Cohomology theories on locally conformally symplectic manifolds

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- Motivations

- Primitive forms and primitive (co)homology

- Primitive cohomology and Lichnerowicz-Novikov cohomology

- Examples and historical backgrounds

- Open problems

• I. Motivations

A differentiable manifold (M^{2n}, ω, θ) provided with a non-degenerate 2-form ω and a closed 1-form θ is called a locally conformally symplectic (l.c.s.) manifold, if $d\omega = -\omega \wedge \theta$, $d\theta = 0$. The 1-form θ is called the Lee form of ω. Locally $\theta = df$ and $\omega = e^{-f}\omega_0$, where $d\omega_0 = 0$.

L.c.s. forms were introduced by Lee, and have been extensively studied by Vaisman.

L.c.s. manifolds are phase spaces for a natural generalization of Hamiltonian dynamics, mapping torus of a contactomorphism, simple model for twisted symplectic geometry. They contain the subclass of L.C. K. manifolds.

The Lichnerowicz deformed differential $d_\theta : \Omega^*(M^{2n}) \to \Omega^*(M^{2n})$ is defined by $d_\theta(\alpha) := d\alpha + \theta \wedge \alpha$.

Note that $d_\theta^2 = 0$ and $d_\theta(\omega) = 0$. The resulting Lichnerowicz cohomology groups, (Novikov cohomology groups) are important conformal invariants of l.c.s. manifolds.

Two l.c.s. forms ω and ω' on M^{2n} are conformally equivalent, if $\omega' = \pm(e^f)\omega$ for some $f \in C^\infty(M^{2n})$. In this case $\theta' = \theta \mp df$, hence d_θ and $d_{\theta'}$ are *gauge equivalent*:

$$d_{\theta'}(\alpha) = (d_\theta \mp df\wedge)\alpha = e^{\pm f} d_\theta(e^{\mp f}\alpha).$$

$$H^*(\Omega^*(M^{2n}), d_\theta) = H^*(\Omega^*(M^{2n}), d_{\theta'}).$$

Remark: By the Darboux theorem there is no local conformal invariant of l.c.s. manifolds. AIM: construct new cohomological invariants for l.c.s. manifolds.

$L : \Omega^*(M^{2n}) \to \Omega^*(M^{2n}),\ \alpha \mapsto \omega \wedge \alpha.$

$d_\theta L = Ld.$

$d_k := d_{k\theta}.$

$d_k L^p = L^p d_{k-p}.$

$I_\omega : T_x M^{2n} \to T_x^* M^{2n},\ V \mapsto i_V \omega.$

$G_\omega \in \Gamma(\Lambda^2 T M^{2n})$ s.t. $i_{G_\omega} I_\omega = Id$, where

$i_{G_\omega} : T_x^* M^{2n} \to T_x M^{2n},\ V \mapsto i_V(G_\omega(x)).$

$*_\omega : \Omega^p(M^{2n}) \to \Omega^{2n-p}(M^{2n}),$

$\beta \wedge *_\omega \alpha := \Lambda^p G_\omega(\beta, \alpha) \wedge \frac{\omega^n}{n!}.$

$*_\omega^2 = Id.$

$L^* : \Omega^p(M^{2n}) \to \Omega^{p-2}(M^{2n}),$

$\alpha^p \mapsto - *_\omega L *_\omega \alpha^p.$

$(d_k)^*_\omega : \Omega^p(M^{2n}) \to \Omega^{p-1}(M^{2n})$,
$\alpha^p \mapsto (-1)^p *_\omega d_{n+k-p} *_\omega (\alpha^p)$.
$\pi_k : \Omega^*(M^{2n}) \to \Omega^k(M^{2n})$ be the projection.
$L^* = i(G_\omega)$,
$[L^*, L] = A$, $[A, L] = -2L$, $[A, L^*] = 2L^*$.

II Primitive forms and primitive (co)homology

$\alpha \in \Lambda^k T^*_x M^{2n}$, $0 \le k \le n$, is called primitive, if $L^{n-k+1}\alpha = 0$. $\alpha \in \Lambda^k T^*_x M^{2n}$, $n+1 \le k \le 2n$, is called primitive, if $\alpha = 0$. $\beta \in \Lambda^k T^*_x M^{2n}$ is called coeffective, if $L\beta = 0$.

$P_x^k(M^{2n})$: = the set of primitive elements in $\Lambda^k T_x^* M^{2n}$.

