On Lagrangian submanifolds in complex hyperquadrics and Hamiltonian volume variational problem

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Contents

Hamiltonian minimality and Hamiltonian stability (Y.-G. Oh (1990))

(M, ω, J, g) : Kähler manifold, $\varphi : L \longrightarrow M$ Lagr. imm.

$$\begin{array}{ll} \mathrm{H} : & \text{mean curvature vector field of } \varphi \\ \updownarrow & \\ \alpha_{\mathrm{H}} := \omega(\mathrm{H}, \cdot) : & \text{“mean curvature form” of } \varphi \end{array}$$

- $d\alpha_{\mathrm{H}} = \varphi^* \rho_M$ where ρ_M : Ricci form of M. (Dazord)
- If M is Einstein-Kähler, then $d\alpha_{\mathrm{H}} = 0$.

Suppose L : compact without boundary
φ : “Hamiltonian minimal” (or “H-minimal ”)
$\underset{\text{def}}{\Longleftrightarrow}$ ${}^{\forall}\varphi_t : L \longrightarrow M$ Hamil. deform. with $\varphi_0 = \varphi$

$$\frac{d}{dt}\mathrm{Vol}\,(L, \varphi_t^* g)|_{t=0} = 0$$

$\Longleftrightarrow$ $\delta\alpha_{\mathrm{H}} = 0$

- minimal $\Longrightarrow$ H-minimal

Assume φ : H-minimal.

φ : "Hamiltonian stable" $\underset{\text{def}}{\Longleftrightarrow}$ $^\forall\{\varphi_t\}$: Hamil. deform. of $\varphi_0 = \varphi$

$$\frac{d^2}{dt^2}\mathrm{Vol}\,(L, \varphi_t^* g)|_{t=0} \geq 0$$

The Second Variational Formula

$$\frac{d^2}{dt^2}\mathrm{Vol}\,(L, \varphi_t^* g)|_{t=0} =$$
$$\int_L \left(\langle \triangle_L^1 \alpha, \alpha\rangle - \langle \overline{R}(\alpha), \alpha\rangle - 2\langle \alpha \otimes \alpha \otimes \alpha_{\mathrm{H}}, S\rangle + \langle \alpha_{\mathrm{H}}, \alpha\rangle^2\right) dv$$

where

- $\alpha := \alpha_{\frac{\partial \varphi_t}{\partial t}\big|_{t=0}} \in B^1(L)$
- $\langle \overline{R}(\alpha), \alpha\rangle := \sum_{i,j=1}^n \mathrm{Ric}^M(e_i, e_j)\alpha(e_i)\alpha(e_j)$ $\{e_i\}$: o.n.b. of T_pL
- $S(X, Y, Z) := \omega(h(X, Y), Z)$ sym. 3-tensor field on L

Corollary

M : Einstein-Kähler manifold with Einstein constant κ.

$L \hookrightarrow M$: compact minimal Lagr. submfd. (i.e. $\alpha_{\mathrm{H}} \equiv 0$)

Then

$$L \text{ is Hamiltonian stable} \iff \lambda_1 \geq \kappa.$$

Here

λ_1 : the first (positive) eigenvalue of the Laplacian of L on $\mathcal{C}^\infty(L)$.

(B. Y. Chen - P. F. Leung - T. Nagano , Y. G. Oh)

Fact (H. Ono, Amarzaya-Ohnita)

Assume M : compact homogeneous Einstein - Kähler mfd. with $\kappa > 0$.
$L \hookrightarrow M$: compact minimal Lagr. submfd.
Then

$$\lambda_1 \leq \kappa.$$

$$\lambda_1 = \kappa \Longleftrightarrow L \text{ is Hamiltonian stable.}$$

Trivial Hamiltonian deformations

X : holomorphic Killing vector field of M
$\Longrightarrow \quad \alpha_X = \omega(X, \cdot)$ is closed
$\Longrightarrow \quad \alpha_X = \omega(X, \cdot)$ is exact if $H^1(M, \mathbb{R}) = \{0\}$.

If M is simply connected, more generally $H^1(M, \mathbb{R}) = \{0\}$, each holomorphic Killing vector field of M generates a volume-preserving Hamiltonian deformation of φ.

Def. Such a Hamiltonian deformation of φ is called trivial.

Strictly Hamiltonian stability

Assume $\varphi : L \to (M, \omega, J, g)$: H-minimal.
φ : "strictly Hamiltonian stable "
$\underset{\text{def}}{\Longleftrightarrow}$
(1) φ is Hamiltonian stable
(2) The null space of the second variation on Hamiltonian deformations coincides with the vector subspace induced by trivial Hamiltonian deformations of φ, i.e., $n(\varphi) = n_{hk}(\varphi)$.

Here, $n(\varphi) :=$ dim[the null space] and
$n_{hk}(\varphi) := \dim\{\varphi^* \alpha_X | X$ is a holomorphic Killing vector field of $M\}$.

If L is strictly Hamiltonian stable, then L has local minimum volume under each Hamiltonian deformation.

Elementary examples

Circles on a plane

$$S^1 \subset \mathbb{R}^2 \cong \mathbb{C},$$

great circles and small circles on a sphere

$$S^1 \subset S^2 \cong \mathbb{C}P^1,$$

are compact Hamiltonian stable H-minimal Lagrangian submanifolds.

(Oh)

The real projective space totally geodesic embedded in the complex projective space

$$\mathbb{R}P^n \subset \mathbb{C}P^n$$

is strictly Hamiltonian stable.

- It is Hamiltonian volume minimizing (Kleiner-Oh).

(Oh)

The $(n+1)$-torus

$$T^{n+1}_{r_0,\cdots,r_n} = S^1(r_0) \times \cdots \times S^1(r_n) \subset \mathbb{C}^{n+1}$$

is strictly Hamiltonian stable H-minimal Lagrangian submanifold in $\mathbb{C}^{n+1}$.

- $T^{n+1}_{r_0,\cdots,r_n}$ is not minimal in $\mathbb{C}^{n+1}$ ($\nexists$ closed minimal submanifolds in $\mathbb{C}^{n+1}$).
  $\Rightarrow$ It is not stable under arbitrary deformation of $T^{n+1}_{r_0,\cdots,r_n}$.
- It is H-minimal in $\mathbb{C}^{n+1}$.
- It is strictly Hamiltonian stable.
- Is it Hamiltonian volume minimizing? (Oh's conjecture, still open)

(Oh, H. Ono)

The quotient space by S^1-action

$$T^{n+1}_{r_0,\cdots,r_n}/S^1 \subset \mathbb{C}P^n$$

is strictly Hamiltonian stable H-minimal Lagrangian submanifold in $\mathbb{C}P^n$.

