun-Mason (2007)	LeBrun-Mason (2009)

Results in hyperbolic PDE and integral transforms are obtained in the way of constructing explicit examples of twistor correspondences.

Introduction

Twistor correspondence is a correspondence between

- complex manifolds with a family of $\mathbb{CP}^1$ or holomorphic disks, and
- manifolds equipped with a certain integrable structure.

	self-dual conformal 4-mfd	Einstein-Weyl 3-mfd
complex	Penrose (1976)	Hitchin (1982)
Riemannian	Atiyah-Hitchin-Singer (1978)	Hitchin (1982) Pedersen-Tod (1993)
indefinite	LeBrun-Mason (2007)	LeBrun-Mason (2009)

Results in hyperbolic PDE and integral transforms are obtained in the way of constructing explicit examples of twistor correspondences.

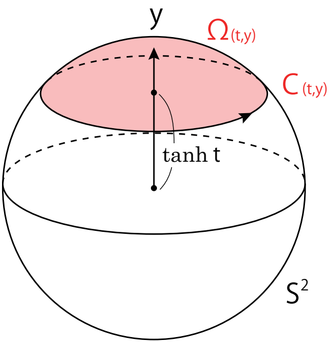
1. Integral transforms on 2-sphere
2. Integral transforms on a cylinder
3. Minitwistor theory
4. LeBrun-Mason twistor theory
   - general theory
   - S^1-invariant case
   - $\mathbb{R}$-invariant case

1. Integral transforms on 2-sphere

Small circles

Let us define

$$M = \{\text{oriented } \text{small circles} \text{ on } S^2\}$$
$$\cong \left\{ \begin{array}{l} \text{domain on } S^2 \\ \text{bouded by a small circle} \end{array} \right\}$$



Each domain is described as

$$\Omega_{(t,y)} = \{u \in S^2 \mid u \cdot y > \tanh t\}$$

by using $(t, y) \in \mathbb{R} \times S^2$.

Hence $M \cong \mathbb{R} \times S^2$.

Small circles

Let us define

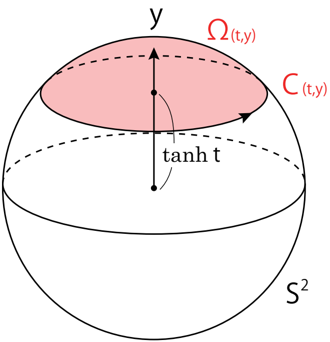
$$M = \{\text{oriented } \textcolor{red}{\text{small circles}} \text{ on } S^2\}$$
$$\cong \left\{ \begin{array}{l} \text{domain on } S^2 \\ \text{bouded by a small circle} \end{array} \right\}$$

Each domain is described as

$$\Omega_{(t,y)} = \{u \in S^2 \mid u \cdot y > \tanh t\}$$

by using $(t, y) \in \mathbb{R} \times S^2$.

Hence $M \cong \mathbb{R} \times S^2$.



Small circles

Let us define

$$M = \{\text{oriented } \textcolor{red}{\text{small circles}} \text{ on } S^2\}$$

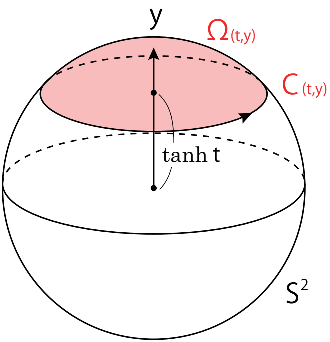
$$\cong \left\{ \begin{array}{l} \text{domain on } S^2 \\ \text{bouded by a small circle} \end{array} \right\}$$

Each domain is described as

$$\Omega_{(t,y)} = \{u \in S^2 \mid u \cdot y > \tanh t\}$$

by using $(t, y) \in \mathbb{R} \times S^2$.

Hence $M \cong \mathbb{R} \times S^2$.



Small circles

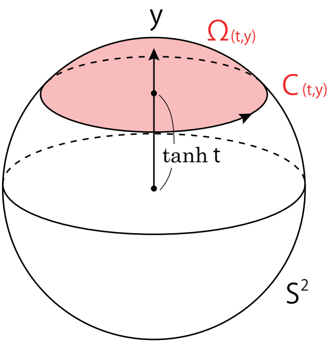
$$M = \{(t,y) \in \mathbb{R} \times S^2\}$$

Let us introduce an indefinite metric on M by

$$g = -dt^2 + \cosh^2 t \cdot g_{S^2}.$$

(M,g) is identified with the de Sitter 3-space $(S^3_1, g_{S^3_1})$

This identification $M \cong S^3_1$ arises from minitwistor correspondence.



Small circles

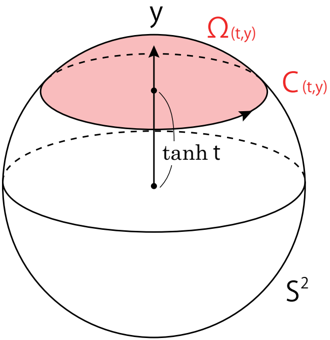
$M = \{(t, y) \in \mathbb{R} \times S^2\}$

Let us introduce an indefinite metric on M by

$$g = -dt^2 + \cosh^2 t \cdot g_{S^2}.$$

(M, g) is identified with the de Sitter 3-space $(S^3_1, g_{S^3_1})$

This identification $M \cong S^3_1$ arises from minitwistor correspondence.



Small circles

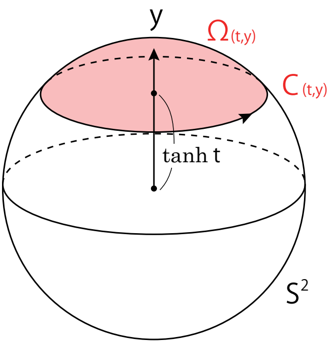
$$M = \{(t, y) \in \mathbb{R} \times S^2\}$$

Let us introduce an indefinite metric on M by

$$g = -dt^2 + \cosh^2 t \cdot g_{S^2}.$$

(M, g) is identified with the de Sitter 3-space $(S^3_1, g_{S^3_1})$

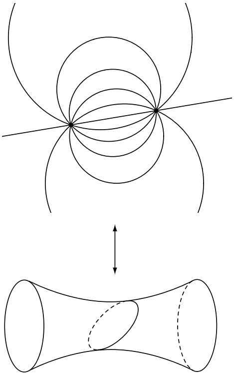
This identification $M \cong S^3_1$ arises from minitwistor correspondence.



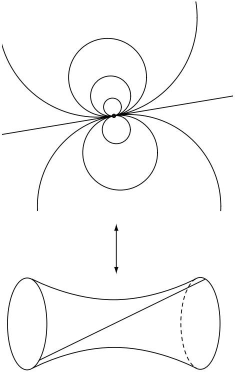
Geodesics

There are subfamilies of small circles known as "Apollonian circles".

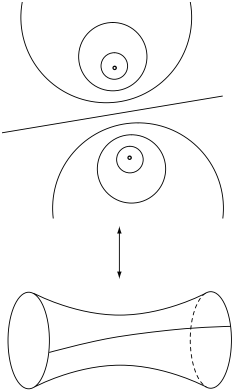
These families corresponde to geodesics on $(S^3_1, g_{S^3_1})$.



space-like geodesic



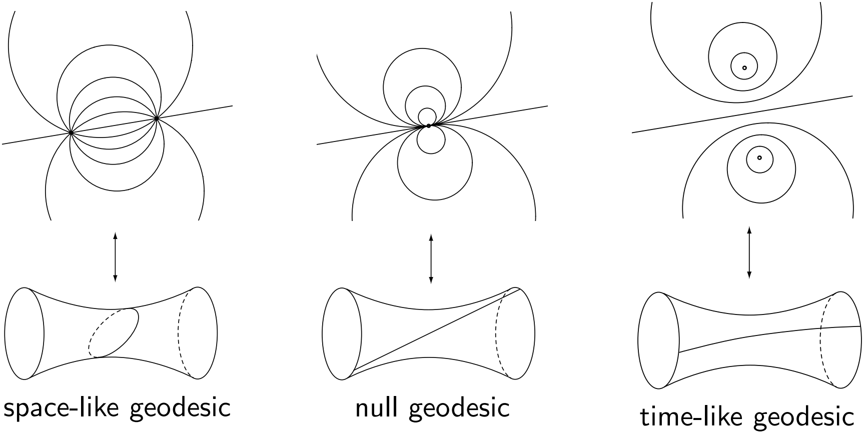
null geodesic



time-like geodesic

Geodesics

There are subfamilies of small circles known as "Apollonian circles".
These families corresponde to geodesics on $(S^3_1, g_{S^3_1})$.