Lemma An element $\alpha \in \Lambda^k T_x^* M^{2n}$, is primitive, if and only if $L^* \alpha = 0$.
2. An element $\beta \in \Lambda^k T_x^* M^{2n}$ is coeffective, if and only if $*_\omega \beta$ is primitive.
3. Lefschetz decomposition $\Lambda^{n-k} T_x^* M^{2n} = P_x^{n-k}(M^{2n}) \oplus L P_x^{n-k-2}(M^{2n}) \oplus L^2 P_x^{n-k-4}(M^{2n}) \cdots$,
$\Lambda^{n+k} T_x^* M^{2n} = L^k P_x^{n-k}(M^{2n}) \oplus L^{k+1} P_x^{n-k-2}(M^{2n}) \cdots$,
for $n \geq k \geq 0$.
4. $L^k : \Lambda^{n-k} T_x^* M^{2n} \to \Lambda^{n+k} T_x^* M^{2n}$ is an isomorphism, for $0 \leq k \leq n$.

5. $L : \Lambda^{n-k-2} T_x^* M^{2n} \to \Lambda^{n-k} T_x^* M^{2n}$ is injective, for $k = -1, 0, 1, \cdots, n-2$.

$K_p^* := (\Omega^*(M^{2n}), d_p)$.

$$F^0 K_p^* := K_p^* \supset F^1 K_p^* := L K_{p-1}^* \supset \cdots$$

$$\supset F^k K_p^* := L^k K_{p-k}^* \supset \cdots \supset F^{n+1} K_p^* = \{0\}.$$

$d_k^+ := \Pi_{pr} d_k : \Omega^q(M^{2n}) \to \mathcal{P}^{q-1}(M^{2n})$.

$$d_k = d_k^+ + L d_k^-,$$

$$d_k^-: \Omega^q(M^{2n}) \to \Omega^{q-1}(M^{2n}),\ 0 \le q \le n.$$

$$\begin{aligned}
(d_k^+)^2(\alpha^q) &= 0,\\
d_{k-1}^- d_k^-(\alpha^q) &= 0,\ q \le n,\\
(d_k^- d_k^+ + d_{k-1}^+ d_k^-)\alpha^q &= 0,\ q \le n-1,\\
(d_{k-1})^*_\omega (d_k)^*_\omega(\alpha^q) &= 0.
\end{aligned}$$

Assume that $0 \le q \le n-1$.

$$H^q(\mathcal{P}^*(M^{2n}), d_k^+) := \frac{\ker d_k^+ \cap \mathcal{P}^q(M^{2n})}{d_k^+(\mathcal{P}^{q-1}(M^{2n}))}.$$

$$H_q(\mathcal{P}^*(M^{2n}), (d_k)^*_\omega) := \frac{\ker (d_k)^*_\omega \cap \mathcal{P}^q(M^{2n})}{(d_{k+1})^*_\omega(\mathcal{P}^{q+1}(M^{2n}))}.$$

$$H_q(\mathcal{P}^*(M^{2n}), d_k^-) := \frac{\ker d_k^- \cap \mathcal{P}^q(M^{2n})}{d_{k+1}^-(\mathcal{P}^{q+1}(M^{2n}))}.$$

Proposition Assume $\dim(M^{2n}, \omega, \theta) \geq 2$.

1. If $[(k-1)\theta] \neq 0 \in H^1(M^{2n}, \mathbf{R})$ then

$$H^1(\mathcal{P}^*(M^{2n}), d_k^+) = H^1(\Omega^*(M^{2n}), d_k).$$

2. If $[(k-1)\theta] = 0 \in H^1(M^{2n}, \mathbf{R})$ then

$$H^1(\mathcal{P}^*(M^{2n}), d_k^+) = H^1(\Omega^*(M^{2n}), d_\theta)$$

if $[\omega] \neq 0 \in H^2(\Omega^*(M^{2n}), d_\theta)$

$$H^1(\mathcal{P}^*(M^{2n}), d_k^+) = H^1(\Omega^*(M^{2n}), d_\theta) \oplus \mathbf{R}$$

if $[\omega] = 0 \in H^2(\Omega^*(M^{2n}), d_\theta)$.

Proposition Assume that $0 \leq k \leq n$. If $\alpha \in \mathcal{P}^k(M^{2n})$, then for all l

$$d_l^-(\alpha^k) = \frac{(d_l)^*_\omega(\alpha^k)}{n-k+1}.$$

Hence $H_k(\mathcal{P}^*(M^{2n}), d_l^-) = H_k(\mathcal{P}^*(M^{2n}), (d_l)^*_\omega)$.

