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**Floer homology and Hamiltonian volume
minimizing properties of real forms of complex
hyperquadric**

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What is the Hamiltonian volume minimization problem?

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Definition (Y.-G. Oh, 1990).

(M, ω, J) : Kähler manifold

$L \subset M$: closed Lagrangian submanifold

• L : **Hamiltonian volume minimizing**

$$\stackrel{\text{def}}{\iff} \text{vol}(\phi L) \geq \text{vol}(L)$$

for any Hamiltonian diffeomorphism

$\phi \in \text{Ham}(M, \omega)$.

Example.

$$M = (\mathbb{C}, dx \wedge dy, J_0)$$

$$L = S^1 : \text{round circle}$$

$$\phi \in \text{Ham}(\mathbb{C}, dx \wedge dy)$$

$\iff \phi : \text{volume preserving diffeomorphism}$

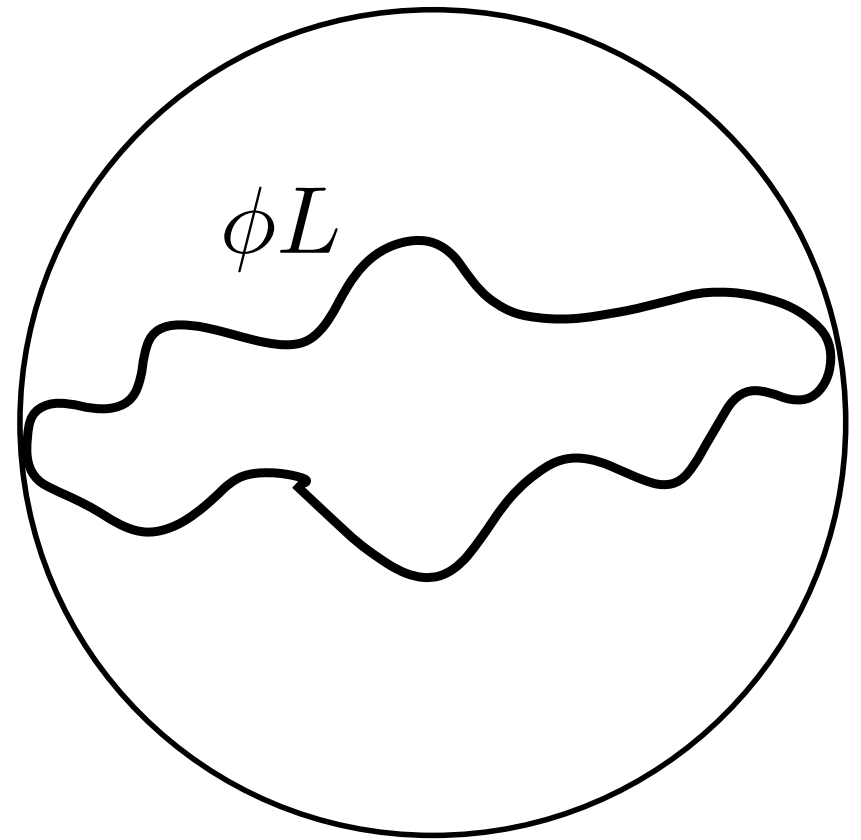
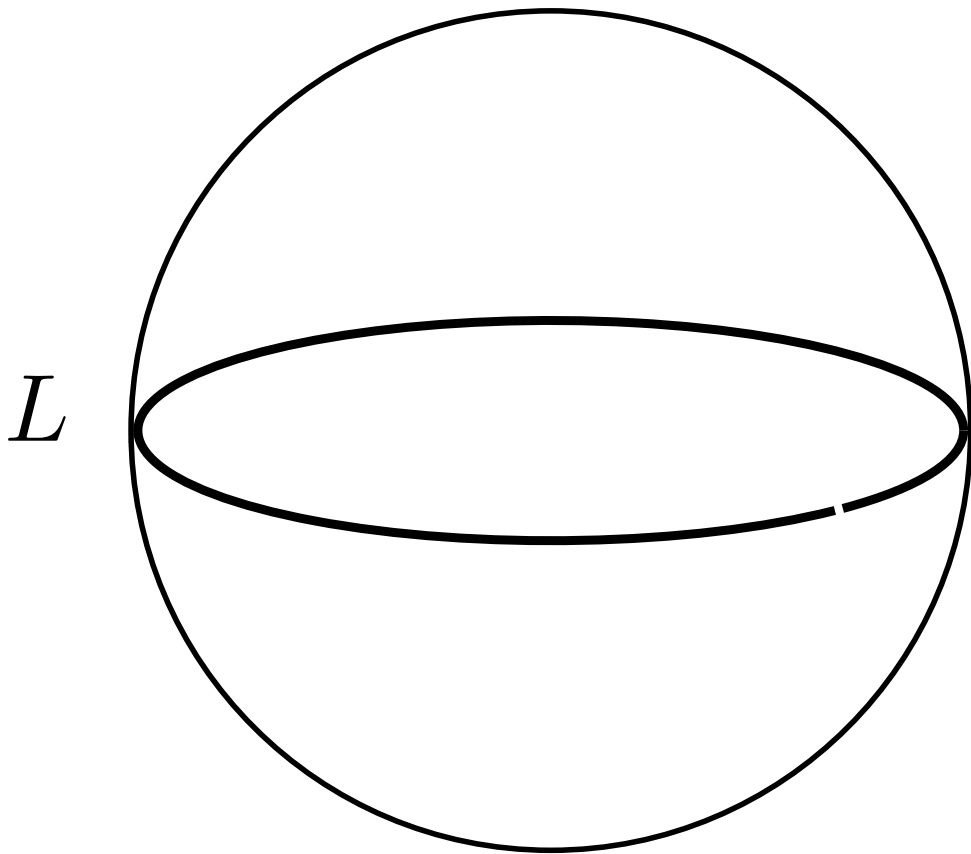
Then for any ϕ ,

$$\text{vol}(\phi S^1) \geq \text{vol}(S^1)$$

“Isoperimetric inequality”

Example.