- If $r_0 = \cdots = r_n = \frac{1}{\sqrt{n+1}}$, then it is minimal ("Clifford torus "), otherwise, not minimal but H-minimal.
- It is strictly Hamiltonian stable for any $(r_0, \cdots, r_n)$
- Is the Clifford torus Hamiltonian volume minimizing?
  (Oh's conjecture, still open)

(Amarzaya-Ohnita)

Compact irreducible minimal Lagrangian submanifolds

$$SU(p)/SO(p)\cdot \mathbf{Z}_p \subset \mathbb{C}P^{\frac{(p-1)(p+2)}{2}}$$
$$SU(p)/\mathbf{Z}_p \subset \mathbb{C}P^{p^2-1}$$
$$SU(2p)/Sp(p)\cdot \mathbf{Z}_{2p} \subset \mathbb{C}P^{(p-1)(2p+1)}$$
$$E_6/F_4\cdot \mathbf{Z}_3 \subset \mathbb{C}P^{26}$$

embedded in complex projective spaces are strictly Hamiltonian stable.

- They are not totally geodesic but their second fundamental forms are parallel.

(R. Chiang,Bedulli-Gori, Ohnita)

The minimal Lagrangian orbit

$$\rho_3(SU(2))[z_0^3 + z_1^3] \subset \mathbb{C}P^3$$

is a compact embedded Hamiltonian stable submanifold with non-parallel second fundamental form.

(M. Takeuchi, Oh, Amarzaya-Ohnita)

M : cpt. irred. Herm. sym. sp.
L : cpt. totally geodesic Lagr. submfd embedded in M.

$$\underset{\substack{\text{tot. geod.}\\ \text{Lagr. submfd.}}}{(L, M)} = \begin{cases} (Q_{p,q}(\mathbb{R}) = (S^{p-1} \times S^{q-1})/\mathbb{Z}_2, \\ \quad Q_{p+q-2}(\mathbb{C}))(p \geq 2, q-p \geq 3) \\ (U(2p)/Sp(p), SO(4p)/U(2p))(p \geq 3), \\ (T \cdot E_6/F_4, E_7/T \cdot E_6). \end{cases}$$

$\Longleftrightarrow$ L is NOT Hamiltonian stable.

Takeuchi:

All cpt. totally geodesic Lagr. submfds in cpt. irred. Herm. sym. sp. are real forms,

i.e., the fixed point subset of involutive anti-holomorphic isometries.

- Let (M, ω, J, g) be an Einstein-Kähler manifold with an involutive anti-holomorphic isometry τ and $L := \mathrm{Fix}(\tau)$, Einstein, positive Ricci curvature. Is L Hamiltonian volume minimizing? (Oh's conjecture, still open)

(Iriyeh-H. Ono-Sakai)

$$S^1(1) \times S^1(1) \xrightarrow[\text{totally geodesic}]{\text{Lagr.}} S^2(1) \times S^2(1)$$

is Hamiltonian volume minimizing.

Complex Hyperquadrics

$$Q_n(\mathbb{C}) \cong \widetilde{Gr}_2(\mathbb{R}^{n+2}) \cong SO(n+2)/SO(2) \times SO(n)$$

a compact Hermitian symmetric space of rank 2

$$Q_n(\mathbb{C}) := \{[\mathbf{z}] \in \mathbb{C}P^{n+1} \mid z_0^2 + z_1^2 + \cdots + z_{n+1}^2 = 0\}$$
$$\widetilde{Gr}_2(\mathbb{R}^{n+2}) := \{W \mid \text{oriented 2-dimensional vector subspace of } \mathbb{R}^{n+2}\}$$

$$Q_n(\mathbb{C}) \ni \ [\mathbf{a} + \sqrt{-1}\mathbf{b}] \longleftrightarrow \mathbf{a} \wedge \mathbf{b} \ \in \widetilde{Gr}_2(\mathbb{R}^{n+2})$$

Here $\{\mathbf{a}, \mathbf{b}\}$ is an orthonormal basis of W compatible with its orientation.

- $(Q_n(\mathbb{C}) \cong \widetilde{Gr}_2(\mathbb{R}^{n+2}), g^{\mathrm{std}}_{Q_n(\mathbb{C})})$ is Einstein-Kähler with Einstein constant $\kappa = n$.
- $Q_1(\mathbb{C}) \cong S^2$
- $Q_2(\mathbb{C}) \cong S^2 \times S^2$
- $n \geq 3$, $Q_n(\mathbb{C})$ is irreducible.

Conormal bundle construction

Given an oriented submanifold $N^m \subset S^{n+1}(1)$

$$p_1 : V_2(\mathbb{R}^{n+2}) \ni (\mathbf{a}, \mathbf{b}) \mapsto \mathbf{a} \in S^{n+1}(1)$$
$$p_2 : V_2(\mathbb{R}^{n+2}) \ni (\mathbf{a}, \mathbf{b}) \mapsto \mathbf{a} \wedge \mathbf{b} \in Q_n(\mathbb{C})$$

$$\begin{array}{ccccccc}
\nu_N^* & \xrightarrow{\text{Lag.}} & T^*S^{n+1}(1) & & & & \\
\downarrow & & \downarrow & & & & \\
\mathrm{U}\nu_N^* & \xrightarrow{\text{Leg.}} & \mathrm{U}(T^*S^{n+1}(1)) & \cong & V_2(\mathbb{R}^{n+2}) & & \\
\downarrow & & {\scriptstyle p_2}\downarrow{\scriptstyle S^1} & & {\scriptstyle p_1}\downarrow{\scriptstyle S^n} & & \\
p_2(U(\nu_N^*)) & \xrightarrow{\text{Lag.imm.}} & Q_n(\mathbb{C}) & & S^{n+1}(1) & \xleftarrow[\text{imm.}]{} & N^m
\end{array}$$

$N^n \subset S^{n+1}$ hypersurface

$\Rightarrow$ This construction is nothing but the following Gauss map.