Integral transforms

For given function $h \in C^\infty(S^2)$, we define functions $Rh, Qh \in C^\infty(S^3_1)$ by

[mean value] $$Rh(t,y) = \frac{1}{2\pi}\int_{\partial\Omega_{(t,y)}} h\, dS^1$$

[area integral] $$Qh(t,y) = \frac{1}{2\pi}\int_{\Omega_{(t,y)}} h\, dS^2$$

where dS^1 is the standard measure on $\partial\Omega_{(t,y)}$ of total length 2π, and dS^2 is the standard measure on S^2.

Integral transforms

For given function $h \in C^\infty(S^2)$, we define functions $Rh, Qh \in C^\infty(S^3_1)$ by

[mean value] $$Rh(t,y) = \frac{1}{2\pi}\int_{\partial\Omega_{(t,y)}} h\, dS^1$$

[area integral] $$Qh(t,y) = \frac{1}{2\pi}\int_{\Omega_{(t,y)}} h\, dS^2$$

where dS^1 is the standard measure on $\partial\Omega_{(t,y)}$ of total length 2π,
and dS^2 is the standard measure on S^2.

Wave equation on de Sitter 3-space

Wave equation on $(S_1^3, g_{S_1^3})$: $\square V := *d * dV = 0.$

Let us put $C_*^\infty(S^2) = \left\{ h \in C^\infty(S^2) \,\middle|\, \int_{S^2} h dS^2 = 0 \right\}.$

Theorem (N. '09)

For each function $h \in C_*^\infty(S^2)$, the function $V := Qh \in C^\infty(S_1^3)$ satisfies

$$\text{(i)} \quad \square V = 0, \qquad \text{(ii)} \quad \lim_{t \to \infty} V(t, y) = \lim_{t \to \infty} V_t(t, y) = 0.$$

Conversely, if $V \in C^\infty(S_1^3)$ satisfies (i) and (ii), then there exists unique $h \in C_*^\infty(S^2)$ such that $V = Qh$.

Remark A similar type theorem for the transform R is also obtained.

Wave equation on de Sitter 3-space

Wave equation on $(S^3_1, g_{S^3_1})$: $\Box V := *d*dV = 0.$

Let us put $$C^\infty_*(S^2) = \left\{ h \in C^\infty(S^2) \;\middle|\; \int_{S^2} h dS^2 = 0 \right\}.$$

Theorem (N. '09)

For each function $h \in C^\infty_*(S^2)$, the function $V := Qh \in C^\infty(S^3_1)$ satisfies

$$\text{(i)}\;\; \Box V = 0, \qquad \text{(ii)}\;\; \lim_{t\to\infty} V(t,y) = \lim_{t\to\infty} V_t(t,y) = 0.$$

Conversely, if $V \in C^\infty(S^3_1)$ satisfies (i) and (ii), then there exists unique $h \in C^\infty_*(S^2)$ such that $V = Qh$.

Remark A similar type theorem for the transform R is also obtained.

Wave equation on de Sitter 3-space

Wave equation on $(S^3_1, g_{S^3_1})$: $\quad \Box V := *d*dV = 0.$

Let us put $\quad C^\infty_*(S^2) = \left\{ h \in C^\infty(S^2) \,\middle|\, \int_{S^2} h dS^2 = 0 \right\}.$

Theorem (N. '09)

For each function $h \in C^\infty_*(S^2)$, the function $V := Qh \in C^\infty(S^3_1)$ satisfies

$$\text{(i)} \ \Box V = 0, \qquad \text{(ii)} \ \lim_{t\to\infty} V(t,y) = \lim_{t\to\infty} V_t(t,y) = 0.$$

Conversely, if $V \in C^\infty(S^3_1)$ satisfies (i) and (ii), then there exists unique $h \in C^\infty_*(S^2)$ such that $V = Qh$.

Remark A similar type theorem for the transform R is also obtained.

Wave equation on de Sitter 3-space

Wave equation on $(S^3_1, g_{S^3_1})$: $\Box V := *d*dV = 0.$

Let us put
$$C^\infty_*(S^2) = \left\{ h \in C^\infty(S^2) \;\middle|\; \int_{S^2} h dS^2 = 0 \right\}.$$

Theorem (N. '09)

For each function $h \in C^\infty_*(S^2)$, the function $V := Qh \in C^\infty(S^3_1)$ satisfies

$$\text{(i)}\ \ \Box V = 0, \qquad \text{(ii)}\ \ \lim_{t\to\infty} V(t,y) = \lim_{t\to\infty} V_t(t,y) = 0.$$

Conversely, if $V \in C^\infty(S^3_1)$ satisfies (i) and (ii), then there exists unique $h \in C^\infty_*(S^2)$ such that $V = Qh$.

Remark A similar type theorem for the transform R is also obtained.

2. Integral transforms on a cylinder

Planar circles

Let $\mathscr{C} = \{(\theta, v) \in S^1 \times \mathbb{R}\}$ be the 2-dimensional cylinder.

Let us define

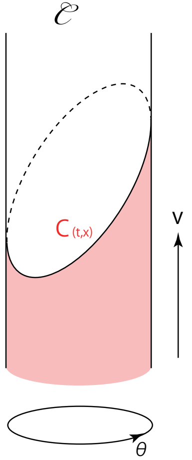
$$M' = \{\text{planar circles on } \mathscr{C}\}$$

Each planar circle is described as

$$C_{(t,x)} = \{(\theta, v) \mid v = t + x_1 \cos\theta + x_2 \sin\theta\}$$

using $(t, x) \in \mathbb{R} \times \mathbb{R}^2$.

Hence $M' \cong \mathbb{R}^3$



Planar circles

Let $\mathscr{C} = \{(\theta, v) \in S^1 \times \mathbb{R}\}$ be the 2-dimensional cylinder.

Let us define

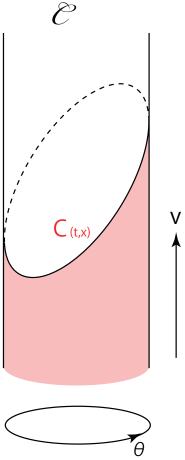
$$M' = \{\text{planar circles on } \mathscr{C}\}$$

Each planar circle is described as

$$C_{(t,x)} = \{(\theta, v) \mid v = t + x_1 \cos\theta + x_2 \sin\theta\}$$

using $(t, x) \in \mathbb{R} \times \mathbb{R}^2$.

Hence $M' \cong \mathbb{R}^3$



Planar circles

Let $\mathscr{C} = \{(\theta, v) \in S^1 \times \mathbb{R}\}$ be the 2-dimensional cylinder.

Let us define

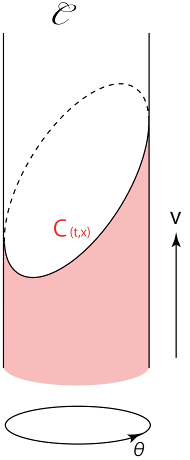
$$M' = \{\text{planar circles on } \mathscr{C}\}$$

Each planar circle is described as

$$C_{(t,x)} = \{(\theta, v) \mid v = t + x_1 \cos\theta + x_2 \sin\theta\}$$

using $(t, x) \in \mathbb{R} \times \mathbb{R}^2$.

Hence $M' \cong \mathbb{R}^3$



Planar circles

Let $\mathscr{C} = \{(\theta, v) \in S^1 \times \mathbb{R}\}$ be the 2-dimensional cylinder.

Let us define

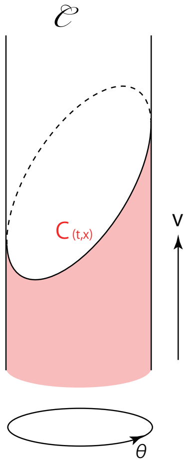
$$M' = \{\text{planar circles on } \mathscr{C}\}$$

Each planar circle is described as

$$C_{(t,x)} = \{(\theta, v) \mid v = t + x_1 \cos\theta + x_2 \sin\theta\}$$

using $(t, x) \in \mathbb{R} \times \mathbb{R}^2$.

Hence $M' \cong \mathbb{R}^3$



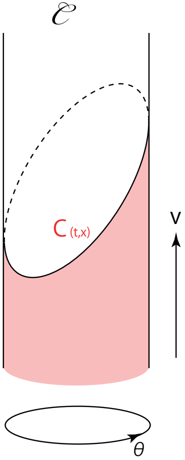
Planar circles

$$M' = \{\text{planar circles on } S^2\}$$
$$= \{C_{(t,x)} \mid (t,x) \in \mathbb{R} \times \mathbb{R}^2\}$$

Let us introduce an indefinite metric on M' by

$$g = -dt^2 + |dx|^2$$

(M', g) is identified with the flat Lorentz 3-space $\mathbb{R}^3_1$.