A great circle $L = \mathbb{R}P^1$ in $(\mathbb{C}P^1, \omega_{FS}, J_0)$ is Hamiltonian volume minimizing.



Theorem 1 (Kleiner-Oh, 1990)

$\mathbb{R}P^n$ in $\mathbb{C}P^n$ is Hamiltonian volume minimizing.

Conjecture (Oh, 1990).

M : Kähler-Einstein manifold with $\text{Ric} > 0$

σ : involutive anti-holomorphic isometry

$L = \text{Fix } \sigma$ “real form”

(totally geodesic Lagrangian submfd)

Assume that L : Einstein with $\text{Ric} > 0$

$\implies L$ is Hamiltonian volume minimizing.

Theorem 2 (Sakai-Tasaki-I., 2011)

$M = Q_n(\mathbb{C})$: n -dim complex hyperquadric

$L = S^n$: totally geodesic Lagrangian sphere

$\implies L$ is Hamiltonian volume minimizing.

Note that

Theorem 3 (Gluck-Morgan-Ziller, 1989)

If n is **even**, then $S^n \subset Q_n(\mathbb{C})$ is homologically volume minimizing.

Contents

1. $Q_n(\mathbb{C})$ and its real forms
2. Floer homology for monotone Lagrangian submanifolds
3. Antipodal sets and Floer homology
4. Proof of Theorem 2
5. A generalization to contact isotopies

1 $Q_n(\mathbb{C})$ and its real forms

- $Q_n(\mathbb{C}) = \{[z] \in \mathbb{C}P^{n+1} \mid z_0^2 + \cdots + z_{n+1}^2 = 0\}$

Hermitian symmetric space of cpt type

Kähler-Einstein manifold with $\text{Ric} > 0$

$$\pi_1(Q_n(\mathbb{C})) = 0$$

$$\text{Ham}(Q_n(\mathbb{C}), \omega_0) = \text{Symp}_0(Q_n(\mathbb{C}), \omega_0)$$

- $Q_n(\mathbb{C}) \cong SO(n+2)/SO(2) \times SO(n)$

- If $z_i = x_i + \sqrt{-1}y_i$, then

$$0 = \sum_{i=0}^{n+1} (x_i^2 - y_i^2) + 2\sqrt{-1} \sum_{i=0}^{n+1} x_i y_i$$
- $x = (x_0, \dots, x_{n+1}), y = (y_0, \dots, y_{n+1}) \in \mathbb{R}^{n+2}$, then $|x| = |y|$, $x \cdot y = 0$.
- $\{x, y\}$ defines an oriented two-plane in \mathbb{R}^{n+2} .

$$Q_n(\mathbb{C}) \cong \tilde{Gr}_2(\mathbb{R}^{n+2}) \subset \wedge^2 \mathbb{R}^{n+2}$$

$$[z] \mapsto x \wedge y$$

real forms of $Q_n(\mathbb{C})$

- $S^{k,n-k} = \{[x] \in \mathbb{R}P^{n+1} \mid x_0^2 + \cdots + x_k^2 - (x_{k+1}^2 + \cdots + z_{n+1}^2) = 0\} \cong (S^k \times S^{n-k})/\mathbb{Z}_2$

- $u_1, u_2, e_1, \dots, e_n$: oriented o.n.b. of \mathbb{R}^{n+2} ,

$$S^{k,n-k} = S^k(\mathbb{R}u_1 + \mathbb{R}e_1 + \cdots + \mathbb{R}e_k) \wedge S^{n-k}(\mathbb{R}u_2 + \mathbb{R}e_{k+1} + \cdots + \mathbb{R}e_n).$$