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\downarrow & & p_2 \downarrow S^1 & & p_1 \downarrow S^n & & \\
p_2(U(\nu_N^*)) & \xrightarrow{\text{Lag.imm.}} & Q_n(\mathbb{C}) & & S^{n+1}(1) & \xleftarrow[\text{imm.}]{} & N^m
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Oriented hypersurface in a sphere

$N^n \hookrightarrow S^{n+1}(1) \subset \mathbb{R}^{n+2}$
$\mathbf{x}$: the position vector of points of N^n
$\mathbf{n}$: the unit normal vector field of N^n in $S^{n+1}(1)$

"Gauss map"

$$\mathcal{G} : N^n \ni p \longmapsto [\mathbf{x}(p) + \sqrt{-1}\mathbf{n}(p)] = \mathbf{x}(p) \wedge \mathbf{n}(p) \in Q_n(\mathbb{C})$$

is a Lagrangian immersion.

- Oriented hypersurfaces N_1, N_2 are parallel to each other in $S^{n+1}(1)$ $\Longleftrightarrow \mathcal{G}(N_1) = \mathcal{G}(N_2)$.
- Choose an orthonormal frame $\{e_i\}$ of N w.r.t. the induced metric from $S^{n+1}(1)$ s.t. $h(e_i, e_j) = \kappa_i \delta_{ij}$ and let θ_i be the dual frame. Then the induced metric on N by the Gauss map $\mathcal{G}$ is

$$\mathcal{G}^* g^{\mathrm{std}}_{Q_n(\mathbb{C})} = \sum (1 + \kappa_i^2)\theta_i \otimes \theta_i.$$

The Mean Curvature Formula (B. Palmer, 1997)

$$\alpha_{\mathrm{H}} = d\left(\mathrm{Im}\left(\log \prod_{i=1}^{n}(1+\sqrt{-1}\kappa_i)\right)\right),$$

where H denotes the mean curvature vector field of $\mathcal{G}$ and κ_i $(i = 1, \cdots, n)$ denote the principal curvatures of $N^n \subset S^{n+1}(1)$.

1. When $n = 2$, if $N^2 \subset S^3(1)$ is a minimal surface, then

$$(1+\sqrt{-1}\kappa_1)(1+\sqrt{-1}\kappa_2) = 1 - \mathrm{K}_N + \sqrt{-1}\mathrm{H}_N,$$

   $\mathcal{G} : N^2 \longrightarrow \widetilde{Gr}_2(\mathbb{R}^4) \cong Q_2(\mathbb{C}) \cong S^2 \times S^2$ is a minimal Lagrangian immersion.

2. If $N^n \subset S^{n+1}(1)$ ia an oriented *austere* hypersurface in $S^{n+1}(1)$ (Harvey-Lawson, 1982), then $\mathcal{G} : N^n \longrightarrow Q_n(\mathbb{C})$ is a minimal Lagrangian immersion.

3. If $N^n \to S^{n+1}(1)$ is an isoparametric hypersurface (i.e., κ_i are constant), then $\mathcal{G} : N^n \longrightarrow Q_n(\mathbb{C})$ is a minimal Lagrangian immersion.

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3. If $N^n \to S^{n+1}(1)$ is an isoparametric hypersurface (i.e., κ_i are constant), then $\mathcal{G} : N^n \longrightarrow Q_n(\mathbb{C})$ is a minimal Lagrangian immersion.

Definition of austere submanifold (Harvey-Lawson)

$N \subset M$: **austere submanifold** in a Riem. mfd. M
$\overset{\text{def}}{\Longleftrightarrow}$ for all $\eta \in T_x^\perp N$, the set of eigenvalues with their multiplicities of the shape operator A_η of N are invariant under the multiplication by -1.

- A minimal surface is an austere submanifold.
- An austere submanifold is a minimal submanifold.

Oriented hypersurface in a sphere

$N^n \hookrightarrow S^{n+1}(1) \subset \mathbb{R}^{n+2}$ with constant principal curvatures
("isoparametric hypersurface")

"Gauss map"

$$\mathcal{G} : N^n \ni p \underset{\text{Larg. imm.}}{\longmapsto} \mathbf{x}(p) \wedge \mathbf{n}(p) \in \widetilde{Gr}_2(\mathbb{R}^{n+2}) \cong Q_n(\mathbb{C})$$

Here $g := \#\{\text{distinct principal curvatures of } N^n\}$
$m_1, \cdots, m_g$: multiplicities of the principal curvatures.

(Münzner, 1980,1981):

- $m_i = m_{i+2}$ for each i;
- g must be $1, 2, 3, 4$ or 6;
- N is defined by a certain real homogeneous polynomial of degree g, called "Cartan-Münzner polynomial ".

$N^n \hookrightarrow S^{n+1}(1) \subset \mathbb{R}^{n+2}$ isoparametric hypersurface

$$\mathcal{G} : N^n \ni p \underset{\text{Lag. imm.}}{\longmapsto} \mathbf{x}(p) \wedge \mathbf{n}(p) \in \widetilde{Gr}_2(\mathbb{R}^{n+2}) \cong Q_n(\mathbb{C})$$

At $p \in N^n$, a normal geodesic γ defined by $\mathbf{x}_{\theta(p)} = \cos\theta\mathbf{x}(p) + \sin\theta\mathbf{n}(p)$ has intersection with N^n at $2g$ points as

$$\gamma \cap N = \{\mathbf{x}_\theta(p) | \theta = \frac{2\pi(\alpha-1)}{g} \text{ or } 2\theta_1 + \frac{2\pi(\alpha-1)}{g} \text{ for some } \alpha = 1, \cdots, g\}$$

For each $\mathbf{x}_\theta(p) \in \gamma \cap N^n$, let $p_\theta \in N$ be a point with $\mathbf{x}_\theta(p) = \mathbf{x}(p_\theta)$.
$\mathcal{G}(p) = \mathcal{G}(q)$ for $p, q \in N^n \Leftrightarrow q = p_\theta$ for some $\theta = \frac{2\pi(\alpha-1)}{g}$ $(\alpha = 1, 2, \cdots, g)$.