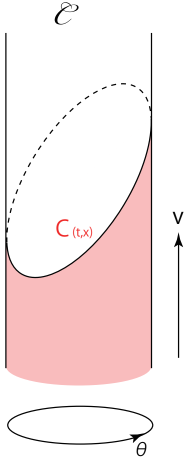
Planar circles

$$M' = \{\text{planar circles on } S^2\}$$
$$= \{C_{(t,x)} \mid (t,x) \in \mathbb{R} \times \mathbb{R}^2\}$$

Let us introduce an indefinite metric on M' by

$$g = -dt^2 + |dx|^2$$

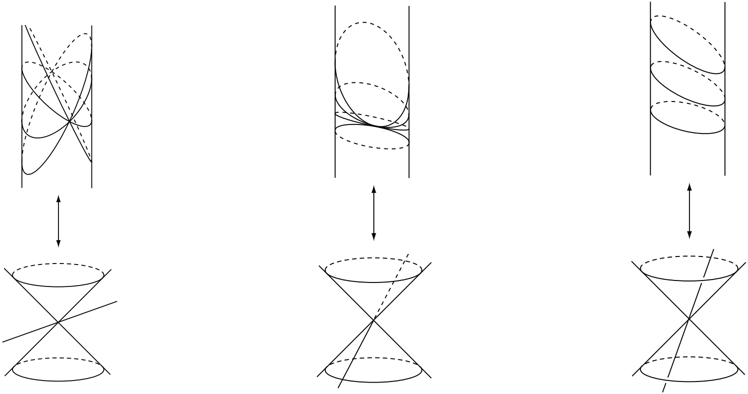
(M', g) is identified with the flat Lorentz 3-space $\mathbb{R}^3_1$.



Geodesics

There are three types of subfamilies of planar circles.

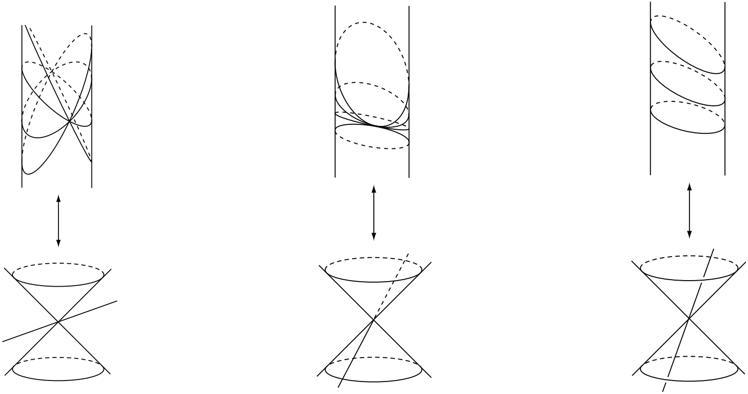
These families corresponde to geodesics on $\mathbb{R}^3_1$.



space-like geodesic　　null geodesic　　time-like geodesic

Geodesics

There are three types of subfamilies of planar circles.
These families corresponde to geodesics on $\mathbb{R}^3_1$.



space-like geodesic   null geodesic   time-like geodesic

Integral transform

For given function $h \in C^\infty(\mathscr{C})$, we define a function $R'h \in \mathbb{C}^\infty(\mathbb{R}^3_1)$ by

$$R'h(t,x) = \frac{1}{2\pi}\int_{C_{(t,x)}} h\, d\theta$$
$$= \frac{1}{2\pi}\int_0^{2\pi} h(\theta, t + x_1\cos\theta + x_2\sin\theta)\, d\theta.$$

Observation

For each function $h \in C^\infty(\mathscr{C})$, the function $V = R'h \in C^\infty(\mathbb{R}^3_1)$ satisfies the wave equation

$$\square V = *d*dV = -\frac{\partial^2 V}{\partial t^2} + \frac{\partial^2 V}{\partial x_1^2} + \frac{\partial^2 V}{\partial x_2^2} = 0.$$

Integral transform

For given function $h \in C^\infty(\mathscr{C})$, we define a function $R'h \in \mathbb{C}^\infty(\mathbb{R}^3_1)$ by

$$R'h(t,x) = \frac{1}{2\pi}\int_{C_{(t,x)}} h\,d\theta$$
$$= \frac{1}{2\pi}\int_0^{2\pi} h(\theta, t + x_1\cos\theta + x_2\sin\theta)\,d\theta.$$

Observation

For each function $h \in C^\infty(\mathscr{C})$, the function $V = R'h \in C^\infty(\mathbb{R}^3_1)$ satisfies the wave equation

$$\Box V = *d*dV = -\frac{\partial^2 V}{\partial t^2} + \frac{\partial^2 V}{\partial x_1^2} + \frac{\partial^2 V}{\partial x_2^2} = 0.$$

Integral transform

For given function $h \in C^\infty(\mathscr{C})$, we define a function $R'h \in \mathbb{C}^\infty(\mathbb{R}^3_1)$ by

$$\begin{aligned} R'h(t,x) &= \frac{1}{2\pi}\int_{C_{(t,x)}} h\,d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} h(\theta, t + x_1\cos\theta + x_2\sin\theta)\,d\theta. \end{aligned}$$

Observation

For each function $h \in C^\infty(\mathscr{C})$, the function $V = R'h \in C^\infty(\mathbb{R}^3_1)$ satisfies the wave equation

$$\Box V = *d*dV = -\frac{\partial^2 V}{\partial t^2} + \frac{\partial^2 V}{\partial x_1^2} + \frac{\partial^2 V}{\partial x_2^2} = 0.$$

3. Minitwistor theory

Minitwistor correspondence (Hitchin correspondence)

S : complex surface (called minitwistor space)
$Y \subset S$: nonsingular rational curve (= holomorphically embedded $\mathbb{CP}^1$)

- Y is called a minitwistor line if the self-intersection number $Y^2 = 2$.
- A small deformation of a minitwistor line Y in S is also a minitwistor line.
- Minitwistor lines are parametrized by a complex 3-manifold M.

Theorem (Hitchin '82)

M has a natural torsion-free complex Einstein-Weyl structure. Conversely, any complex 3-dimensional torsion-free Einstein-Weyl manifold is always obtained in this way locally.

- Einstein-Weyl structure is the conformal version of Einstein metric.
- In dimension 3, Einstein-Weyl condition is an integrable condition.

Minitwistor correspondence (Hitchin correspondence)

S : complex surface (called minitwistor space)
$Y \subset S$: nonsingular rational curve (= holomorphically embedded $\mathbb{CP}^1$)

- Y is called a minitwistor line if the self-intersection number $Y^2 = 2$.
- A small deformation of a minitwistor line Y in S is also a minitwistor line.
- Minitwistor lines are parametrized by a complex 3-manifold M.

Theorem (Hitchin '82)

M has a natural torsion-free complex Einstein-Weyl structure. Conversely, any complex 3-dimensional torsion-free Einstein-Weyl manifold is always obtained in this way locally.

- Einstein-Weyl structure is the conformal version of Einstein metric.
- In dimension 3, Einstein-Weyl condition is an integrable condition.

Minitwistor correspondence (Hitchin correspondence)

S : complex surface (called minitwistor space)

$Y \subset S$: nonsingular rational curve ($=$ holomorphically embedded $\mathbb{CP}^1$)

- Y is called a minitwistor line if the self-intersection number $Y^2 = 2$.
- A small deformation of a minitwistor line Y in S is also a minitwistor line.
- Minitwistor lines are parametrized by a complex 3-manifold M.

Theorem (Hitchin '82)

M has a natural torsion-free complex Einstein-Weyl structure. Conversely, any complex 3-dimensional torsion-free Einstein-Weyl manifold is always obtained in this way locally.

- Einstein-Weyl structure is the conformal version of Einstein metric.
- In dimension 3, Einstein-Weyl condition is an integrable condition.

Minitwistor correspondence (Hitchin correspondence)

S : complex surface (called minitwistor space)

$Y \subset S$: nonsingular rational curve ($=$ holomorphically embedded $\mathbb{CP}^1$)

- Y is called a minitwistor line if the self-intersection number $Y^2 = 2$.
- A small deformation of a minitwistor line Y in S is also a minitwistor line.
- Minitwistor lines are parametrized by a complex 3-manifold M.

Theorem (Hitchin '82)

M has a natural torsion-free complex Einstein-Weyl structure. Conversely, any complex 3-dimensional torsion-free Einstein-Weyl manifold is always obtained in this way locally.

- Einstein-Weyl structure is the conformal version of Einstein metric.
- In dimension 3, Einstein-Weyl condition is an integrable condition.