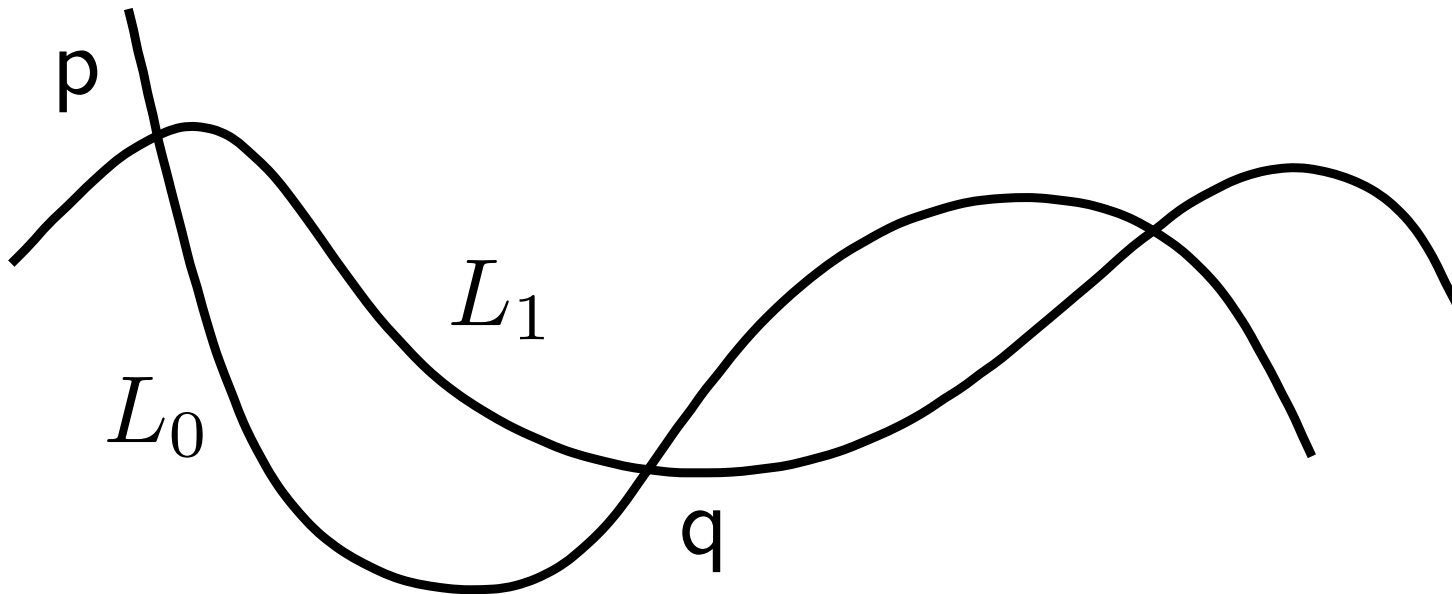
- $S^{0,n} = S^n$

2 Lagrangian Floer homology

(M, ω) : closed symplectic manifold

J_t : ω -compatible almost complex structures

L_0, L_1 : closed Lagrangian submfd, $L_0 \pitchfork L_1$

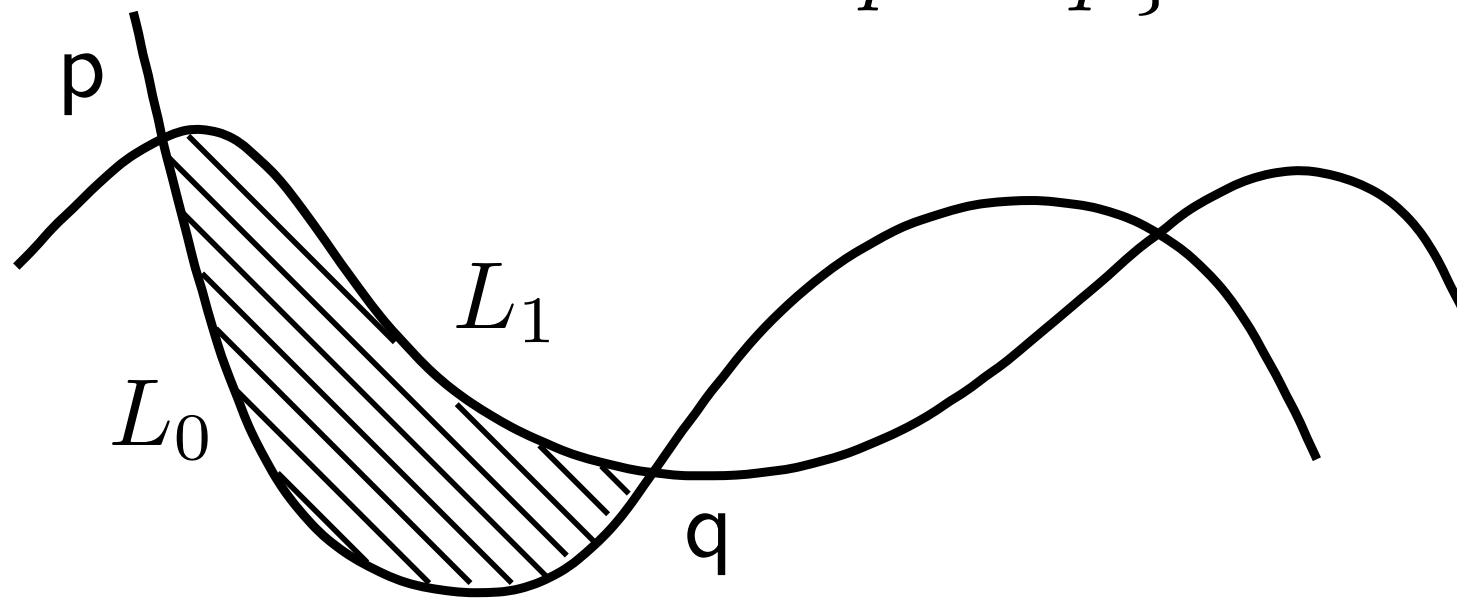


$CF(L_0, L_1)$: free \mathbb{Z}_2 -module gen. by $L_0 \cap L_1$

$$\partial : CF(L_0, L_1) \longrightarrow CF(L_0, L_1)$$

$$\partial(p) = \sum_{q \in L_0 \cap L_1} n(p, q) \cdot q$$

$n(p, q) := \#_2 \{ \text{isolated } \mathbf{J}\text{-holomorphic strip} \\ \text{from } p \text{ to } q \}$



- $u : \mathbb{R} \times [0, 1] \rightarrow M$ satisfying

$$\begin{cases} \frac{\partial u}{\partial s} + J_t(u) \frac{\partial u}{\partial t} = 0, \\ u(\cdot, 0) \in L_0, \quad u(\cdot, 1) \in L_1, \\ u(-\infty, \cdot) = p, \quad u(+\infty, \cdot) = q. \end{cases}$$

- $\partial^2 = 0 \implies HF(L_0, L_1) := \ker \partial / \text{im} \partial$

Floer homology of the pair (L_0, L_1) with \mathbb{Z}_2 -coefficients

$L \subset M$: closed Lagrangian submfd

• $I_\mu : \pi_2(M, L) \rightarrow \mathbb{Z}$: Maslov index

• $I_\omega : \pi_2(M, L) \rightarrow \mathbb{R}$,

$$I_\omega([u]) := \int_{D^2} u^* \omega \quad \text{for } u : D^2 \rightarrow (M, L).$$

Definition.