Then

$$\nu : N \ni p \mapsto \cos\frac{2\pi}{g}\mathbf{x}(p) + \sin\frac{2\pi}{g}\mathbf{n}(p) \in N$$

is a diffeomorphism of N onto itself of order g and $\{\mathrm{Id}, \nu, \cdots, \nu^{g-1}\}$ is a cyclic group of order g acting freely on N.

$$\boxed{\mathcal{G}(N^n) \cong N^n/\mathbb{Z}_g}$$

H. Ono's integral formula of Maslov index

Let L be a Lagrangian submanifold in a Kähler manifold (M, ω, J, g). For each smooth map of pairs $w : (D^2, \partial D^2) \to (M, L)$, it holds

$$I_{\mu,L}([w]) = \frac{1}{\pi}\int_{D^2} w^*\rho_M + \frac{1}{\pi}\int_{\partial D^2} w^*|_{\partial D^2}\alpha_H.$$

Proposition (H. Ono)

Suppose that (M, ω, J, g) is Einstein-Kähler with positive Einstein constant and L is a compact Lagrangian embedded submanifold in M. Then L is monotone $\Leftrightarrow$ $[\alpha_{\mathrm{H}}] = 0$ in $H^1(L, \mathbb{R})$.

Proposition (H. Ono)

Let (M, ω, J, g) be a simply connected Einstein-Kähler manifold with positive Einstein constant. If L is a compact monotone Lagrangian embedded submanifold in M, then L is cyclic and

$$n_L \Sigma_L = 2\gamma_{c_1}.$$

- $\gamma_{c_1}(Q_n(\mathbb{C})) = n$ for $n \geq 2$.

Proposition (M.-Ohnita)

The Gauss image of an isoparametric hypersurface $N^n \subset S^{n+1}(1)$

$$L^n = \mathcal{G}(N^n) \xrightarrow[\text{embedd.}]{\text{cpt. min. Lag.}} Q_n(\mathbb{C})$$

is a compact monotone and cyclic embedded Lagrangian submanifold and its minimal Maslov number Σ_L is given by

$$\Sigma_L = 2n/g = \begin{cases} m_1 + m_2, & \text{if } g \text{ is even;} \\ 2m_1, & \text{if } g \text{ is odd.} \end{cases}$$

$\Longrightarrow$

g	1	2	3	4	6
Σ_L	$2n$	n	$\frac{2n}{3}$	$\frac{n}{2}$	$\frac{n}{3}$

Isoparametric hypersurfaces in $S^{n+1}(1)$ I

All isoparametric hypersurfaces in $S^{n+1}(1)$ are classified into

- Homogeneous ones (Hsiang-Lawson, R. Takagi-T. Takahashi) can be obtained as principal orbits of the linear isotropy representations of Riemannian symmetric pairs (U, K) of rank 2.
  - $g = 1 : N^n = S^n$, a great or small sphere;
  - $g = 2, N^n = S^{m_1} \times S^{m_2}$, $(n = m_1 + m_2, 1 \leq m_1 \leq m_2)$, the Clifford hypersurfaces;
  - $g = 3$, N^n is homog., $N^n = \frac{SO(3)}{\mathbb{Z}_2+\mathbb{Z}_2}, \frac{SU(3)}{T^2}, \frac{Sp(3)}{Sp(1)^3}, \frac{F_4}{Spin(8)}$;
  - $g = 6$: homogenous
    - $g = 6, m_1 = m_2 = 1$: homog. (Dorfmeister-Neher, R. Miyaoka)
    - $g = 6, m_1 = m_2 = 2$: homog. (R. Miyaoka)
- Non-homogenous ones exist (H.Ozeki- M.Takeuchi) and are almost classified (Ferus-Karcher-Münzner, Cecil-Chi-Jensen, Immervoll, Chi).
  - $g = 4$: except for $(m_1, m_2) = (7, 8)$, either homog. or OT-FKM type.

Isoparametric hypersurfaces in $S^{n+1}(1)$ II

There exists only one minimal isoparametric hypersurface N^n in each isoparametric family of $S^{n+1}(1)$. Its principal curvatures are

- If $g = 1$, then $k_1 = 0$
- If $g = 2$, then $k_1 = \sqrt{\frac{m_2}{m_1}}$, $k_2 = -\sqrt{\frac{m_1}{m_2}}$
- If $g = 3$, then $k_1 = \sqrt{3}$, $k_2 = 0$, $k_3 = -\sqrt{3}$
- If $g = 4$, then

$$k_1 = \frac{\sqrt{m_1+m_2}+\sqrt{m_2}}{\sqrt{m_2}}, \qquad k_2 = \frac{\sqrt{m_1+m_2}-\sqrt{m_2}}{\sqrt{m_1}},$$
$$k_3 = -\frac{\sqrt{m_1+m_2}-\sqrt{m_2}}{\sqrt{m_1}}, \qquad k_4 = -\frac{\sqrt{m_1+m_2}+\sqrt{m_1}}{\sqrt{m_2}}$$

- If $g = 6$, then $m_1 = m_2 = 1$ or 2,

$$k_1 = 2+\sqrt{3}, \qquad k_2 = 1, \qquad k_3 = 2-\sqrt{3},$$
$$k_4 = -(2-\sqrt{3}), \quad k_5 = -1, \quad k_6 = -(2+\sqrt{3}).$$

Oriented hypersurface in a sphere

$N^n \hookrightarrow S^{n+1}(1) \subset \mathbb{R}^{n+2}$ with constant principal curvatures
("isoparametric hypersurface")

"Gauss map" and Gauss image

$$\mathcal{G} : N^n \ni p \underset{\text{min. Larg. imm.}}{\longmapsto} x(p) \wedge n(p) \in Q_n(\mathbb{C})$$

$$N^n \underset{\mathbb{Z}_g}{\longrightarrow} L^n = \mathcal{G}(N^n) \cong N^n/\mathbb{Z}_g \hookrightarrow Q_n(\mathbb{C})$$

cpt. embedded minimal Lagr. submfd

Proposition 2.1.

An isoparametric hypersurface $N^n \subset S^{n+1}(1)$ is homogeneous $\Longleftrightarrow$ $L^n = \mathcal{G}(N^n)$ is a compact homogeneous Lagrangian submanifold in $Q_n(\mathbb{C})$.

$N^n \hookrightarrow S^{n+1}(1)$: compact embedded isoparametric hypersurface

H-stability of the Gauss map. (Palmer)

Its Gauss map $\mathcal{G} : N \to Q_n(\mathbb{C})$ is H-stable $\Longleftrightarrow N^n = S^n \subset S^{n+1}$ $(g = 1)$.

Question

Hamiltonian stability of its Gauss image $\mathcal{G}(N^n) \subset Q_n(\mathbb{C})$?

We determine the Hamiltonian stability of Gauss images of ALL homogeneous isoparametric hypersurfaces.