Proposition Let (M^{2n}, ω, θ) be a compact l.c.s manifold. Then
$H^k(\mathcal{P}^*(M^{2n}), d_l^+) = H_k(\mathcal{P}^*(M^{2n}), (d_{-l+k-n})^*_\omega)$
for all l and $0 \leq k \leq n-1$.

III The relations between primitive cohomology and Lichnerowicz-Novikov cohomology

The spectral sequence $\{E_{k,r}^{p,q}, d_{k,r} : E_{k,r}^{p,q} \to E_{k,r}^{p+r,q-r+1}\}$, $r \geq 0$, is associated to the filtration $(F^* K_k^*, d_k)$.

$$E_{k,0}^{p,q} \cong \mathcal{P}^{q-p}(M^{2n}) \text{ if } n \geq q \geq p$$

$$E_{k,0}^{p,q} = 0 \text{ otherwise .}$$

$$E_{k,1}^{p,q} = H^{q-p}(\mathcal{P}^*(M^{2n}), d_{k-p}^+) \text{ if } 0 \leq p \leq q \leq n-1,$$

$$E_{k,1}^{p,n} = \frac{\mathcal{P}^{n-p}(M^{2n})}{d_{k-p}^{+}(\mathcal{P}^{n-p-1}(M^{2n}))}, \text{ if } 0 \le p \le n,$$

$$E_{k,1}^{p,q} = 0 \text{ otherwise .}$$

$$d_{l+p,1} : E_{l+p,1}^{p,q} \longrightarrow E_{l+p,1}^{p+1,q}$$

is defined for $0 \le p \le q \le n$ by

$$H^{q-p}(\mathcal{P}^*(M^{2n}), d_l^+) \longrightarrow H^{q-p-1}(\mathcal{P}^*(M^{2n}), d_{l-1}^+),$$

$$[\tilde{\alpha}] \mapsto [d_l^- \tilde{\alpha}].$$

Corollary Assume that $1 \le p \le q \le n-1$. Then $E_{l,2}^{p,q} = E_{l,2}^{p-1,q-1}$.

Theorem The spectral sequences $E_{k,r}^{p,q}$ on (M^{2n}, ω, θ) and on $(M^{2n}, \omega', \theta')$ are isomorphic, if ω and ω' are conformal equivalent. Furthermore, the $E_{k,1}$-terms of the spectral sequences on (M, ω, θ) and (M, ω', θ') are isomorphic, if $\omega' = \omega + d_\theta \rho$ for some $\rho \in \Omega^1(M^{2n})$.

Theorem Assume that $\omega = d_1 \tau$.

1. $E_{l+p,1}^{p,q} = H_l^{q-p}(M^{2n}) \oplus H_{l-1}^{q-p-1}(M^{2n})$ for $0 \leq p \leq q \leq n-1$.
2. $E_{l,2}^{p,q} = 0$, if $1 \leq p \leq q \leq n-1$.
3. If $0 \leq q \leq n$, then $E_{l,2}^{0,q} = H_l^q(M^{2n})$.

4. If $0 \le p \le n$ then $E^{p,n}_{l+p,2} = H^{n+p}_{l+p}(M^{2n})$.
5. The spectral sequence $\{E^{p,q}_{l,r}, d_{l,r}\}$ stabilizes at the term $E_{l,2}$.

$$C^k_l := \frac{\ker d^-_l \cap \Omega^k(M^{2n})}{d_l(\Omega^{k-1}(M^{2n}))}.$$

Lemma For $0 \le p \le q \le n-1$ the following sequences is exact

$$0 \to (\Omega^{q-(p+1)}(M^{2n}), d_{l-1}) \xrightarrow{L} (\Omega^{q+1-p}(M^{2n}), d_l) \to$$

$$\xrightarrow{\Pi L^p} (E^{p,q+1}_{l+p,0}, d_{l+p,0}) \to 0.$$

$$\cdots \to H_l^{q-p}(M^{2n}) \xrightarrow{\bar{L}^p} E_{l+p,1}^{p,q} \xrightarrow{\delta_{p,q}} H_{l-1}^{q-(p+1)}(M^{2n}) \to$$

$$\xrightarrow{\bar{L}} H_l^{q+1-p}(M^{2n}) \xrightarrow{\bar{L}^p} E_{l+p,1}^{p,q+1} \to \cdots$$

$$\cdots \to E_{l+p,1}^{p,n-1} \xrightarrow{\delta_{p,n-1}} H_{l-1}^{n-(p+2)}(M^{2n}) \xrightarrow{[L]} C_l^{n-p} \to .$$