• L : **monotone**

$$\stackrel{\text{def}}{\iff} \exists \alpha > 0 : \text{const. s.t. } I_\omega = \alpha I_\mu.$$

• $\Sigma_L \geq 0$: **min. Maslov number** of L

$$\iff \{I_\mu(u) \mid [u] \in \pi_2(M, L)\} = \Sigma_L \cdot \mathbb{Z}$$

Theorem 4 (Oh)

L_0, L_1 : monotone with $\Sigma_{L_0}, \Sigma_{L_1} \geq 3$

\implies

- ∂ : well-defined.
- $\partial^2 = 0$.
- $HF(L_0, L_1 : \mathbb{Z}_2) \cong HF(L_0, \phi L_1 : \mathbb{Z}_2)$
for $\phi \in \text{Ham}(M, \omega)$.

Hence if $L_0 \pitchfork \phi L_1$,

$$\#(L_0 \cap \phi L_1) \geq \text{rank } HF(L_0, L_1 : \mathbb{Z}_2).$$

Remark.

(M, ω, J_0) : irreducible Hermitian symmetric space (**HSS**) of compact type

\implies a real form $L \subset M$: monotone

Theorem 5 (Oh)

(M, ω, J_0) : **HSS** of compact type

L_0, L_1 : real forms of M

$\implies J_0$ is regular.

We can use J_0 for calculation of HF .

Theorem 6 (Sakai-Tasaki- I)

(M, ω, J_0) : monotone **HSS** of compact type

L_0, L_1 : real forms of M with $\Sigma_{L_0}, \Sigma_{L_1} \geq 3$

$$L_0 \pitchfork L_1$$

$$\implies HF(L_0, L_1 : \mathbb{Z}_2) \cong \bigoplus_{p \in L_0 \cap L_1} \mathbb{Z}_2[p].$$

Remark.

- $L_0 \cap L_1$: **antipodal set**
- $L_0 \cong L_1 \implies HF \cong H_*(L_0 : \mathbb{Z}_2)$ **(Oh)**

3 Antipodal sets and HF

Definition (Chen-Nagano, 1988).

M : Riemannian symmetric space

s_x : geodesic symmetry at $x \in M$ ($s_x^2 = id$)

$S \subset M$: subset

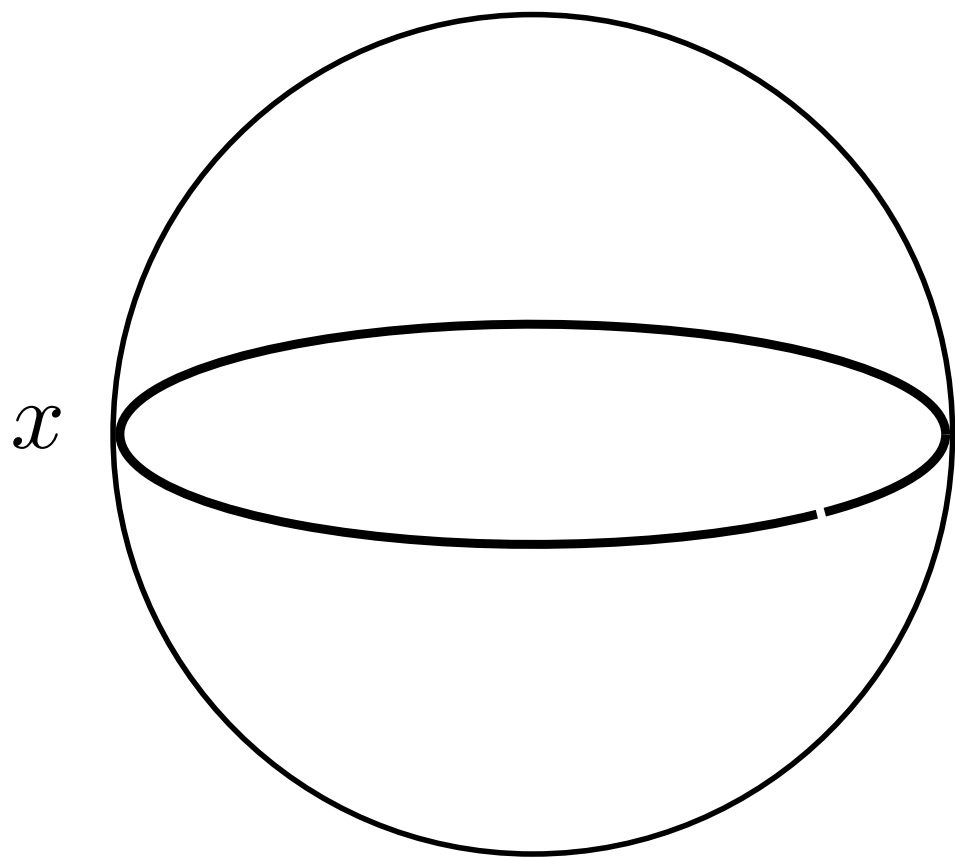
• S : **antipodal set**

$\stackrel{\text{def}}{\iff}$ For any $x, y \in S$, we have $s_x y = y$.