Minitwistor correspondence (Hitchin correspondence)

S : complex surface (called minitwistor space)
$Y \subset S$: nonsingular rational curve (= holomorphically embedded $\mathbb{CP}^1$)

- Y is called a minitwistor line if the self-intersection number $Y^2 = 2$.
- A small deformation of a minitwistor line Y in S is also a minitwistor line.
- Minitwistor lines are parametrized by a complex 3-manifold M.

Theorem (Hitchin '82)

M has a natural torsion-free complex Einstein-Weyl structure. Conversely, any complex 3-dimensional torsion-free Einstein-Weyl manifold is always obtained in this way locally.

- Einstein-Weyl structure is the conformal version of Einstein metric.
- In dimension 3, Einstein-Weyl condition is an integrable condition.

Minitwistor correspondence (Hitchin correspondence)

S : complex surface (called minitwistor space)
$Y \subset S$: nonsingular rational curve (= holomorphically embedded $\mathbb{CP}^1$)

- Y is called a minitwistor line if the self-intersection number $Y^2 = 2$.
- A small deformation of a minitwistor line Y in S is also a minitwistor line.
- Minitwistor lines are parametrized by a complex 3-manifold M.

Theorem (Hitchin '82)

M has a natural torsion-free complex Einstein-Weyl structure. Conversely, any complex 3-dimensional torsion-free Einstein-Weyl manifold is always obtained in this way locally.

- Einstein-Weyl structure is the conformal version of Einstein metric.
- In dimension 3, Einstein-Weyl condition is an integrable condition.

Minitwistor correspondence (Hitchin correspondence)

S : complex surface (called minitwistor space)
$Y \subset S$: nonsingular rational curve (= holomorphically embedded $\mathbb{CP}^1$)

- Y is called a minitwistor line if the self-intersection number $Y^2 = 2$.
- A small deformation of a minitwistor line Y in S is also a minitwistor line.
- Minitwistor lines are parametrized by a complex 3-manifold M.

Theorem (Hitchin '82)

M has a natural torsion-free complex Einstein-Weyl structure. Conversely, any complex 3-dimensional torsion-free Einstein-Weyl manifold is always obtained in this way locally.

- Einstein-Weyl structure is the conformal version of Einstein metric.
- In dimension 3, Einstein-Weyl condition is an integrable condition.

Minitwistor correspondence (Hitchin correspondence)

S : complex surface (called minitwistor space)

$Y \subset S$: nonsingular rational curve ($=$ holomorphically embedded $\mathbb{CP}^1$)

- Y is called a minitwistor line if the self-intersection number $Y^2 = 2$.
- A small deformation of a minitwistor line Y in S is also a minitwistor line.
- Minitwistor lines are parametrized by a complex 3-manifold M.

Theorem (Hitchin '82)

M has a natural torsion-free complex Einstein-Weyl structure. Conversely, any complex 3-dimensional torsion-free Einstein-Weyl manifold is always obtained in this way locally.

- Einstein-Weyl structure is the conformal version of Einstein metric.
- In dimension 3, Einstein-Weyl condition is an integrable condition.

Characterization of Einstein-Weyl 3-space

S : minitwistor space
$\{Y_x\}_{x \in M}$: family of minitwistor lines

The Einstein-Weyl structure $([g], \nabla)$ on M is determined so that

- for distinct two points $p, q \in S$,
  $\{x \in M \mid p, q \in Y_x\}$ is a geodesic,
- for a point $p \in S$ and a direction $0 \neq v \in T_pS$,
  $\{x \in M \mid p \in Y_x, v \in T_pY_x\}$ is a null geodesic.

Characterization of Einstein-Weyl 3-space

S : minitwistor space
$\{Y_x\}_{x \in M}$: family of minitwistor lines

The Einstein-Weyl structure $([g], \nabla)$ on M is determined so that

- for distinct two points $p, q \in S$,

  $\{x \in M \mid p, q \in Y_x\}$ is a geodesic,

- for a point $p \in S$ and a direction $0 \neq v \in T_pS$,

  $\{x \in M \mid p \in Y_x, v \in T_pY_x\}$ is a null geodesic.

Standard examples

There are two standard examples of minitwistor spaces:

$$\mathscr{S}_1 = \left\{ [z_0 : z_1 : z_2 : z_3] \in \mathbb{CP}^3 \;\middle|\; z_0 z_1 = z_2 z_3 \right\} \quad \text{nonsingular quadric}$$
$$\mathscr{S}_2 = \left\{ [z_0 : z_1 : z_2 : z_3] \in \mathbb{CP}^3 \;\middle|\; z_1^2 = z_0 z_2 \right\} \quad \text{degenerated quadric}$$

In both cases, the minitwistor lines are nonsingular plane sections.

The corresponding complex Einstein-Weyl spaces are

$$\mathscr{S}_1 \longrightarrow M_1 = \mathbb{CP}^3 \setminus Q \quad \text{with an EWstr. of const. curv.}$$
$$\mathscr{S}_2 \longrightarrow M_2 = \mathbb{C}^3 \quad \text{with a flat EWstr.}$$

where Q is a nonsingular quadric.

Standard examples

There are two standard examples of minitwistor spaces:

$$\mathscr{S}_1 = \left\{ [z_0 : z_1 : z_2 : z_3] \in \mathbb{CP}^3 \;\middle|\; z_0 z_1 = z_2 z_3 \right\} \quad \text{nonsingular quadric}$$
$$\mathscr{S}_2 = \left\{ [z_0 : z_1 : z_2 : z_3] \in \mathbb{CP}^3 \;\middle|\; z_1^2 = z_0 z_2 \right\} \quad \text{degenerated quadric}$$

In both cases, the minitwistor lines are nonsingular plane sections.

The corresponding complex Einstein-Weyl spaces are

$$\mathscr{S}_1 \longrightarrow M_1 = \mathbb{CP}^3 \setminus Q \quad \text{with an EWstr. of const. curv.}$$
$$\mathscr{S}_2 \longrightarrow M_2 = \mathbb{C}^3 \quad \text{with a flat EWstr.}$$

where Q is a nonsingular quadric.

Standard examples

There are two standard examples of minitwistor spaces:

$$\mathscr{S}_1 = \left\{ [z_0 : z_1 : z_2 : z_3] \in \mathbb{CP}^3 \;\middle|\; z_0 z_1 = z_2 z_3 \right\} \quad \text{nonsingular quadric}$$
$$\mathscr{S}_2 = \left\{ [z_0 : z_1 : z_2 : z_3] \in \mathbb{CP}^3 \;\middle|\; z_1^2 = z_0 z_2 \right\} \quad \text{degenerated quadric}$$

In both cases, the minitwistor lines are nonsingular plane sections.

The corresponding complex Einstein-Weyl spaces are

$$\mathscr{S}_1 \longrightarrow M_1 = \mathbb{CP}^3 \setminus Q \quad \text{with an EWstr. of const. curv.}$$
$$\mathscr{S}_2 \longrightarrow M_2 = \mathbb{C}^3 \quad \text{with a flat EWstr.}$$

where Q is a nonsingular quadric.

Real slices

Standard examples of real minitwistor correspondences are obtained as the real slices of $\mathscr{S}_i$.

Let us define anti-holomorphic involutions σ_i by

$$\sigma_1([z_0 : z_1 : z_2 : z_3]) = [\bar{z}_1 : \bar{z}_0 : \bar{z}_2 : \bar{z}_3] \quad \text{on} \quad \mathscr{S}_1,$$

$$\sigma_2([z_0 : z_1 : z_2 : z_3]) = [\bar{z}_2 : \bar{z}_1 : \bar{z}_0 : \bar{z}_3] \quad \text{on} \quad \mathscr{S}_2.$$

Then σ_i induces an involution on the Einstein-Weyl space M_i.
The fixed point set $M_i^{\sigma_i}$ corresponde with σ_i-invariant minitwistor lines.

The real 3-folds $M_i^{\sigma_i}$ have real indefinite Einstein-Weyl structures:

$M_1^{\sigma_1} \cong S_1^3/\mathbb{Z}_2$	$\mathbb{Z}_2$-quotient of de Sitter 3-space
$M_2^{\sigma_2} \cong \mathbb{R}_1^3$	Lorentz 3-space

Real slices

Standard examples of real minitwistor correspondences are obtained as the real slices of $\mathscr{S}_i$.
Let us define anti-holomorphic involutions σ_i by

$$\sigma_1([z_0 : z_1 : z_2 : z_3]) = [\bar{z}_1 : \bar{z}_0 : \bar{z}_2 : \bar{z}_3] \quad \text{on} \quad \mathscr{S}_1,$$

$$\sigma_2([z_0 : z_1 : z_2 : z_3]) = [\bar{z}_2 : \bar{z}_1 : \bar{z}_0 : \bar{z}_3] \quad \text{on} \quad \mathscr{S}_2.$$

Then σ_i induces an involution on the Einstein-Weyl space M_i.
The fixed point set $M_i^{\sigma_i}$ corresponde with σ_i-invariant minitwistor lines.