$g = 1$: $N^n = S^n$ a great or small sphere
$L = \mathcal{G}(N^n) = Q_{1,n+1}(\mathbb{R}) \cong S^n$ is strictly H-stable

$g = 2$: $N^n = S^{m_1}(r_1) \times S^{m_2}(r_2)$, $(1 \leq m_1 \leq m_2, r_1^2 + r_2^2 = 1)$
$L = \mathcal{G}(N^n) = Q_{m_1+1,m_2+1}(\mathbb{R}) \cong (S^{m_1} \times S^{m_2})/\mathbb{Z}_2$ is H-stable
$\iff m_2 - m_1 < 3$

- If $m_2 - m_1 \geq 3$, then the spherical harmonics of degree 2 on $S^{m_1} \subset \mathbb{R}^{m_1+1}$ of smaller dimension give volume-decreasing Hamiltonian deformations of $\mathcal{G}(N^n)$.
- If $m_1 - m_2 = 2$, then it is H-stable but not strictly H-stable.
- If $m_1 - m_2 = 0$ or 1, then it is strictly H-stable.

Remark: $\mathcal{G}(N^n) = Q_{p,q}(\mathbb{R})$ totally geodesic for $g = 1, 2$.

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$\Sigma_L = 2n$

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- If $m_2 - m_1 \geq 3$, then the spherical harmonics of degree 2 on $S^{m_1} \subset \mathbb{R}^{m_1+1}$ of smaller dimension give volume-decreasing Hamiltonian deformations of $\mathcal{G}(N^n)$.
- If $m_1 - m_2 = 2$, then it is H-stable but not strictly H-stable.
- If $m_1 - m_2 = 0$ or 1, then it is strictly H-stable.

$\Sigma_L = n$

Remark: $\mathcal{G}(N^n) = Q_{p,q}(\mathbb{R})$ totally geodesic for $g = 1, 2$.

Theorem 3.1 (M.-Ohnita).

$$g = 3: \quad L = \mathcal{G}(N^n) = \begin{array}{l} SO(3)/(\mathbb{Z}_2 + \mathbb{Z}_2) \cdot \mathbb{Z}_3 \quad (m_1 = m_2 = 1) \\ SU(3)/T^2 \cdot \mathbb{Z}_3 \quad (m_1 = m_2 = 2) \\ Sp(3)/Sp(1)^3 \cdot \mathbb{Z}_3 \quad (m_1 = m_2 = 4) \\ F_4/Spin(8) \cdot \mathbb{Z}_3 \quad (m_1 = m_2 = 8) \end{array}$$

$\Longrightarrow L$ is strictly H-stable.

Theorem 3.2 (M.-Ohnita).

$$g = 6: \quad L = \mathcal{G}(N^n) = \begin{array}{l} SO(4)/(\mathbb{Z}_2 + \mathbb{Z}_2) \cdot \mathbb{Z}_6 \quad (m_1 = m_2 = 1) \\ G_2/T^2 \cdot \mathbb{Z}_6 \quad (m_1 = m_2 = 2) \end{array}$$

$\Longrightarrow L$ is strictly H-stable.

Theorem 3.1 (M.-Ohnita).

$$g=3: \quad L=\mathcal{G}(N^n)= \begin{array}{l} SO(3)/(\mathbb{Z}_2+\mathbb{Z}_2)\cdot\mathbb{Z}_3 \quad (m_1=m_2=1,\, \Sigma_L=2) \\ SU(3)/T^2\cdot\mathbb{Z}_3 \quad (m_1=m_2=2,\, \Sigma_L=4) \\ Sp(3)/Sp(1)^3\cdot\mathbb{Z}_3 \quad (m_1=m_2=4,\, \Sigma_L=8) \\ F_4/Spin(8)\cdot\mathbb{Z}_3 \quad (m_1=m_2=8,\, \Sigma_L=16) \end{array}$$

$\Longrightarrow$ L is strictly H-stable.

Theorem 3.2 (M.-Ohnita).

$$g=6: \quad L=\mathcal{G}(N^n)= \begin{array}{l} SO(4)/(\mathbb{Z}_2+\mathbb{Z}_2)\cdot\mathbb{Z}_6 \quad (m_1=m_2=1,\, \Sigma_L=2) \\ G_2/T^2\cdot\mathbb{Z}_6 \quad (m_1=m_2=2,\, \Sigma_L=4) \end{array}$$

$\Longrightarrow$ L is strictly H-stable.

Theorem 3.3 (M.-Ohnita).

$g = 4$, N^n homogeneous, $L = \mathcal{G}(N^n)$:

1. $L = SO(5)/T^2 \cdot \mathbb{Z}_4$ $(m_1 = m_2 = 2)$ is strictly H-stable.
2. $L = \frac{U(5)}{(SU(2)\times SU(2)\times U(1))\cdot\mathbb{Z}_4}$ $(m_1 = 4, m_2 = 5)$ is strictly H-stable.
3. $L = \frac{SO(2)\times SO(m)}{(\mathbb{Z}_2\times SO(m-2))\cdot\mathbb{Z}_4}$
   $$(m_1 = 1, m_2 = m-2, m \geq 3)$$
   L is NOT H-stable $\Longleftrightarrow m_2 - m_1 \geq 3$, i.e., $m \geq 6$.
4. $L = \frac{S(U(2)\times U(m))}{S(U(1)\times U(1)\times U(m-2))\cdot\mathbb{Z}_4}$
   $$(m_1 = 2, m_2 = 2m-3, m \geq 2)$$
   L is NOT H-stable $\Longleftrightarrow m_2 - m_1 \geq 3$, i.e., $m \geq 4$.
5. $L = \frac{Sp(2)\times Sp(m)}{(Sp(1)\times Sp(1)\times Sp(m-2))\cdot\mathbb{Z}_4}$
   $$(m_1 = 4, m_2 = 4m-5, m \geq 2)$$
   L is NOT H-stable $\Longleftrightarrow m_2 - m_1 \geq 3$, i.e., $m \geq 3$.
6. $L = \frac{U(1)\cdot Spin(10)}{(S^1\cdot Spin(6))\cdot\mathbb{Z}_4}$, $(m_1 = 6, m_2 = 9)$ is strictly H-stable.

Theorem 3.3 (M.-Ohnita).