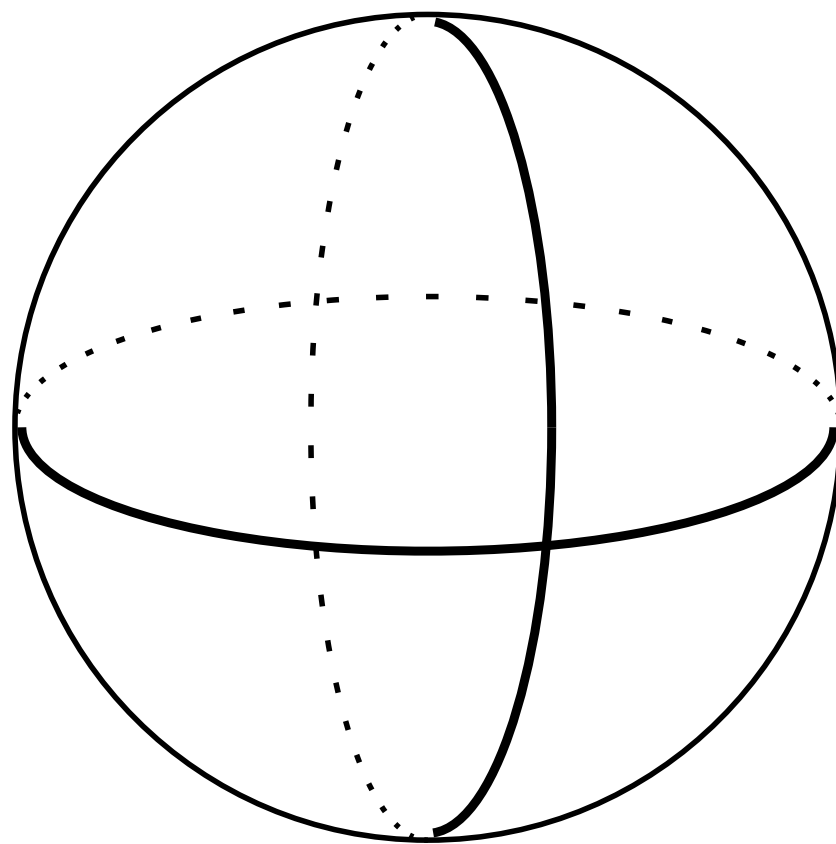
Example.

Fix $x \in S^2 = \mathbb{C}P^1$. $s_x(x) = x$, $s_x(-x) = -x$.

$\{x, -x\}$ is an antipodal set of S^2 .



— x



Theorem 7 (Tanaka-Tasaki, to appear)

(M, ω, J_0) : **HSS** of compact type

L_0, L_1 : real forms of M , $L_0 \pitchfork L_1$

$\implies L_0 \cap L_1$: antipodal set of M .