The real 3-folds $M_i^{\sigma_i}$ have real indefinite Einstein-Weyl structures:

$M_1^{\sigma_1} \cong S_1^3/\mathbb{Z}_2$ $\mathbb{Z}_2$-quotient of de Sitter 3-space

$M_2^{\sigma_2} \cong \mathbb{R}_1^3$ Lorentz 3-space

Real slices

Standard examples of real minitwistor correspondences are obtained as the real slices of $\mathscr{S}_i$.
Let us define anti-holomorphic involutions σ_i by

$$\sigma_1([z_0 : z_1 : z_2 : z_3]) = [\bar{z}_1 : \bar{z}_0 : \bar{z}_2 : \bar{z}_3] \quad \text{on} \quad \mathscr{S}_1,$$

$$\sigma_2([z_0 : z_1 : z_2 : z_3]) = [\bar{z}_2 : \bar{z}_1 : \bar{z}_0 : \bar{z}_3] \quad \text{on} \quad \mathscr{S}_2.$$

Then σ_i induces an involution on the Einstein-Weyl space M_i.
The fixed point set $M_i^{\sigma_i}$ corresponde with σ_i-invariant minitwistor lines.

The real 3-folds $M_i^{\sigma_i}$ have real indefinite Einstein-Weyl structures:

$M_1^{\sigma_1} \cong S_1^3/\mathbb{Z}_2$ $\quad$ $\mathbb{Z}_2$-quotient of de Sitter 3-space

$M_2^{\sigma_2} \cong \mathbb{R}_1^3$ $\quad$ Lorentz 3-space

Real slices

Standard examples of real minitwistor correspondences are obtained as the real slices of $\mathscr{S}_i$.

Let us define anti-holomorphic involutions σ_i by

$$\sigma_1([z_0 : z_1 : z_2 : z_3]) = [\bar{z}_1 : \bar{z}_0 : \bar{z}_2 : \bar{z}_3] \quad \text{on} \quad \mathscr{S}_1,$$

$$\sigma_2([z_0 : z_1 : z_2 : z_3]) = [\bar{z}_2 : \bar{z}_1 : \bar{z}_0 : \bar{z}_3] \quad \text{on} \quad \mathscr{S}_2.$$

Then σ_i induces an involution on the Einstein-Weyl space M_i.

The fixed point set $M_i^{\sigma_i}$ corresponde with σ_i-invariant minitwistor lines.

The real 3-folds $M_i^{\sigma_i}$ have real indefinite Einstein-Weyl structures:

$M_1^{\sigma_1} \cong S_1^3/\mathbb{Z}_2$	$\mathbb{Z}_2$-quotient of de Sitter 3-space
$M_2^{\sigma_2} \cong \mathbb{R}_1^3$	Lorentz 3-space

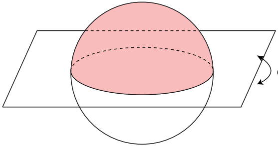
de Sitter 3-space as small circles

$\mathscr{S}_1$: nonsingular quadric equipped with an involution σ_1

$M_1 = \mathbb{CP}^3 \setminus Q$: the space of minitwistor lines on $\mathscr{S}_1$

$S^3_1/\mathbb{Z}_2 = M_1^{\sigma_1}$ $\mathbb{Z}_2$-quotient of de Sitter 3-space

$= \{\sigma_1\text{-invariant minitwistor line on } \mathscr{S}_1\}$



$S^3_1 = \{$holomorphic disks on $(\mathscr{S}_1, \mathscr{S}_1^{\sigma_1})\}$,

$= \{$oriented small circles on $S^2\}$

$\mathscr{S}_1^{\sigma_1} = \left\{[s : \bar{s} : 1 : |s|^2] \in \mathbb{CP}^3 \mid s \in \mathbb{C} \cup \{\infty\}\right\}$

$= S^2$: 2 sphere

de Sitter 3-space as small circles

$\mathscr{S}_1$: nonsingular quadric equipped with an involution σ_1

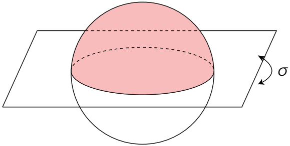
$M_1 = \mathbb{CP}^3 \setminus Q$: the space of minitwistor lines on $\mathscr{S}_1$

$S^3_1/\mathbb{Z}_2 = M_1^{\sigma_1}$ $\mathbb{Z}_2$-quotient of de Sitter 3-space

$= \{\sigma_1$-invariant minitwistor line on $\mathscr{S}_1\}$

$S^3_1 = \{$holomorphic disks on $(\mathscr{S}_1, \mathscr{S}_1^{\sigma_1})\}$,

$= \{$oriented small circles on $S^2\}$



$$\mathscr{S}_1^{\sigma_1} = \left\{ [s : \bar{s} : 1 : |s|^2] \in \mathbb{CP}^3 \mid s \in \mathbb{C} \cup \{\infty\} \right\}$$

$= S^2$: 2 sphere

Lorentz 3-space as planar circles

$\mathscr{S}_2$: degenerated quadric equipped with an involution σ_2

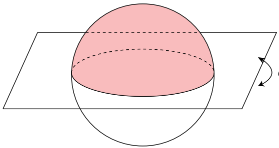
$M_2 = \mathbb{C}^3$: the space of minitwistor lines on $\mathscr{S}_2$

$\mathbb{R}^3_1 = M_2^{\sigma_2}$ Lorentz 3-space

$= \{\sigma_2\text{-invariant minitwistor line on } \mathscr{S}_2\}$

$= \{$holomorphic disks on $(\mathscr{S}_2, \mathscr{S}_2^{\sigma_2})\}$,

$= \{$planner circles on $\mathscr{C}\}$



$$\mathscr{S}_2^{\sigma_2} = \left\{ [e^{-i\theta} : 1 : e^{i\theta} : v] \in \mathbb{CP}^3 \;\middle|\; \theta \in S^1, v \in \mathbb{R} \cup \{\infty\} \right\}$$

$= \mathscr{C} \cup \{\infty\}$: 1 point compactification of the cylinder $\mathscr{C}$

Lorentz 3-space as planar circles

$\mathscr{S}_2$: degenerated quadric equipped with an involution σ_2

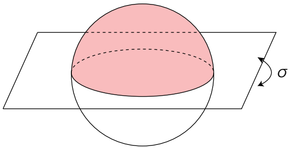
$M_2 = \mathbb{C}^3$: the space of minitwistor lines on $\mathscr{S}_2$

$\mathbb{R}^3_1 = M_2^{\sigma_2}$ Lorentz 3-space

$= \{\sigma_2\text{-invariant minitwistor line on } \mathscr{S}_2\}$

$= \{$holomorphic disks on $(\mathscr{S}_2, \mathscr{S}_2^{\sigma_2})\}$,

$= \{$planner circles on $\mathscr{C}\}$



$$\mathscr{S}_2^{\sigma_2} = \left\{ [e^{-i\theta} : 1 : e^{i\theta} : v] \in \mathbb{CP}^3 \;\middle|\; \theta \in S^1, v \in \mathbb{R} \cup \{\infty\} \right\}$$

$= \mathscr{C} \cup \{\infty\}$: 1 point compactification of the cylinder $\mathscr{C}$

4. LeBrun-Mason twistor theory

general theory

LM correspondence for self-dual 4-fold

Theorem (LeBrun-Mason '07)

There is a natural one-to-one correspondence between

- self-dual Zollfrei conformal structures $[g]$ on $S^2 \times S^2$ of signature $(- - ++)$, and
- pairs $(\mathbb{CP}^3, P)$ where P is an embedded $\mathbb{RP}^3$,

on the neighborhoods of the standard objects.

$(M, [g])$ is Zollfrei $\Longleftrightarrow$ every maximal null geodesic is closed.

- standard SD metric on $S^2 \times S^2$ is the product $g_0 = (-g_{S^2}) \oplus g_{S^2}$,
- standard embedding $\mathbb{RP}^3 \subset \mathbb{CP}^3$ is the fixed point set of the complex conjugation.

LM correspondence for self-dual 4-fold

Theorem (LeBrun-Mason '07)

There is a natural one-to-one correspondence between

- self-dual Zollfrei conformal structures $[g]$ on $S^2 \times S^2$ of signature $(--++)$, and
- pairs $(\mathbb{CP}^3, P)$ where P is an embedded $\mathbb{RP}^3$,

on the neighborhoods of the standard objects.