If moreover $\omega = d_1\tau$ the following sequences are exact

$$0 \to H_l^{q-p}(M^{2n}) \xrightarrow{\bar{L}^p} E_{l+p,1}^{p,q} \to H_{l-1}^{q-(p+1)}(M^{2n}) \to 0,$$

$$\to E_{l+p,2}^{p-1,q} \to H_l^{q-p}(M^{2n}) \xrightarrow{\delta} H_l^{q-p}(M^{2n}) \to$$

$$\to E_{l+p,2}^{p,q} \to H_{l-1}^{q-(p+1)}(M^{2n}) \xrightarrow{\delta} .$$

For $0 \le p \le n-1$ we have

$$0 \to C_l^{n-p} \stackrel{[L^p]}{\longrightarrow} E_{l+p,1}^{p,n} \stackrel{\delta_{p,n}}{\longrightarrow} T_{l-1}^{n-(p+1)} \to 0,$$

$$T_{l-1}^{n-(p+1)} := \ker[L^{p+1}] : C_{l-1}^{n-(p+1)} \to H_{l+p}^{n+p+1}.$$

Theorem Assume that $\omega^T = d_T \rho$ and $T \ge 2$. Then the spectral sequence $(E_{l,r}^{p,q}, d_{l,r})$ stabilizes at terms $E_{l,T+1}^{*,*}$.

The main idea is to find a short exact sequence, whose middle term is $E_{l,T}^{*,*}$, and moreover, this short exact sequence is induced from the trivial action of the operator L^T on (a part of) complexes entering in the derived exact couples.

Theorem Assume that (M^{2n}, ω, θ) is a compact connected globally conformally symplectic manifold. Then the spectral sequence $(E_{k,r}^{p,q}, d_{k,r})$ stabilizes at the $E_{k,2}^{*,*}$-term.

The main idea: For symplectic manifolds (M^{2n}, ω) the term $E_k^{p,p}$, $0 \leq p \leq 1$ and $k \geq 1$, is generated by ω^p, which acts on $E_k^{0,r}$ injectively, if $p + r \leq n$.

IV Examples and historical backgrounds

• For $\theta = 0$ there is known construction of coeffective cohomology groups (Bouche, Fernandez, De Leon) which are dual to the primitive cohomology groups (Tseng-Yau) via the the symplectic star operators.

• There is a compact 6-dimensional nilmanifold M^6 equipped with a family of symplectic forms ω_t, $t \in [0, 1]$, with varying cohomology classes $[\omega_t] \in H^2(M^6, \mathbf{R})$. Fernandez at all. showed that the coeffective cohomology groups associated to ω_1 and ω_2 have different Betti number b_4.

• The filtration on the symplectization of a contact manifold gives rise to a filtration on the contact manifold, which have been discovered by Lychagin and Rumin.

• For compact Kähler manifolds the spectral sequence converges at the term $E_1^{*,*}$, hence there is no trivial primitive cohomology groups.

• If $\omega = d_\theta \tau$ all the primitive cohomology groups are parts of the Lichnerowicz-Novikov cohomology groups of M^{2n}.

• A generalization of the symplectization is the notion of mapping torus of a contactomorphism, which has a l.c.s. structure. The primitive cohomology of the associated l.c.s. is a part of the associated Lichnerowicz-Novikov cohomology.

V Open questions

• Understand the behaviour of the primitive cohomology groups (and the whole spectral sequences) under l.c.s. surgery.

• Investigate the associated cohomology

$$\mathcal{P}H_l^k = \frac{\ker(d_l^+ + d_l^-) \cap \mathcal{P}^k(M^{2n})}{\mathrm{im} d_{l+1}^- d_{l+1}^+ \cap \mathcal{P}^k(M^{2n})}$$

since $d_l^- d_{l+1}^- = 0$ and $d_{l+1}^- d_{l+1}^+ + d_l^+ d_{l+1}^- = 0$, which implies that
$\mathrm{im}(d_{l+1}^- d_{l+1}^+) \subset \ker d_l^+ \cap d_l^- = \ker(d_l^+ + d_l^-)$.

- And more cohomology to play with (see Tseng-Yau: Cohomology and Hodge Theory on Symplectic Manifolds, I, II arXiv:0909.5418, arXiv:1011.1250.)

- Is it possible to use this technique to distinguish the L.C.K. manifolds among l.c.s. manifolds?

- Applications for coistropic and Lagrangian submanifolds.

- Develop the elliptic cohomology theory for l.c.s. manifolds.

Hong Van Le and Jiri Vanzura, Cohomology theories on locally conformally symplectic manifolds, arXiv:1111.3841

Thank you!