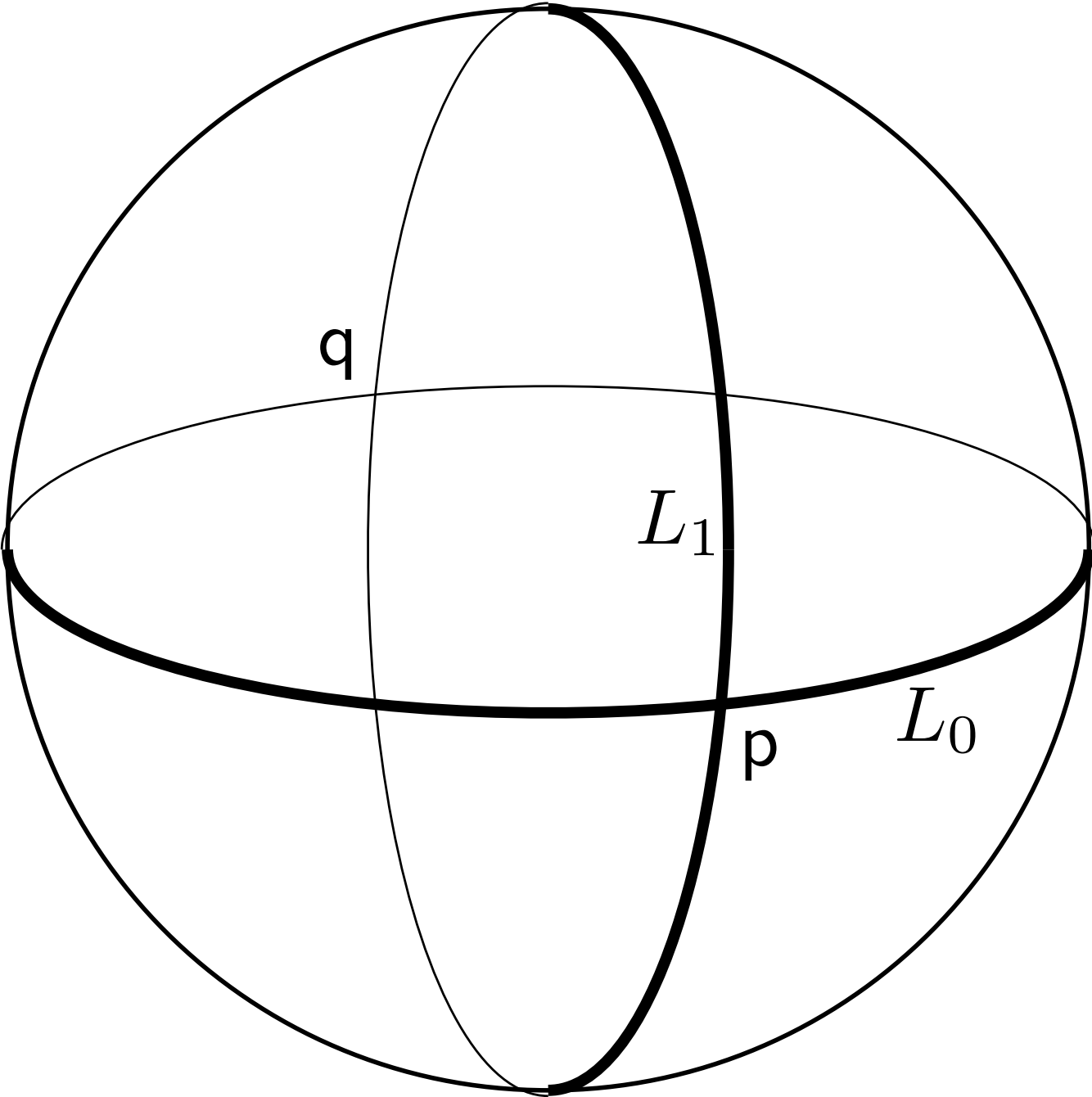
Outline of the Proof of Theorem 6

We want to show

$$\partial(p) = \sum_{q \in L_0 \cap L_1} n(p, q) \cdot q = 0$$

for any $p \in L_0 \cap L_1$.

- Assume $\exists J_0$ -holomorphic strip u .
- Consider s_p . Since $L_0 \cap L_1$: antipodal set,
 $s_p(p) = p, s_p(q) = q$.
- Consider $s_p \circ u$.