$(M, [g])$ is Zollfrei $\iff$ every maximal null geodesic is closed.

- standard SD metric on $S^2 \times S^2$ is the product $g_0 = (-g_{S^2}) \oplus g_{S^2}$,
- standard embedding $\mathbb{RP}^3 \subset \mathbb{CP}^3$ is the fixed point set of the complex conjugation.

LM correspondence for self-dual 4-fold

Theorem (LeBrun-Mason '07)

There is a natural one-to-one correspondence between

- self-dual Zollfrei conformal structures $[g]$ on $S^2 \times S^2$ of signature $(--++)$, and
- pairs $(\mathbb{CP}^3, P)$ where P is an embedded $\mathbb{RP}^3$,

on the neighborhoods of the standard objects.

$(M, [g])$ is Zollfrei $\iff$ every maximal null geodesic is closed.

- standard SD metric on $S^2 \times S^2$ is the product $g_0 = (-g_{S^2}) \oplus g_{S^2}$,
- standard embedding $\mathbb{RP}^3 \subset \mathbb{CP}^3$ is the fixed point set of the complex conjugation.

LM correspondence (Rough sketch)

$(\mathbb{CP}^3, P) \Rightarrow (S^2 \times S^2, [g])$ SD

P : small deformation of $\mathbb{RP}^3$ in $\mathbb{CP}^3$,

$\Rightarrow$ There exist $S^2 \times S^2$-family of holomorphic disks in $\mathbb{CP}^3$ with boundaries lying on P representing the generator of $H_2(\mathbb{CP}^3, P; \mathbb{Z}) \simeq \mathbb{Z}$.

$\Rightarrow$ The self-dual conformal structure $[g]$ on $S^2 \times S^2$ is recovered so that

$$\mathfrak{S}_q = \{x \in S^2 \times S^2 \mid q \in \partial D_x\} \qquad (D_x : \text{holomorphic disk})$$

is a null-surface for each $q \in P$.

$(S^2 \times S^2, [g])$ SD $\Rightarrow (\mathbb{CP}^3, P)$: omitted (Key is the Zollfrei condition) □

LM correspondence (Rough sketch)

$(\mathbb{CP}^3, P) \Rightarrow (S^2 \times S^2, [g])$ SD

P : small deformation of $\mathbb{RP}^3$ in $\mathbb{CP}^3$,

$\Rightarrow$ There exist $S^2 \times S^2$-family of holomorphic disks in $\mathbb{CP}^3$ with boundaries lying on P representing the generator of $H_2(\mathbb{CP}^3, P; \mathbb{Z}) \simeq \mathbb{Z}$.

$\Rightarrow$ The self-dual conformal structure $[g]$ on $S^2 \times S^2$ is recovered so that

$$\mathfrak{S}_q = \{x \in S^2 \times S^2 \mid q \in \partial D_x\} \qquad (D_x : \text{holomorphic disk})$$

is a null-surface for each $q \in P$.

$(S^2 \times S^2, [g])$ SD $\Rightarrow (\mathbb{CP}^3, P)$: omitted (Key is the Zollfrei condition) □

LM correspondence (Rough sketch)

$(\mathbb{CP}^3, P) \Rightarrow (S^2 \times S^2, [g])$ SD

P : small deformation of $\mathbb{RP}^3$ in $\mathbb{CP}^3$,

$\Rightarrow$ There exist $S^2 \times S^2$-family of holomorphic disks in $\mathbb{CP}^3$ with boundaries lying on P representing the generator of $H_2(\mathbb{CP}^3, P; \mathbb{Z}) \simeq \mathbb{Z}$.

$\Rightarrow$ The self-dual conformal structure $[g]$ on $S^2 \times S^2$ is recovered so that

$$\mathfrak{S}_q = \{x \in S^2 \times S^2 \mid q \in \partial D_x\} \qquad (D_x : \text{holomorphic disk})$$

is a null-surface for each $q \in P$.

$(S^2 \times S^2, [g])$ SD $\Rightarrow (\mathbb{CP}^3, P)$: omitted (Key is the Zollfrei condition) □

LM correspondence (Rough sketch)

$(\mathbb{CP}^3, P) \Rightarrow (S^2 \times S^2, [g])$ SD

P : small deformation of $\mathbb{RP}^3$ in $\mathbb{CP}^3$,

$\Rightarrow$ There exist $S^2 \times S^2$-family of holomorphic disks in $\mathbb{CP}^3$ with boundaries lying on P representing the generator of $H_2(\mathbb{CP}^3, P; \mathbb{Z}) \simeq \mathbb{Z}$.

$\Rightarrow$ The self-dual conformal structure $[g]$ on $S^2 \times S^2$ is recovered so that

$$\mathfrak{S}_q = \{x \in S^2 \times S^2 \mid q \in \partial D_x\} \qquad (D_x : \text{holomorphic disk})$$

is a null-surface for each $q \in P$.

$(S^2 \times S^2, [g])$ SD $\Rightarrow (\mathbb{CP}^3, P)$: omitted (Key is the Zollfrei condition) □

4. LeBrun-Mason twistor theory

S^1-invariant case

Standard model

$$\mathbb{RP}^3 = \left\{ [z_0 : z_1 : z_2 : z_3] \in \mathbb{CP}^3 \;\middle|\; z_3 = \bar{z}_0, z_2 = \bar{z}_1 \right\}$$

We notice to a $(\mathbb{C}^*, U(1))$-action on $(\mathbb{CP}^3, \mathbb{RP}^3)$ defined by

$$\mu \cdot [z_0 : z_1 : z_2 : z_3] = [\mu^{\frac{1}{2}} z_0 : \mu^{\frac{1}{2}} z_1 : \mu^{-\frac{1}{2}} z_2 : \mu^{-\frac{1}{2}} z_3] \qquad \mu \in \mathbb{C}^*.$$

Its free quotient is the minitwistor space $(\mathscr{S}_1, S^2)$.

Correspondingly, an S^1-action on the standard SD Zollfrei space $S^2 \times S^2$ is induced, and its free quotient is the de Sitter 3-space S^3_1.

Standard model

$$\mathbb{RP}^3 = \left\{ [z_0 : z_1 : z_2 : z_3] \in \mathbb{CP}^3 \;\middle|\; z_3 = \bar{z}_0, z_2 = \bar{z}_1 \right\}$$

We notice to a $(\mathbb{C}^*, U(1))$-action on $(\mathbb{CP}^3, \mathbb{RP}^3)$ defined by

$$\mu \cdot [z_0 : z_1 : z_2 : z_3] = [\mu^{\frac{1}{2}} z_0 : \mu^{\frac{1}{2}} z_1 : \mu^{-\frac{1}{2}} z_2 : \mu^{-\frac{1}{2}} z_3] \qquad \mu \in \mathbb{C}^*.$$

Its free quotient is the minitwistor space $(\mathscr{S}_1, S^2)$.

Correspondingly, an S^1-action on the standard SD Zollfrei space $S^2 \times S^2$ is induced, and its free quotient is the de Sitter 3-space S^3_1.

Standard model

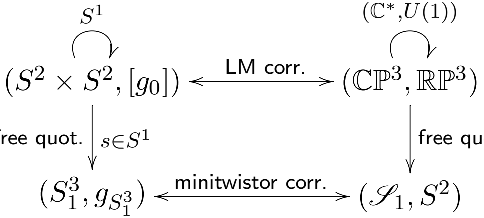
$$\mathbb{RP}^3 = \left\{ [z_0 : z_1 : z_2 : z_3] \in \mathbb{CP}^3 \;\middle|\; z_3 = \bar{z}_0, z_2 = \bar{z}_1 \right\}$$

We notice to a $(\mathbb{C}^*, U(1))$-action on $(\mathbb{CP}^3, \mathbb{RP}^3)$ defined by

$$\mu \cdot [z_0 : z_1 : z_2 : z_3] = [\mu^{\frac{1}{2}} z_0 : \mu^{\frac{1}{2}} z_1 : \mu^{-\frac{1}{2}} z_2 : \mu^{-\frac{1}{2}} z_3] \qquad \mu \in \mathbb{C}^*.$$

Its free quotient is the minitwistor space $(\mathscr{S}_1, S^2)$.

de Sitter sp



quadric

Correspondingly, an S^1-action on the standard SD Zollfrei space $S^2 \times S^2$ is induced, and its free quotient is the de Sitter 3-space S^3_1.