$g = 4$, N^n homogeneous, $L = \mathcal{G}(N^n)$:

1. $L = SO(5)/T^2 \cdot \mathbb{Z}_4$ $(m_1 = m_2 = 2,\ \Sigma_L = 4)$ is strictly H-stable.
2. $L = \frac{U(5)}{(SU(2)\times SU(2)\times U(1))\cdot \mathbb{Z}_4}$ $(m_1 = 4, m_2 = 5,\ \Sigma_L = 9)$ is strictly H-stable.
3. $L = \frac{SO(2)\times SO(m)}{(\mathbb{Z}_2\times SO(m-2))\cdot \mathbb{Z}_4}$
$$(m_1 = 1, m_2 = m-2, m \geq 3,\ \Sigma_L = m-1)$$
L is NOT H-stable $\Longleftrightarrow m_2 - m_1 \geq 3$, i.e., $m \geq 6$.
4. $L = \frac{S(U(2)\times U(m))}{S(U(1)\times U(1)\times U(m-2))\cdot \mathbb{Z}_4}$
$$(m_1 = 2, m_2 = 2m-3, m \geq 2,\ \Sigma_L = 2m-1)$$
L is NOT H-stable $\Longleftrightarrow m_2 - m_1 \geq 3$, i.e., $m \geq 4$.
5. $L = \frac{Sp(2)\times Sp(m)}{(Sp(1)\times Sp(1)\times Sp(m-2))\cdot \mathbb{Z}_4}$
$$(m_1 = 4, m_2 = 4m-5, m \geq 2,\ \Sigma_L = 4m-1)$$
L is NOT H-stable $\Longleftrightarrow m_2 - m_1 \geq 3$, i.e., $m \geq 3$.
6. $L = \frac{U(1)\cdot Spin(10)}{(S^1\cdot Spin(6))\cdot \mathbb{Z}_4}$, $(m_1 = 6, m_2 = 9,\ \Sigma_L = 15)$ is strictly H-stable.

Summarize,

Theorem 3.4 (M.- Ohnita).

Suppose that (U, K) is not of type EIII,
then $L = \mathcal{G}(N)$ is not Hamiltonian stable if and only if $m_2 - m_1 \geq 3$.

Moreover, if (U, K) is of type EIII, that is, $(U, K) = (E_6, U(1) \cdot Spin(10))$,
then $(m_1, m_2) = (6, 9)$ but $L = \mathcal{G}(N)$ is strictly Hamiltonian stable.

Sketch of our proof

- $N^n \subset S^{n+1}(1)$ cpt. homog. isop. hypersurface
- $L = \mathcal{G}(N^n) \cong K/K_{[\mathfrak{a}]} \longrightarrow (Q_n(\mathbb{C}), g^{\mathrm{std}}_{Q_n(\mathbb{C})})$ cpt min. Lagr.
- $(Q_n(\mathbb{C}), g^{\mathrm{std}}_{Q_n(\mathbb{C})})$ cpt sym sp, E-K, $\kappa = n$
- In order to determine the Hamiltonian stability of $L = \mathcal{G}(N^n)$,
  we need to determine λ_1 of the Laplacian of L
  w.r.t. the induced metric from $(Q_n(\mathbb{C}), g^{\mathrm{std}}_{Q_n(\mathbb{C})})$
  based on the spherical function theory of compact homogeneous spaces
  and fibrations on homogeneous isoparametric hypersurfaces.

Homogeneous isoparametric hypersurfaces in $S^{n+1}(1)$

- (U, K): cpt. Riem. sym. pair of rank 2
- $\mathfrak{u} = \mathfrak{k} + \mathfrak{p}$, $\mathfrak{a} \subset \mathfrak{p}$: a maximal abelian subspace
- $\langle \, , \, \rangle_{\mathfrak{u}}$: Ad$U$-inv. inner product of $\mathfrak{u}$ defined by the Killing-Cartan form of $\mathfrak{u}$
- For each regular element H of $\mathfrak{a} \cap S^{n+1}(1)$, we have a homog. isop. hyp. in the unit sphere

$$N^n := (\mathrm{Ad}_{\mathfrak{p}} K)H \subset S^{n+1}(1) \subset \mathbb{R}^{n+2} \cong (\mathfrak{p}, \langle \, , \, \rangle_{\mathfrak{u}}|_{\mathfrak{p}}).$$

- Its Gauss image is

$$\mathcal{G}(N^n) = [(\mathrm{Ad}_{\mathfrak{p}} K)\mathfrak{a}] \subset \widetilde{\mathrm{Gr}}_2(\mathfrak{p}) \cong Q_n(\mathbb{C}).$$

Homogeneous spaces expressions:

$$\begin{aligned} N^n &\cong K/K_0 \\ L^n &= \mathcal{G}(N^n) \cong K/K_{[\mathfrak{a}]} \end{aligned}$$

where

$$K_0 := \{k \in K | \mathrm{Ad}_{\mathfrak{p}}(k)H = H\},$$
$$K_{\mathfrak{a}} := \{k \in K | \mathrm{Ad}_{\mathfrak{p}}(k)\mathfrak{a} = \mathfrak{a}\},$$
$$K_{[\mathfrak{a}]} := \{k \in K_{\mathfrak{a}} | \mathrm{Ad}_{\mathfrak{p}}(k) : \mathfrak{a} \to \mathfrak{a} \text{ preserves the orientation of } \mathfrak{a}\}.$$

The deck transformation group of the covering map $\mathcal{G} : N^n \to \mathcal{G}(N)$ equals to

$$K_{[\mathfrak{a}]}/K_0 = W(U,K)/\mathbb{Z}_2 \cong \mathbb{Z}_g,$$

where $W(U,K) = K_{\mathfrak{a}}/K_0$ is the Weyl group of (U,K).