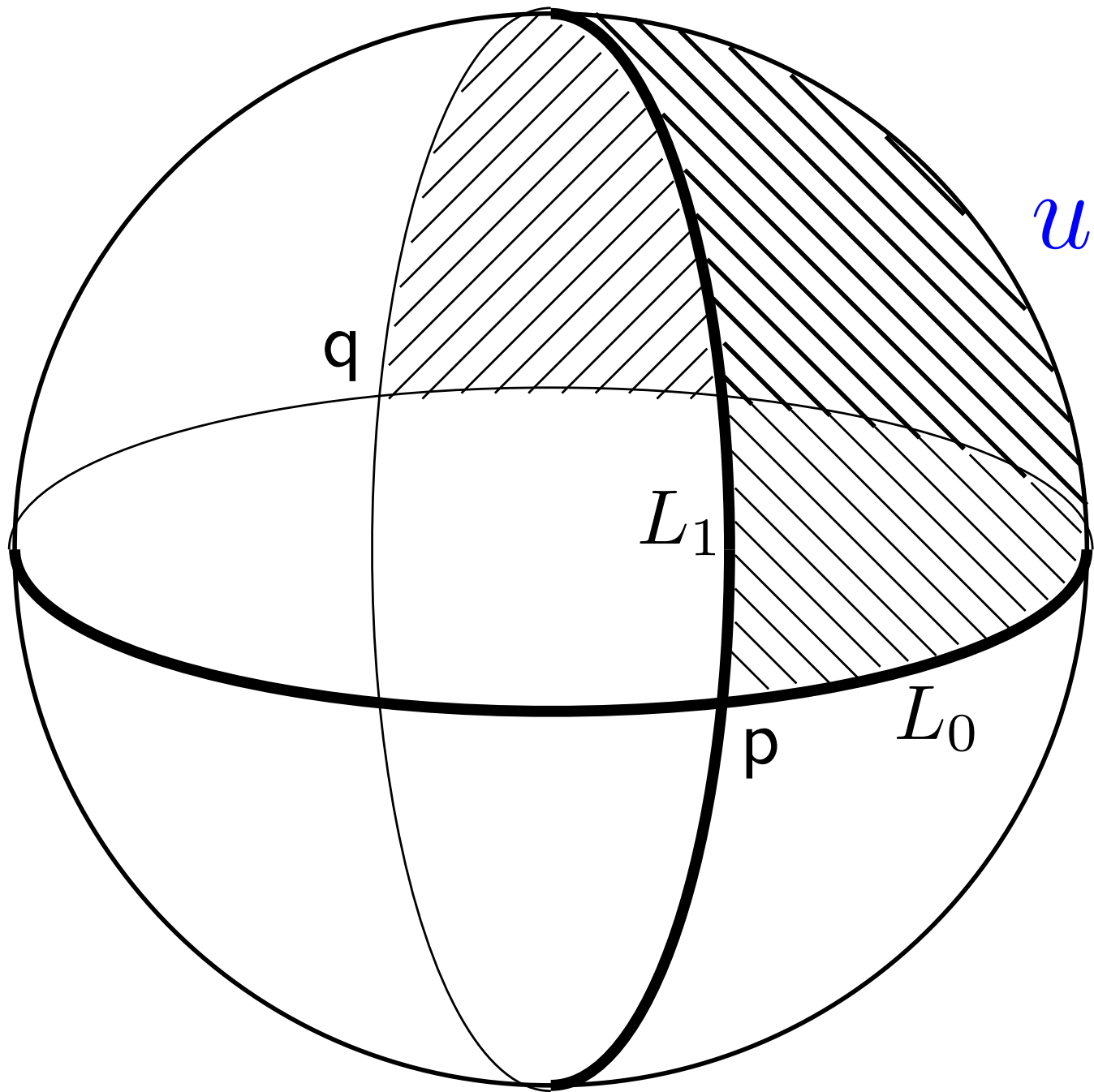


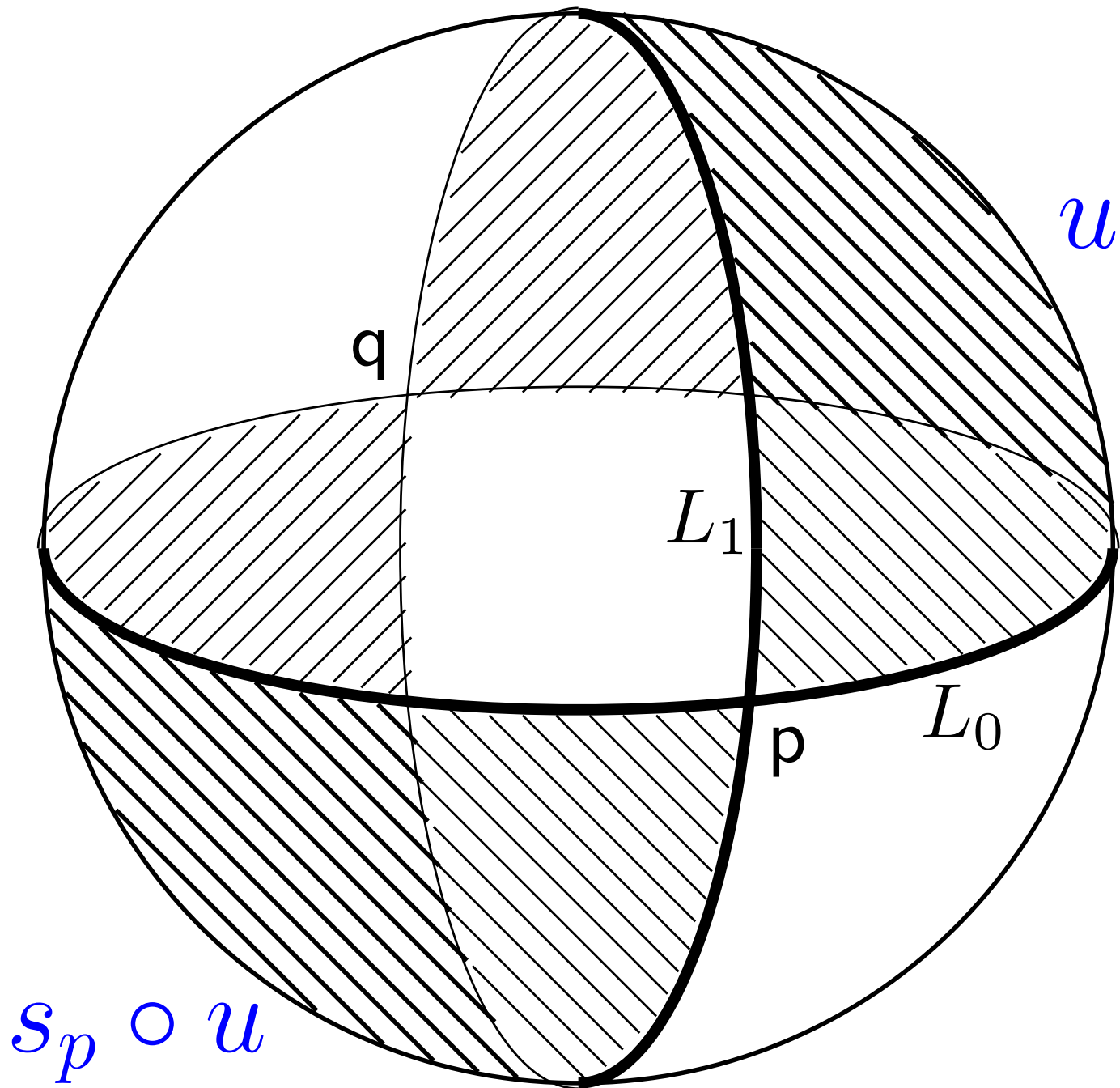
q

L_1

p

L_0





Hence, $\partial(p) = 0$.

$$HF(L_0, L_1 : \mathbb{Z}_2) \cong \bigoplus_{p \in L_0 \cap L_1} \mathbb{Z}_2[p].$$

4 Proof of Theorem 2

Theorem 8 (Tasaki, 2010)

L_0, L_1 : real forms of $Q_n(\mathbb{C})$

$$L_0 \cong S^{k, n-k}, L_1 \cong S^{l, n-l} \quad (k \leq l \leq [n/2])$$

$$L_0 \pitchfork L_1$$

\implies

$$L_0 \cap L_1 \cong \{\pm u_1 \wedge u_2, \pm e_1 \wedge e_{k+1}, \dots, \pm e_k \wedge e_{2k}\}.$$

Remark.

$$\begin{aligned} S^{k, n-k} &= S^k(\mathbb{R}u_1 + \mathbb{R}e_1 + \dots + \mathbb{R}e_k) \\ &\quad \wedge S^{n-k}(\mathbb{R}u_2 + \mathbb{R}e_{k+1} + \dots + \mathbb{R}e_n). \end{aligned}$$

- $L_0 \cong S^{k, n-k}, L_1 \cong S^{l, n-l} \quad (k \leq l \leq \lfloor n/2 \rfloor)$

By Theorem 6 and Theorem 8, we have

$$HF(L_0, L_1 : \mathbb{Z}_2) \cong (\mathbb{Z}_2)^{2(k+1)}.$$

Hence if $L_0 \cap \phi L_1$,

$$\#(L_0 \cap \phi L_1) \geq 2(k+1).$$

Here, put $k = 0$,

$$\#(S^n \cap \phi S^{l, n-l}) \geq 2.$$

Integrating over $SO(n+2)$,

$$\int_{SO(n+2)} \#(S^n \cap g\phi S^{l,n-l}) d\mu \geq 2 \text{vol}(SO(n+2)).$$

Crofton type formula (Lê Hồng Vân, 1993).