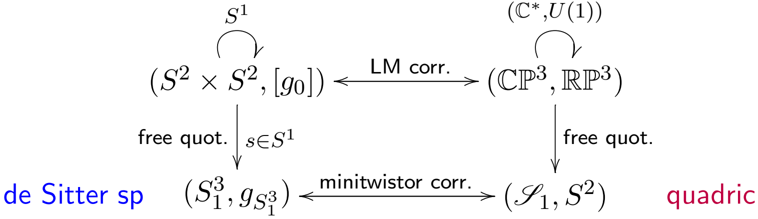
Standard model

$$\mathbb{RP}^3 = \left\{ [z_0 : z_1 : z_2 : z_3] \in \mathbb{CP}^3 \;\middle|\; z_3 = \bar{z}_0, z_2 = \bar{z}_1 \right\}$$

We notice to a $(\mathbb{C}^*, U(1))$-action on $(\mathbb{CP}^3, \mathbb{RP}^3)$ defined by

$$\mu \cdot [z_0 : z_1 : z_2 : z_3] = [\mu^{\frac{1}{2}} z_0 : \mu^{\frac{1}{2}} z_1 : \mu^{-\frac{1}{2}} z_2 : \mu^{-\frac{1}{2}} z_3] \qquad \mu \in \mathbb{C}^*.$$

Its free quotient is the minitwistor space $(\mathscr{S}_1, S^2)$.



Correspondingly, an S^1-action on the standard SD Zollfrei space $S^2 \times S^2$ is induced, and its free quotient is the de Sitter 3-space S^3_1.

S^1-invariant deformation

The $(\mathbb{C}^*, U(1))$-invariant deformations of $(\mathbb{CP}^3, \mathbb{RP}^3)$ fixing the quotient is written as

$$P_h = \left\{ [z_i] \in \mathbb{CP}^3 \;\middle|\; z_3 = e^{h(z_1/z_0)} \bar{z}_0,\ z_2 = e^{h(z_1/z_0)} \bar{z}_1 \right\}.$$

by using the parameter h is contained in the function space

$$C^\infty_*(S^2) := \left\{ h \in C^\infty(S^2) \;\middle|\; \int_{S^2} h\, dS^2 = 0 \right\}$$

S^1-invariant deformation

The $(\mathbb{C}^*, U(1))$-invariant deformations of $(\mathbb{CP}^3, \mathbb{RP}^3)$ fixing the quotient is written as

$$P_h = \left\{ [z_i] \in \mathbb{CP}^3 \;\middle|\; z_3 = e^{h(z_1/z_0)}\bar{z}_0,\;\; z_2 = e^{h(z_1/z_0)}\bar{z}_1 \right\}.$$

by using the parameter h is contained in the function space

$$C^\infty_*(S^2) := \left\{ h \in C^\infty(S^2) \;\middle|\; \int_{S^2} h\, dS^2 = 0 \right\}$$

$$\begin{array}{ccccc}
 & \overset{S^1}{\circlearrowright} & & \overset{(\mathbb{C}^*, U(1))}{\circlearrowright} & \\
 & (S^2 \times S^2, [g_{(V,A)}]) & \xleftrightarrow{\text{LM corr.}} & (\mathbb{CP}^3, P_h) & \\
 & \text{free quot.} \big\downarrow s \in S^1 & & \big\downarrow \text{free quot.} & \\
\text{de Sitter sp} & (S^3_1, g_{S^3_1}) & \xleftrightarrow{\text{minitwistor corr.}} & (\mathscr{S}_1, S^2) & \text{quadric}
\end{array}$$

S^1-invariant deformation

The corresponding SD Zollfrei metric on $S^2 \times S^2$ is written as

$$g_{(V,A)} \sim -V^{-1}(ds + A)^2 + V g_{S^3_1} \qquad \text{conformally}$$

where $(V, A) \in C^\infty(S^3_1) \times \Omega^1(S^3_1)$ is defined by

$$V = 1 - Q\Delta_{S^2} h, \qquad A = -\check{*}\check{d}Rh$$

by using the integral transforms R and Q.

Here Δ_{S^2} is the Laplacian on S^2, $\check{*}$ and $\check{d}$ are the Hodge-$*$ and exterior differential on the S^2-direction of $S^3_1 \simeq \mathbb{R} \times S^2$.

S^1-invariant deformation

The corresponding SD Zollfrei metric on $S^2 \times S^2$ is written as

$$g_{(V,A)} \sim -V^{-1}(ds + A)^2 + V g_{S^3_1} \qquad \text{conformally}$$

where $(V, A) \in C^\infty(S^3_1) \times \Omega^1(S^3_1)$ is defined by

$$V = 1 - Q\Delta_{S^2} h, \qquad A = -\check{*}\check{d}Rh$$

by using the integral transforms R and Q.

Here Δ_{S^2} is the Laplacian on S^2, $\check{*}$ and $\check{d}$ are the Hodge-$*$ and exterior differential on the S^2-direction of $S^3_1 \simeq \mathbb{R} \times S^2$.

Monopole equation & Wave equation

$$g_{(V,A)} \sim -V^{-1}(ds+A)^2 + V g_{S^3_1} \qquad \text{conformally}$$

The self-duality of $g_{(V,A)}$ is equivalent to the monopole equation :

$$*dV = dA.$$

Hence we obtain solutions of monopole equation written as

$$V = 1 - Q\Delta_{S^2}h, \qquad A = -\check{*}\check{d}Rh.$$

In particular, V satisfies the wave equation : $\Box V = *d*dV = 0.$

$$\Longrightarrow \qquad \Box Q\tilde{h} = 0 \qquad \text{for} \quad \tilde{h} \in C^\infty_*(S^2) = \Delta_{S^2}C^\infty(S^2).$$

Monopole equation & Wave equation

$$g_{(V,A)} \sim -V^{-1}(ds+A)^2 + V g_{S^3_1} \qquad \text{conformally}$$

The self-duality of $g_{(V,A)}$ is equivalent to the monopole equation :

$$*dV = dA.$$

Hence we obtain solutions of monopole equation written as

$$V = 1 - Q\Delta_{S^2} h, \qquad A = -\check{*}\check{d}Rh.$$

In particular, V satisfies the wave equation : $\Box V = *d*dV = 0.$

$$\Longrightarrow \qquad \Box Q\tilde{h} = 0 \qquad \text{for} \quad \tilde{h} \in C^\infty_*(S^2) = \Delta_{S^2} C^\infty(S^2).$$

Monopole equation & Wave equation

$$g_{(V,A)} \sim -V^{-1}(ds+A)^2 + V g_{S^3_1} \qquad \text{conformally}$$

The self-duality of $g_{(V,A)}$ is equivalent to the monopole equation :

$$*dV = dA.$$

Hence we obtain solutions of monopole equation written as

$$V = 1 - Q\Delta_{S^2}h, \qquad A = -\check{*}\check{d}Rh.$$

In particular, V satisfies the wave equation : $\Box V = *d*dV = 0$.

$$\Longrightarrow \qquad \Box Q\tilde{h} = 0 \qquad \text{for} \quad \tilde{h} \in C^\infty_*(S^2) = \Delta_{S^2}C^\infty(S^2).$$

Monopole equation & Wave equation

$$g_{(V,A)} \;\sim\; -V^{-1}(ds+A)^2 + V g_{S^3_1} \qquad \text{conformally}$$

The self-duality of $g_{(V,A)}$ is equivalent to the monopole equation :

$$*dV = dA.$$

Hence we obtain solutions of monopole equation written as

$$V = 1 - Q\Delta_{S^2}h, \qquad A = -\check{*}\check{d}Rh.$$

In particular, V satisfies the wave equation : $\Box V = *d*dV = 0$.

$$\Longrightarrow \qquad \Box Q\tilde{h} = 0 \qquad \text{for} \quad \tilde{h} \in C^\infty_*(S^2) = \Delta_{S^2}C^\infty(S^2).$$

Monopole equation & Wave equation

Theorem (N. '09) revisit

For each function $h \in C^\infty_*(S^2)$, the function $V := Qh \in C^\infty(S^3_1)$ satisfies

$$\text{(i)} \quad \Box V = 0, \qquad \text{(ii)} \quad \lim_{t\to\infty} V(t,y) = \lim_{t\to\infty} V_t(t,y) = 0.$$

Conversely, if $V \in C^\infty(M)$ satisfies (i) and (ii), then there exists unique $h \in C^\infty_*(S^2)$ such that $V = Qh$.

Tod-Kamada metric

$$g_{(V,A)} \sim -V^{-1}(ds+A)^2 + V g_{S^3_1} \qquad \text{conformally}$$

$$(V,A) : \text{monopole} \quad \text{i.e.} \quad * dV = dA$$

This indefinite self-dual meteric $g_{(V,A)}$ on $S^2 \times S^2$ is constructed by K. P. Tod ('95) and is also rediscovered by H. Kamada ('05) in more general investigation.