Fibrations on homogenous isoparametric hypersurfaces by homogeneous isoparametric hypersurfaces

For $g = 4, 6$, (U, K) are of $\mathfrak{b}_2$, $\mathfrak{bc}_2$ or $\mathfrak{g}_2$ type.
In the case when (U, K) is of $\mathfrak{b}_2$ or $\mathfrak{g}_2$, we have one fibration as follows:

$$\begin{array}{c} N^n = K/K_0 \\ \Big\downarrow {\scriptstyle K_1/K_0} \\ K/K_1 \end{array}$$

When (U, K) is of type $\mathfrak{bc}_2$, we have the following two fibrations:

$$\begin{array}{ccc} N^n = K/K_0 & \xrightarrow{=} & K/K_0 \\ \Big\downarrow {\scriptstyle K_1/K_0} & & \Big\downarrow {\scriptstyle K_2/K_0} \\ K/K_1 & \xrightarrow{K_2/K_1} & K/K_2 \end{array}$$

In case $g = 6$ and $(U, K) = (G_2, SO(4))$, $(m_1, m_2) = (1, 1)$

$$N^6 = K/K_0 = SO(4)/\mathbf{Z}_2 + \mathbf{Z}_2 \subset S^7$$

$$\Big\downarrow {\scriptstyle K_1/K_0 = SO(3)/\mathbf{Z}_2 + \mathbf{Z}_2 \subset S^4}$$

$$K/K_1 = SO(4)/SO(3) \cong S^3$$

$$U/K = G_2/SO(4) \supset U_1/K_1 = SU(3)/SO(3)$$
$$K/K_0 = SO(4)/(\mathbf{Z}_2 + \mathbf{Z}_2) : g = 6, m_1 = m_2 = 1,$$
$$K_1/K_0 = SO(3)/(\mathbf{Z}_2 + \mathbf{Z}_2) : g = 3, m_1 = m_2 = 1.$$

In case $g = 6$ and $(U, K) = (G_2 \times G_2, G_2)$, $(m_1, m_2) = (2, 2)$

$$\begin{array}{c} N^{12} = K/K_0 = G_2/T^2 \subset S^{13} \\ \Big\downarrow {\scriptstyle K_1/K_0 = SU(3)/T^2 \subset S^7} \\ K/K_1 = G_2/SU(3) \cong S^6 \end{array}$$

$$U/K = (G_2 \times G_2)/G_2 \supset U_1/K_1 = (SU(3) \times SU(3))/SO(3)$$

$$K/K_0 = G_2/T^2 : g = 6, m_1 = m_2 = 2,$$

$$K_1/K_0 = SU(3)/T^2 : g = 3, m_1 = m_2 = 2.$$

In case $g=4$ and $(U,K)=(SO(10),U(5))$, $(m_1,m_2)=(4,5)$

$$\begin{array}{ccc}
N^{18}=\frac{U(5)}{SU(2)\times SU(2)\times U(1)} & \xrightarrow{\quad = \quad} & K/K_0=\frac{U(5)}{SU(2)\times SU(2)\times U(1)}\subset S^{19} \\
\Big\downarrow {\scriptstyle K_1/K_0=\frac{U(2)\times U(2)\times U(1)}{SU(2)\times SU(2)\times U(1)}\subset S^3} & & \Big\downarrow {\scriptstyle K_2/K_0=\frac{U(4)\times U(1)}{SU(2)\times SU(2)\times U(1)}\subset S^{11}} \\
K/K_1=\frac{U(5)}{U(2)\times U(2)\times U(1)} & \xrightarrow{K_2/K_1=\frac{U(4)\times U(1)}{U(2)\times U(2)\times U(1)}} & K/K_2=\frac{U(5)}{U(4)\times U(1)}
\end{array}$$

$$\frac{U}{K}=\frac{SO(10)}{U(5)}\supset_{\max}\frac{U_2}{K_2}=\frac{SO(8)\times SO(2)}{U(4)\times U(1)}\cong\widetilde{Gr}_2(\mathbf{R}^8)\quad(DIII(4)=BDI)$$

$$\supset_{\text{not max}}\frac{U_1}{K_1}=\frac{SO(4)\times SO(4)\times SO(2)}{U(2)\times U(2)\times U(1)}\cong S^2\times S^2\cong\widetilde{Gr}_2(\mathbf{R}^4).$$

$$\left(\frac{SO(4)}{U(2)}\cong S^2\right)$$

For cpt. homog. hyp. $N(\cong K/K_0) \subset S^{n+1}(1)$ given by (U, K) and $L = \mathcal{G}(N) \cong K/K_{[\mathfrak{a}]}$,

- Restricted root systems $\Sigma(U, K)$ are of $\mathfrak{a}_2$, $\mathfrak{b}_2$, $\mathfrak{bc}_2$ and $\mathfrak{g}_2$ types when $g = 3, 4$ or 6.
- The Casimir op. on L w.r.t. $\mathcal{G}^* g^{\mathrm{std}}_{Q_n(\mathbb{C})}$ can be split into 1, 2 or 3 Casimir operators on certain cpt. homog. spaces w.r.t. the corresponding invariant metrics.
- Compute the eigenvalues of Casimir op. (thus the Laplacian) by Freudanthal's formula and branching laws of irreducible representations of compact Lie groups.
- Compute $\mathcal{E} := \{\Lambda \in D(K, K_{[\mathfrak{a}]}) | -c(\Lambda) \leq n\}$.
- $L = \mathcal{G}(N^n) \to Q_n(\mathbb{C})$ is H-stable $\Longleftrightarrow \min \mathcal{E} = n$.

Classification of Homogeneous Lagr. submfds. in $\mathbb{C}P^n$ (Bedulli and Gori

16 examples of minimal Lagr. orbits in $\mathbb{C}P^n$
$= [5 \text{ examples with } \nabla S = 0] + [11 \text{ examples with } \nabla S \neq 0]$

$K \subset SU(n+1)$: cpt. simple subgroup

$L = K \cdot [v] \subset \mathbb{C}P^n$ Lagr. submfd.

$\Updownarrow$

complexified orbit (Zariski open)

$$K^{\mathbb{C}} \cdot [v] \subset \mathbb{C}P^n \quad \text{is Stein}$$

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$\Uparrow$

Classification Theory of "Prehomogeneous vector spaces"(Mikio Sato and Tatsuo Kimura)

Classification of Homogeneous Lagrangian submanifolds in complex hyperquadrics $Q_n(\mathbb{C})$ (M. and Ohnita)

Suppose

$$G \subset SO(n+2): \text{ cpt. subgroup },$$

$$L = G \cdot [W] \subset Q_n(\mathbb{C}) \quad \text{Lagr. submfd.}$$

$\Downarrow$

There exists

$$N^n \subset S^{n+1}(1) \subset \mathbb{R}^{n+2}: \text{ cpt. homog. isop. hypersurf.}$$

such that

1. $L = \mathcal{G}(N)$ and L is a cpt. minimal Lagr. submfd., or
2. L is a Lagrangian deformation of $\mathcal{G}(N)$.

Classification of Homogeneous Lagrangian submanifolds in complex hyperquadrics $Q_n(\mathbb{C})$ (M. and Ohnita)

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2. L is a Lagrangian deformation of $\mathcal{G}(N)$.