$N \subset Q_n(\mathbb{C})$: n -dim. submfd

$$\int_{SO(n+2)} \#(S^n \cap gN) d\mu \leq 2 \frac{\text{vol}(SO(n+2))}{\text{vol}(S^n)} \text{vol}(N).$$

Put $N = \phi S^{l,n-l}$,

$$2 \frac{\text{vol}(SO(n+2))}{\text{vol}(S^n)} \text{vol}(\phi S^{l,n-l}) \geq 2 \text{vol}(SO(n+2)).$$

Formula

Any real form $S^{l,n-l} \subset Q_n(\mathbb{C})$ satisfies

$$\text{vol}(\phi S^{l,n-l}) \geq \text{vol}(S^n)$$

for any $\phi \in \text{Ham}(Q_n(\mathbb{C}), \omega)$.

- $l = 0$: $\text{vol}(\phi S^n) \geq \text{vol}(S^n)$. **Ham. vol. min.**

5 A generalization to contact isotopies

Theorem 9

$\pi : P \rightarrow Q_n(\mathbb{C})$: pre-quantization bundle

i.e., (P, θ) : contact manifold, $d\theta = \pi^*\omega_0$

L : a Legendrian lift of a real form S^n

$$n \geq 3 + \sigma$$

$$\implies \text{vol}(\pi(\psi L)) \geq \text{vol}(L)$$

for any $\psi \in \text{Cont}(P)$: contact diffeo. of P

$Q_2(\mathbb{C})$	S^2	$S^1 \times S^1 / \mathbb{Z}_2$		
$Q_3(\mathbb{C})$	S^3	$S^1 \times S^2 / \mathbb{Z}_2$		
$Q_4(\mathbb{C})$	S^4	$S^1 \times S^3 / \mathbb{Z}_2$	$S^2 \times S^2 / \mathbb{Z}_2$	
$Q_5(\mathbb{C})$	S^5	$S^1 \times S^4 / \mathbb{Z}_2$	$S^2 \times S^3 / \mathbb{Z}_2$	
$Q_6(\mathbb{C})$	S^6	$S^1 \times S^5 / \mathbb{Z}_2$	$S^2 \times S^4 / \mathbb{Z}_2$	$S^3 \times S^3 / \mathbb{Z}_2$
$Q_7(\mathbb{C})$	S^7	$S^1 \times S^6 / \mathbb{Z}_2$	$S^2 \times S^5 / \mathbb{Z}_2$	$S^3 \times S^4 / \mathbb{Z}_2$

$Q_2(\mathbb{C})$	S^2	$S^1 \times S^1 / \mathbb{Z}_2$	Hamiltonian volume minimizing (I.-Ono-Sakai)	
$Q_3(\mathbb{C})$	S^3	$S^1 \times S^2 / \mathbb{Z}_2$		
$Q_4(\mathbb{C})$	S^4	$S^1 \times S^3 / \mathbb{Z}_2$	$S^2 \times S^2 / \mathbb{Z}_2$	
$Q_5(\mathbb{C})$	S^5	$S^1 \times S^4 / \mathbb{Z}_2$	$S^2 \times S^3 / \mathbb{Z}_2$	
$Q_6(\mathbb{C})$	S^6	$S^1 \times S^5 / \mathbb{Z}_2$	$S^2 \times S^4 / \mathbb{Z}_2$	$S^3 \times S^3 / \mathbb{Z}_2$
$Q_7(\mathbb{C})$	S^7	$S^1 \times S^6 / \mathbb{Z}_2$	$S^2 \times S^5 / \mathbb{Z}_2$	$S^3 \times S^4 / \mathbb{Z}_2$

Hamiltonian volume minimizing

$Q_2(\mathbb{C})$

S^2

$S^1 \times S^1 / \mathbb{Z}_2$

Hamiltonian volume minimizing
(I.-Ono-Sakai)

$Q_3(\mathbb{C})$

S^3

$S^1 \times S^2 / \mathbb{Z}_2$

H-stable (Oh, Amarzaya-Ohnita)

$Q_4(\mathbb{C})$

S^4

$S^1 \times S^3 / \mathbb{Z}_2$

$S^2 \times S^2 / \mathbb{Z}_2$

$Q_5(\mathbb{C})$

S^5

$S^1 \times S^4 / \mathbb{Z}_2$

$S^2 \times S^3 / \mathbb{Z}_2$

$Q_6(\mathbb{C})$

S^6

$S^1 \times S^5 / \mathbb{Z}_2$

$S^2 \times S^4 / \mathbb{Z}_2$

$S^3 \times S^3 / \mathbb{Z}_2$

$Q_7(\mathbb{C})$

S^7

$S^1 \times S^6 / \mathbb{Z}_2$

$S^2 \times S^5 / \mathbb{Z}_2$

$S^3 \times S^4 / \mathbb{Z}_2$

H-unstable (Oh, A-O)

Hamiltonian volume minimizing