Theorem (N. 2009)

- The twistor correspondence for Tod-Kamada metric is explicitly written down.
- Tod-Kamada metric is Zollfrei.
- Unless $Q\Delta_{S^2}h < 1$, the twistor correspondence fails.

Tod-Kamada metric

$$g_{(V,A)} \sim -V^{-1}(ds + A)^2 + V g_{S^3_1} \qquad \text{conformally}$$

$$(V, A) : \text{monopole} \quad \text{i.e.} \quad * dV = dA$$

This indefinite self-dual meteric $g_{(V,A)}$ on $S^2 \times S^2$ is constructed by K. P. Tod ('95) and is also rediscovered by H. Kamada ('05) in more general investigation.

Theorem (N. 2009)

- The twistor correspondence for Tod-Kamada metric is explicitly written down.
- Tod-Kamada metric is Zollfrei.
- Unless $Q\Delta_{S^2} h < 1$, the twistor correspondence fails.

Tod-Kamada metric

$$g_{(V,A)} \sim -V^{-1}(ds + A)^2 + V g_{S^3_1} \qquad \text{conformally}$$

$$(V, A) : \text{monopole} \quad \text{i.e.} \quad * dV = dA$$

This indefinite self-dual meteric $g_{(V,A)}$ on $S^2 \times S^2$ is constructed by K. P. Tod ('95) and is also rediscovered by H. Kamada ('05) in more general investigation.

Theorem (N. 2009)

- The twistor correspondence for Tod-Kamada metric is explicitly written down.
- Tod-Kamada metric is Zollfrei.
- Unless $Q\Delta_{S^2} h < 1$, the twistor correspondence fails.

Tod-Kamada metric

$$g_{(V,A)} \sim -V^{-1}(ds + A)^2 + V g_{S^3_1} \qquad \text{conformally}$$

$$(V, A) : \text{monopole} \quad \text{i.e.} \quad * dV = dA$$

This indefinite self-dual meteric $g_{(V,A)}$ on $S^2 \times S^2$ is constructed by K. P. Tod ('95) and is also rediscovered by H. Kamada ('05) in more general investigation.

Theorem (N. 2009)

- The twistor correspondence for Tod-Kamada metric is explicitly written down.
- Tod-Kamada metric is Zollfrei.
- Unless $Q\Delta_{S^2} h < 1$, the twistor correspondence fails.

4. LeBrun-Mason twistor theory

$\mathbb{R}$-invariant case

Standard model

$$\mathbb{RP}^3 = \left\{ [z_0 : z_1 : z_2 : z_3] \in \mathbb{CP}^3 \;\middle|\; z_3 = \bar{z}_0, z_2 = \bar{z}_1 \right\}$$

We notice to the $(\mathbb{C}, \mathbb{R})$-action on $(\mathbb{CP}^3, \mathbb{RP}^3)$

$$\mu \cdot [z_0 : z_1 : z_2 : z_3] = [z_0 : z_1 : z_2 + i\mu z_0 : z_3 + i\mu z_1] \qquad \mu \in \mathbb{C}.$$

Its free quotient is the minitwistor space $(\mathscr{S}_2, \mathscr{C})$.

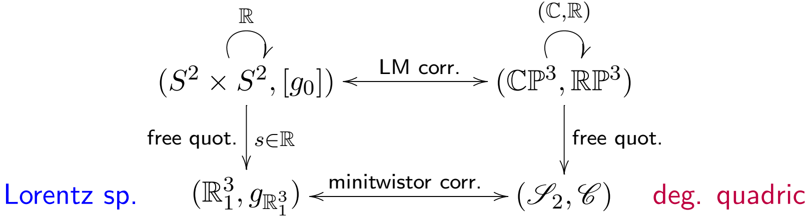
Standard model

$$\mathbb{RP}^3 = \left\{ [z_0 : z_1 : z_2 : z_3] \in \mathbb{CP}^3 \;\middle|\; z_3 = \bar{z}_0, z_2 = \bar{z}_1 \right\}$$

We notice to the $(\mathbb{C}, \mathbb{R})$-action on $(\mathbb{CP}^3, \mathbb{RP}^3)$

$$\mu \cdot [z_0 : z_1 : z_2 : z_3] = [z_0 : z_1 : z_2 + i\mu z_0 : z_3 + i\mu z_1] \qquad \mu \in \mathbb{C}.$$

Its free quotient is the minitwistor space $(\mathscr{S}_2, \mathscr{C})$.



$\mathbb{R}$-invariant deformation

The $\mathbb{R}$-invariant deformations of $(\mathbb{CP}^3, \mathbb{RP}^3)$ fixing the quotient is parametrized by functions $h \in C^\infty(\mathscr{C})$, and the corresponding self-dual metric on $\mathbb{R}^4 \subset S^2 \times S^2$ (one of the two free parts of $\mathbb{R}$-action) is explicitly written as

$$g_{(V,A)} = -V^{-1}(ds + A)^2 + Vg$$

$$V = 1 - \partial_t R'h, \qquad A = -\check{*}\check{d}R'h.$$

Here (V, A) gives a solution of the monopole equation $*dV = dA$.

$$\Longrightarrow \qquad \Box R'h = 0 \qquad \text{for} \qquad h \in C^\infty(\mathscr{C}).$$

$\mathbb{R}$-invariant deformation

The $\mathbb{R}$-invariant deformations of $(\mathbb{CP}^3, \mathbb{RP}^3)$ fixing the quotient is parametrized by functions $h \in C^\infty(\mathscr{C})$, and the corresponding self-dual metric on $\mathbb{R}^4 \subset S^2 \times S^2$ (one of the two free parts of $\mathbb{R}$-action) is explicitly written as

$$g_{(V,A)} = -V^{-1}(ds + A)^2 + Vg$$

$$V = 1 - \partial_t R'h, \qquad A = -\check{*}\check{d}R'h.$$

Here (V, A) gives a solution of the monopole equation $*dV = dA$.

$$\Longrightarrow \qquad \Box R'h = 0 \qquad \text{for} \quad h \in C^\infty(\mathscr{C}).$$

$\mathbb{R}$-invariant deformation

The $\mathbb{R}$-invariant deformations of $(\mathbb{CP}^3, \mathbb{RP}^3)$ fixing the quotient is parametrized by functions $h \in C^\infty(\mathscr{C})$, and the corresponding self-dual metric on $\mathbb{R}^4 \subset S^2 \times S^2$ (one of the two free parts of $\mathbb{R}$-action) is explicitly written as

$$g_{(V,A)} = -V^{-1}(ds + A)^2 + Vg$$

$$V = 1 - \partial_t R'h, \qquad A = -\check{*}\check{d}R'h.$$

Here (V, A) gives a solution of the monopole equation $*dV = dA$.

$$\Longrightarrow \qquad \Box R'h = 0 \qquad \text{for} \qquad h \in C^\infty(\mathscr{C}).$$

Problems

Problems

- Deformed version of minitwistor correspondence,
- Twistor correspondence for connections on vector bundles,
- Higher dimensional version,
- Holomorphic disks $\longrightarrow$ Riemann surfaces with boundary,
- Correspondences for low regularity. Geometry of “shock wave”.

Thank you!

Problems

- Deformed version of minitwistor correspondence,
- Twistor correspondence for connections on vector bundles,
- Higher dimensional version,
- Holomorphic disks → Riemann surfaces with boundary,
- Correspondences for low regularity. Geometry of "shock wave".

Thank you!

Problems

- Deformed version of minitwistor correspondence,
- Twistor correspondence for connections on vector bundles,
- Higher dimensional version,
- Holomorphic disks $\longrightarrow$ Riemann surfaces with boundary,
- Correspondences for low regularity. Geometry of "shock wave".

Thank you!

Problems

- Deformed version of minitwistor correspondence,
- Twistor correspondence for connections on vector bundles,
- Higher dimensional version,
- Holomorphic disks $\rightarrow$ Riemann surfaces with boundary,
- Correspondences for low regularity. Geometry of "shock wave".

Thank you!

Problems

- Deformed version of minitwistor correspondence,
- Twistor correspondence for connections on vector bundles,
- Higher dimensional version,
- Holomorphic disks $\longrightarrow$ Riemann surfaces with boundary,
- Correspondences for low regularity. Geometry of "shock wave".

Thank you!

Problems

- Deformed version of minitwistor correspondence,
- Twistor correspondence for connections on vector bundles,
- Higher dimensional version,
- Holomorphic disks $\longrightarrow$ Riemann surfaces with boundary,
- Correspondences for low regularity. Geometry of "shock wave".

Thank you!