W.Y.Hsiang-H.B.Lawson's theorem (1971)

There is a compact Riemannian symmetric pair (U, K) of rank 2 such that

$$N = \mathrm{Ad}(K)v \subset S^{n+1}(1) \subset \mathbb{R}^{n+2} = \mathfrak{p},$$

where $\mathfrak{u} = \mathfrak{k} + \mathfrak{p}$ is the canonical decomposition of (U, K).

The second case happens only when (U, K) is one of

1. $(S^1 \times SO(3), SO(2))$,
2. $(SO(3) \times SO(3), SO(2) \times SO(2))$,
3. $(SO(3) \times SO(n+1), SO(2) \times SO(n))$ $(n \geq 3)$,
4. $(SO(m+2), SO(2) \times SO(m))$ $(n = 2m-2, m \geq 3)$.

If (U, K) is $(S^1 \times SO(3), SO(2))$,

then L is a small or great circle in $Q_1(\mathbb{C}) \cong S^2$.

If (U, K) is $(SO(3) \times SO(3), SO(2) \times SO(2))$,

then L is a product of small or great circles of S^2 in $Q_2(\mathbb{C}) \cong S^2 \times S^2$.

If (U, K) is $(SO(3) \times SO(n+1), SO(2) \times SO(n))$ $(n \geq 2)$,

then

$$L = K \cdot [W_\lambda] \subset Q_n(\mathbb{C}) \quad \text{for some } \lambda \in S^1 \setminus \{\pm\sqrt{-1}\},$$

where $K \cdot [W_\lambda]$ $(\lambda \in S^1)$ is the S^1-family of Lagr. or isotropic K-orbits satisfying

1. $K \cdot [W_1] = K \cdot [W_{-1}] = \mathcal{G}(N^n)$ is a tot. geod. Lagr. submfd. in $Q_n(\mathbb{C})$.
2. For each $\lambda \in S^1 \setminus \{\pm\sqrt{-1}\}$,
$$K \cdot [W_\lambda] \cong (S^1 \times S^{n-1})/\mathbb{Z}_2 \cong Q_{2,n}(\mathbb{R})$$
is a Lagr. orbit in $Q_n(\mathbb{C})$ with $\nabla S = 0$.
3. $K \cdot [W_{\pm\sqrt{-1}}]$ are isotropic orbits in $Q_n(\mathbb{C})$ with $\dim K \cdot [W_{\pm\sqrt{-1}}] = 0$.

If (U, K) is $(SO(m+2), SO(2) \times SO(m))$ $(n = 2m-2)$,

then

$$L = K \cdot [W_\lambda] \subset Q_n(\mathbb{C}) \quad \text{for some } \lambda \in S^1 \setminus \{\pm\sqrt{-1}\},$$

where $K \cdot [W_\lambda]$ $(\lambda \in S^1)$ is the S^1-family of Lagr. or isotropic orbits satisfying

1. $K \cdot [W_1] = K \cdot [W_{-1}] = \mathcal{G}(N^n)$ is a minimal (NOT tot. geod.) Lagr. submfd. in $Q_n(\mathbb{C})$.
2. For each $\lambda \in S^1 \setminus \{\pm\sqrt{-1}\}$,
$$K \cdot [W_\lambda] \cong (SO(2) \times SO(m))/(\mathbb{Z}_2 \times \mathbb{Z}_4 \times SO(m-2))$$
is a Lagr. orbit in $Q_n(\mathbb{C})$ with $\nabla S \neq 0$.
3. $K \cdot [W_{\pm\sqrt{-1}}] \cong SO(m)/S(O(1) \times O(m-1)) \cong \mathbb{R}P^{m-1}$ are isotropic orbits in $Q_n(\mathbb{C})$ with $\dim K \cdot [W_{\pm\sqrt{-1}}] = m-1$.

Further questions

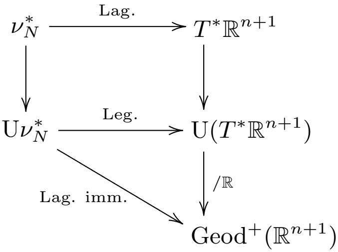
1. Investigate the Hamiltonian stability of the Gauss images of compact non-homogenous isoparametric hypersurfaces (OT-FKM type, embedded in spheres with $g = 4$).

2. Study other properties of the Gauss images in complex hyperquadrics.

3. Investigate the relation between our Gauss image construction and Karigiannis-Min-Oo's results.

4. Investigate further relations between hypersurfaces in M and Lagrangian submanifolds in $\mathrm{Geod}^+(M)$.

$N^m \subset \mathbb{R}^{n+1}$ submanifold



(Harvey-Lawson)

$\nu_N^* \subset T^*\mathbb{R}^{n+1}$ is Special Lagrangian with phase $i^m \;\Leftrightarrow\; N^m \subset \mathbb{R}^{n+1}$ austere.

$N^n \subset S^{n+1}(1)$ oriented hypersurface

$$\begin{array}{ccc}
\nu_N^* & \xrightarrow{\text{min.Lag.}} & T^*S^{n+1}(1) \\
\downarrow & & \downarrow \\
\mathrm{U}\nu_N^* & \xrightarrow{\text{min.Leg.}} & \mathrm{U}(T^*S^{n+1}(1)) \cong V_2(\mathbb{R}^{n+2}) \\
& \searrow^{\text{min. Lag. imm.}} & \downarrow /S^1 \\
& & \mathrm{Geod}^+(S^{n+1}(1)) \cong Q_n(\mathbb{C}) \subset \mathbb{C}P^{n+1}
\end{array}$$

(Karigiannis-Min-Oo)

$\nu_N^* \subset (T^*S^{n+1}, g_{\mathrm{Stenzel}})$ is Special Lagrangian $\Leftrightarrow$ $N^m \subset S^{n+1}$ austere.

- M: a complete Riemannian manifold which is a Hadamard mfd or a mfd with closed geodesics with the same length
- $U(T^*(M))$: the unit cotangent bundle of M
- $\mathrm{Geod}^+(M)$: the space of oriented geodesics of M

$$\begin{array}{ccc} & U(T^*M) & \\ {}^{p_1}\swarrow & & \searrow^{p_2} \\ M & & \mathrm{Geod}^+(M) \end{array}$$

- $\mathrm{Geod}^+(S^{n+1}(1)) \cong \widetilde{Gr}_2(\mathbb{R}^{n+2}) \cong Q_n(\mathbb{C})$.

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Thanks for your attention!